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ORDINARY DIFFERENTIAL EQUATIONS

# Matrix Linearization of Functional-Differential Equations of Point Type and Existence and Uniqueness of Periodic Solutions

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**Abstract**—We study $\omega$-periodic solutions of a functional-differential equation of point type that is $\omega$-periodic in the independent variable. In terms of the right-hand side of the equation, we state easy-to-verify sufficient conditions for the existence and uniqueness of an $\omega$-periodic solution and describe an iteration process for constructing the solution. In contrast to the previously considered scalar linearization, we use a more complicated matrix linearization, which permits extending the class of equations for which one can establish the existence and uniqueness of an $\omega$-periodic solution.

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## 1. INTRODUCTION. STATEMENT OF THE PROBLEM

The problem on the existence of $\omega$-periodic solutions of differential equations $\omega$-periodic in the independent variable is one of the classical problems of the theory of ordinary differential equations. This problem has been studied by various methods such as the Poincaré–Andronov method of point mappings [2, p. 328; 3, p. 66], the topological method, the method of directing functions [4, p. 72; 5, p. 172], variational methods, and so on.

The approach proposed in the present paper is in fact most close to the method of integral equations presented in detail in the monographs [6, p. 146] and [7, p. 26] and several related papers, where periodic and bounded solutions of differential equations were studied. In these papers, the study of the problem on periodic solutions was based on the construction of the operator Green function used then to construct such solutions themselves. The construction of the operator Green function and the verification of the conditions that must be satisfied by this function are rather laborious. To solve any specific problem, a large amount of nontrivial preliminary work is required. The question as to whether the constructed solution is classical should be studied separately.

The approach developed in this paper, which was considered under more restricting assumptions in [1], permits avoiding these difficulties. The conditions ensuring the existence and uniqueness of a classical $\omega$-periodic solution are easily verifiable and can be stated in terms of numerical characteristics of the right-hand side of the differential equation (such as the Lipschitz constant for the residual nonlinear perturbation, the value of deviations in the case of functional-differential equation of point type, and the coefficients of the linearized equation).

One specific characteristic of the approach considered here is the choice of an appropriate linearization of the right-hand side of the equation. As a rule, the most widely used method for isolating the linear part is the Taylor linearization. But there exist examples showing that the Taylor linearization does not always permit establishing the existence of a periodic solution, even though this can be done by other linearization methods. Varying the linear part, one can choose a linearization for which the nonresonance conditions are satisfied, and this permits correctly determining the periodic solution operator and solving the problem of minimizing the Lipschitz constant for the residual nonlinear perturbation if the nonresonance condition is satisfied. The nonresonance condition is "typical" and can be attained by using only the scalar linearization under which the linearized equation has the simplest form, but when the problem of the Lipschitz constant minimization for the residual nonlinear perturbation is further solved in the class of scalar linearizations, to obtain the desired value of this constant, it is necessary to impose rather rigid restrictions on the right-hand side of the equation, because this class of linearizations is narrower. That is why one needs to use the matrix linearization, studied in the present paper, of the original functional-differential equation.

Consider the functional-differential equation of point type

$$\dot{x}(t) = g(t, x(t+\tau_1), \ldots, x(t+\tau_s)), \qquad t \in \mathbb{R}, \tag{1}$$

where the function $g(\cdot)$ belongs to the space $C^{(r)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$, $r \in \mathbb{N}_0 \equiv \mathbb{N} \sqcup \{0\}$. A solution of Eq. (1) is an absolutely continuous function $x(\cdot)$ satisfying this equation. Since the right-hand side $g(\cdot)$ of Eq. (1) belongs to the space $C^{(r)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$, $r \in \mathbb{N}_0$, it follows that each of its solutions $x(\cdot)$ belongs to the space $C^{(r+1)}(\mathbb{R}; \mathbb{R}^n)$. In the paper, for Eq. (1) with right-hand side $2\pi$-periodic in time $t$, sufficient conditions for the existence and uniqueness of a $2\pi$-periodic solution are obtained, the iteration process of construction of such a solution is described, and the rate of convergence of the process is given. Obviously, the equation of any other period $\omega > 0$ can be reduced by the change $t = (2\pi)^{-1}\omega\tau$ of the independent variable to an equation of period $2\pi$.

Since the goal of this study is to investigate $2\pi$-periodic solutions, we can assume without loss of generality that all deviations $\tau_1, \ldots, \tau_s$ belong to the half-interval $[0, 2\pi)$. Indeed, if the deviations are such that $\tau_j \in [2\pi p_j, 2\pi(p_j+1))$, $j \in \{1, \ldots, s\}$, $p_j \in \mathbb{Z}$, then, replacing them with the corresponding deviations $\overline{\tau}_j = \tau_j - 2\pi p_j$ belonging to the half-interval $[0, 2\pi)$, we obtain an equation that obviously has the same $2\pi$-periodic solutions as the original one.

Moreover, in what follows we assume that the numbers $\tau_1, \ldots, \tau_s$ are pairwise commensurable; i.e., for any $\tau_i$ and $\tau_j$, $i, j \in \{1, \ldots, s\}$, there exist $n_{ji}, n_{ij} \in \mathbb{N}_0$ such that $n_{ji} + n_{ij} \neq 0$ and $n_{ji}\tau_i = n_{ij}\tau_j$.

We write Eq. (1) in the form

$$\dot{x}(t) = \sum_{j=1}^{s} A_j x(t+\tau_j) + f(t, x(t+\tau_1), \ldots, x(t+\tau_s)), \qquad t \in \mathbb{R}, \tag{2}$$

where $A_j$ is the real $n \times n$ matrix, $\tau_j \in [0, 2\pi)$, $j \in \{1, \ldots, s\}$, and

$$f(t, x(t+\tau_1), \ldots, x(t+\tau_s)) = g(t, x(t+\tau_1), \ldots, x(t+\tau_s)) - \sum_{j=1}^{s} A_j x(t+\tau_j). \tag{3}$$

In this paper, which continues and develops the study in [1], we obtain sufficient conditions on the matrices $A_j$, the deviations $\tau_j$, $j \in \{1, \ldots, s\}$, and the vector function $f(\cdot)$, for Eq. (1) to have a unique $2\pi$-periodic solution. The main distinction from [1] is that here we consider a more general case of separation of the linearized part; namely, the role of coefficients of the linearized part in Eq. (2) is played by the matrices $A_j$, while the case of separation of a scalar linear part was considered in [1], where Eq. (2) has the form

$$\dot{x}(t) = \sum_{j=1}^{s} a_j x(t+\tau_j) + f(t, x(t+\tau_1), \ldots, x(t+\tau_s)), \qquad t \in \mathbb{R},$$

with $a_j \in \mathbb{R}$. Thus, the results obtained in this paper extend the scope of possible applications of the proposed method.

Functional-differential equations of the form (1) were studied in the monograph [8], where, in particular, conditions [8, p. 45] on Eq. (1) were obtained ensuring the existence and uniqueness of a solution of the Cauchy problem

$$x(\overline{t}) = \overline{x}, \qquad \overline{t} \in \mathbb{R}, \qquad \overline{x} \in \mathbb{R}^n, \tag{4}$$

in a special class of functions. The conditions that must be satisfied by the function $g(\cdot)$ are the following:

I. The inclusion $g(\cdot) \in C^{(r)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$ holds for some $r \in \mathbb{N}_0$.

II. For all $t$, $x_j$, and $\overline{x}_j$, $j = 1, \ldots, s$, the inequalities

$$\|g(t, x_1, \ldots, x_s)\|_{\mathbb{R}^n} \le M_0(t) + M_1 \sum_{j=1}^{s} \|x_j\|_{\mathbb{R}^n}, \quad M_0(\cdot) \in C^{(0)}(\mathbb{R}; \mathbb{R}),$$

$$\|g(t, x_1, \ldots, x_s) - g(t, \overline{x}_1, \ldots, \overline{x}_s)\|_{\mathbb{R}^n} \le L_g \sum_{j=1}^{s} \|x_j - \overline{x}_j\|_{\mathbb{R}^n} \tag{5}$$

are satisfied. Note that the second inequality is the Lipschitz condition.

III. There exists a $\mu^* \in \mathbb{R}_+$ such that the expression

$$\sup_{i\in\mathbb{Z}} M_0(t+i)(\mu^*)^{|i|}$$

is finite for each $t \in \mathbb{R}$ and is a continuous function of the argument $t$.

IV. For the number $\mu^*$ in condition III, the family of functions

$$\tilde{g}_{i,z_1,\ldots,z_s}(t) = g(t+i, z_1, \ldots, z_s)(\mu^*)^{|i|}, \qquad i \in \mathbb{Z}, \qquad (z_1, \ldots, z_s) \in \mathbb{R}^{n\times s},$$

is equicontinuous on any finite interval.

In the above conditions I–IV and everywhere below, by $\|\cdot\|_{\mathbb{R}^n}$ we denote the Euclidean norm on the space $\mathbb{R}^n$.

Obviously, if the function $g(\cdot)$ is periodic, then condition III is automatically satisfied for all $\mu^* \in (0, 1]$.

We define the space

$$\mathcal{L}_\mu^n C^{(r)}(\mathbb{R}) = \Big\{ x(\cdot) : x(\cdot) \in C^{(r)}(\mathbb{R}; \mathbb{R}^n), \ \max_{0\le l\le r} \sup_{t\in\mathbb{R}} \|x^{(l)}(t) e^{-\delta|t|}\|_{\mathbb{R}^n} < +\infty \Big\},$$

where $r \in \mathbb{N}_0$ and $\mu = e^{-\delta}$.

**Theorem 1** [8, p. 45]. *If a function $g(\cdot)$ satisfies conditions* I–IV *and the inequality*

$$L_g \sum_{j=1}^{s} \mu^{-|\tau_j|} < \ln \mu^{-1} \tag{6}$$

*holds for some $\mu \in (0, \mu^*) \cap (0, 1)$, then for any $\overline{x} \in \mathbb{R}^n$ there exits a solution $x(\cdot) \in \mathcal{L}_\mu^n C^{(k)}(\mathbb{R})$ of the Cauchy problem* (1), (4). *This solution is unique and belongs to the class $\mathcal{L}_\mu^n C^{(r+1)}(\mathbb{R})$.*

If the function $g(\cdot) \in C^{(r)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$, $r \in \mathbb{N}_0$, on the right-hand side in Eq. (1) is $\omega$-periodic, then Theorem 1 can be restated as follows with regard to the fact that condition III is satisfied in this case.

**Corollary 1.** *Assume that the function $g(\cdot) \in C^{(r)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$, $r \in \mathbb{N}_0$, in Eq.* (1) *is $\omega$-periodic in time. If the function $g(\cdot)$ satisfies conditions* II *and* IV *and inequality* (6) *holds for some $\mu \in (0, \mu^*) \cap (0, 1)$, then for any $\overline{x} \in \mathbb{R}^n$ there exists a solution $x(\cdot) \in \mathcal{L}_\mu^n C^{(r)}(\mathbb{R})$ of the Cauchy problem* (1), (4). *This solution is unique and belongs to the class $\mathcal{L}_\mu^n C^{(r+1)}(\mathbb{R})$.*

We define the function space

$$V_{\mu^*}(\mathbb{R}\times\mathbb{R}^{n\times s};\mathbb{R}^n)=\{f(\cdot):\text{the function } f(\cdot) \text{ satisfies conditions I–III}\}.$$

For all functions in $V_{\mu^*}(\mathbb{R}\times\mathbb{R}^{n\times s};\mathbb{R}^n)$, the parameter $\mu^*\in\mathbb{R}_+$ coincides with the corresponding constant in condition III. In the space $V_{\mu^*}(\mathbb{R}\times\mathbb{R}^{n\times s};\mathbb{R}^n)$, one can introduce the Lipschitz norm

$$\|g(\cdot)\|_{\mathrm{Lip}}=\sup_{t\in\mathbb{R}}\|f(t,0,\ldots,0)e^{-\delta^*|t|}\|_{\mathbb{R}^n} + \sup_{(t,z_1,\ldots,z_s,\overline{z}_1,\ldots,\overline{z}_s)\in\mathbb{R}\times\mathbb{R}^{n\times s}\times\mathbb{R}^{n\times s}}\frac{\|g(t,z_1,\ldots,z_s)-g(t,\overline{z}_1,\ldots,\overline{z}_s)\|_{\mathbb{R}^n}}{\sum_{j=1}^s\|z_j-\overline{z}_j\|_{\mathbb{R}^n}},$$

where $\mu^*=e^{-\delta^*}$. Obviously, the least value of the constant $L_g$ in the Lipschitz condition (condition II) for the function $g(\cdot)\in V_{\mu^*}(\mathbb{R}\times\mathbb{R}^{n\times s};\mathbb{R}^n)$ coincides with the value of the second term in the definition of the norm $\|g(\cdot)\|_{\mathrm{Lip}}$. In what follows, when speaking about the Lipschitz condition, we understand the constant $L_g$ as this least value. The right-hand side of the functional-differential equation of point type will be viewed as an element of the Banach space $V_{\mu^*}(\mathbb{R}\times\mathbb{R}^{n\times s};\mathbb{R}^n)$.

To indicate that the solution of the Cauchy problem (1), (4) depends on the initial value $\overline{x}$ and on the right-hand side $g(\cdot)$ itself, we denote this solution by $x(t;\overline{t},\overline{x},g)$. The continuous dependence of the solution $x(\cdot)$ on the variables $\overline{t},\overline{x},g$ is understood as its continuous dependence on the variable $(\overline{t},\overline{x},g)\in\mathbb{R}^1\times\mathbb{R}^n\times V_{\mu^*}(\mathbb{R}\times\mathbb{R}^{n\times s};\mathbb{R}^n)$.

**Theorem 2** (“coarseness” theorem) [8, p. 47]. *In Theorem* 1 *and Corollary* 1, *the solution of the main Cauchy problem* (1), (4) *continuously depends on the variables* $\overline{t}$, $\overline{x}$, *and* $g$.

In Theorem 2, if the continuous dependence on the initial condition $\overline{x}$ and the right-hand side $g(\cdot)$ of the differential equation is considered, then one can state somewhat more.

**Remark 1** [8, p. 47]. In Theorem 2, the solution of the Cauchy problem (1), (4) as an element of the space $\mathcal{L}_\mu^nC^{(0)}(\mathbb{R})$ continuously depends on $\bar{x}$ and $g(\cdot)$.

Since we only consider periodic functions in what follows, we use the ordinary spaces $C^{(r)}(\mathbb{R};\mathbb{R}^n)$, $r\in\mathbb{N}_0$, of continuous functions rather than the spaces $\mathcal{L}_\mu^nC^{(r)}(\mathbb{R})$.

## 2. PROPERTIES OF PERIODIC SOLUTIONS OF THE LINEAR HOMOGENEOUS EQUATION

Let us establish several properties of linear functional-differential equations of point type.

Consider the homogeneous functional-differential equation of point type

$$\dot{x}(t)=\sum_{j=1}^s A_jx(t+\tau_j),\qquad t\in\mathbb{R},\tag{7}$$

where $A_j$ is a real $n\times n$ matrix and $\tau_j\in[0,2\pi)$, $j\in\{1,\ldots,s\}$.

Let us describe the set $(A_1,\ldots,A_s)$ of matrices and deviations $(\tau_1,\ldots,\tau_s)\in[0,2\pi)\times\ldots\times[0,2\pi)$ for which Eq. (7) has only the zero $2\pi$-periodic solution (the *nonresonance condition*). To this end, consider the following $2n\times 2n$ matrix consisting of four $n\times n$ blocks:

$$\mathbb{A}_k=\begin{pmatrix}-\sum_{j=1}^s A_j\cos(k\tau_j) & kI-\sum_{j=1}^s A_j\sin(k\tau_j)\\ -kI+\sum_{j=1}^s A_j\sin(k\tau_j) & -\sum_{j=1}^s A_j\cos(k\tau_j)\end{pmatrix},\qquad k\in\mathbb{N},\tag{8}$$

where $I$ is the $n\times n$ identity matrix. We introduce the notation

$$\overline{A}_k=-\sum_{j=1}^s A_j\cos(k\tau_j),\qquad \underline{A}_k=kI-\sum_{j=1}^s A_j\sin(k\tau_j).$$

In this notation, the matrix $\mathbb{A}_k$ becomes

$$\mathbb{A}_k = \begin{pmatrix} \overline{A}_k & \underline{A}_k \\ -\underline{A}_k & \overline{A}_k \end{pmatrix}, \qquad k \in \mathbb{N}.$$

We assume that the matrix $\mathbb{A}_k$ is invertible for all $k \in \mathbb{N}$.

In what follows, by $\|\cdot\|_{\mathbb{R}^n}$ we also denote the matrix spectral norm, i.e, the operator matrix norm induced by the Euclidean vector norm. Obviously, there exist constants $\widetilde{C}_1 > 0$ and $\widetilde{C}_2 > 0$ such that the estimates $\|\overline{A}_k\|_{\mathbb{R}^n} < \widetilde{C}_1$ and $\|\sum_{j=1}^{s} A_j \sin(k\tau_j)\|_{\mathbb{R}^n} < \widetilde{C}_2$ are satisfied for all $k \in \mathbb{N}$. It is also obvious that the norm $\|\underline{A}_k\|_{\mathbb{R}^n}$ tends to infinity at the rate $O(k)$ as $k \to \infty$.

Let us estimate the behavior of the norm $\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}$ as $k \to \infty$. Since, as is easily seen,

$$\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}} = \left( \min_{e \in \mathbb{R}^{2n},\ \|e\|_{\mathbb{R}^{2n}}=1} \|\mathbb{A}_k e\|_{\mathbb{R}^{2n}} \right)^{-1},$$

we see that it is then necessary to estimate the minimum

$$\min_{e \in \mathbb{R}^{2n},\ \|e\|_{\mathbb{R}^{2n}}=1} \|\mathbb{A}_k e\|_{\mathbb{R}^{2n}}. \tag{9}$$

Assume that $e_1, e_2 \in \mathbb{R}^n$. We form a vector $e = (e_1', e_2')'$ of the space $\mathbb{R}^{2n}$. (The prime stands for transposition.) Then

$$\mathbb{A}_k e = \begin{pmatrix} \overline{A}_k & \underline{A}_k \\ -\underline{A}_k & \overline{A}_k \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \overline{A}_k e_1 + \underline{A}_k e_2 \\ -\underline{A}_k e_1 + \overline{A}_k e_2 \end{pmatrix}$$

and

$$\|\mathbb{A}_k e\|_{\mathbb{R}^{2n}}^2 = \|\overline{A}_k e_1 + \underline{A}_k e_2\|_{\mathbb{R}^n}^2 + \|-\underline{A}_k e_1 + \overline{A}_k e_2\|_{\mathbb{R}^n}^2, \qquad \|e\|_{\mathbb{R}^{2n}}^2 = \|e_1\|_{\mathbb{R}^n}^2 + \|e_2\|_{\mathbb{R}^n}^2. \tag{10}$$

By the representations (10), under the assumption that $\|e\|_{\mathbb{R}^{2n}} = 1$, we have the lower bound

$$\begin{aligned} \|\mathbb{A}_k e\|_{\mathbb{R}^{2n}} &\geq 2^{-1}(\|\overline{A}_k e_1 + \underline{A}_k e_2\|_{\mathbb{R}^n} + \|-\underline{A}_k e_1 + \overline{A}_k e_2\|_{\mathbb{R}^n}) \\ &\geq 2^{-1}(\|\overline{A}_k e_2\|_{\mathbb{R}^n} - \|\underline{A}_k e_1\|_{\mathbb{R}^n} + \|\underline{A}_k e_1\|_{\mathbb{R}^n} - \|\overline{A}_k e_2\|_{\mathbb{R}^n}) \\ &\geq 2^{-1}(k(\|e_1\|_{\mathbb{R}^n} + \|e_2\|_{\mathbb{R}^n}) - 4\max\{\widetilde{C}_1, \widetilde{C}_2\}) \geq 2^{-1}k - 2\max\{\widetilde{C}_1, \widetilde{C}_2\}. \end{aligned}$$

On the other hand, the representations (10) obviously imply the upper bound

$$\|\mathbb{A}_k e\|_{\mathbb{R}^{2n}} \leq 2k + 4\max\{\widetilde{C}_1, \widetilde{C}_2\}.$$

Therefore, the minimum (9) tends to infinity at the rate $O(k)$ as $k \to \infty$, and hence the norm $\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}$ tends to zero at the rate $O(1/k)$ as $k \to \infty$. This assertion will be important for us below.

**Lemma 1.** *The homogeneous equation* (7) *has a unique* $2\pi$*-periodic solution if and only if the conditions*

$$\det \sum_{j=1}^{s} A_j \neq 0, \quad \det \mathbb{A}_k \neq 0 \quad \textit{for all} \quad k \in \mathbb{N} \tag{11}$$

*are satisfied. This* $2\pi$*-periodic solution is trivial. Otherwise, the homogeneous equation* (7) *has infinitely many* $2\pi$*-periodic solutions.*

**Proof.** Since the solutions of the homogeneous equation (7) belong to the space $C^{(1)}(\mathbb{R};\mathbb{R}^n)$, it follows that an arbitrary $2\pi$-periodic solution $x(t)$, $t\in\mathbb{R}$, can be represented on the interval $[0,2\pi]$ by a convergent Fourier series

$$x(t)=\alpha_0+\sum_{k=1}^{\infty}\alpha_{2k-1}\cos(kt)+\alpha_{2k}\sin(kt).$$

We substitute this representation into Eq. (7), match the coefficients of each of the basis functions 1, $\cos(kt)$, and $\sin(kt)$, and obtain the relations

$$-\Big(\sum_{j=1}^{s}A_j\Big)\alpha_0=0,\quad -\Big(\sum_{j=1}^{s}A_j\cos(k\tau_j)\Big)\alpha_{2k-1}+\Big(kI-\sum_{j=1}^{s}A_j\sin(k\tau_j)\Big)\alpha_{2k}=0,$$
$$\Big(-kI+\sum_{j=1}^{s}A_j\sin(k\tau_j)\Big)\alpha_{2k-1}-\Big(\sum_{j=1}^{s}A_j\cos(k\tau_j)\Big)\alpha_{2k}=0.$$

Therefore, for the existence of a nonzero $2\pi$-periodic solution of Eq. (7), it is necessary and sufficient to have the following nonstrict alternative: either $\det\sum_{j=1}^{s}A_j=0$ or $\det\mathbb{A}_k=0$ for some $k\in\mathbb{N}$. The proof of the lemma is compete.

## 3. PROPERTIES OF PERIODIC SOLUTIONS OF THE LINEAR INHOMOGENEOUS EQUATION

Consider a well-known property of periodic solutions which we need below.

**Proposition 1.** *Let the assumptions of Corollary* 1 *be satisfied for* $\omega=2\pi$. *Then the solution* $x(\cdot)$ *of Eq.* (1) *is* $2\pi$*-periodic if and only if* $x(0)=x(2\pi)$ *for it.*

**Proof.** The proof of this assertion readily follows from the $2\pi$-periodicity of the functions $g(\cdot)$ in the variable $t$ and the fact that the existence and uniqueness conditions are satisfied for the solution of the Cauchy problem (1), (4) (Corollary 1). The proof of the proposition is complete.

Consider the linear inhomogeneous equation

$$\dot{x}(t)=\sum_{j=1}^{s}A_jx(t+\tau_j)+\xi(t),\qquad t\in\mathbb{R},\tag{12}$$

where $A_j$ is a real $n\times n$ matrix, $\tau_j\in[0,2\pi)$, $j\in\{1,\dots,s\}$, and $\xi(\cdot)\in C^{(1)}(\mathbb{R};\mathbb{R}^n)$ is a $2\pi$-periodic function. Along with Eq. (12), we consider the corresponding linear homogeneous equation (7).

Conditions I–IV for the right-hand sides of Eqs. (7) and (12) are obviously satisfied for $\mu^*=1$. We define the constant

$$M=\max_{1\le j\le s}\|A_j\|_{\mathbb{R}^n}.\tag{13}$$

Then inequality (6) becomes

$$M\sum_{j=1}^{s}\mu^{-|\tau_j|}<\ln\mu^{-1}.\tag{14}$$

**Theorem 3.** *Assume that inequality* (14) *holds for some* $\mu\in(0,1)$. *Then a necessary and sufficient condition for the existence of a unique* $2\pi$*-periodic solution of the inhomogeneous equation* (12) *is that the unique* $2\pi$*-periodic solution of the corresponding homogeneous equation* (7) *is identically zero.*

**Proof.** Consider the fundamental matrix $\varphi(t)$ of Eq. (7) normalized at zero. It is a solution of the matrix equation

$$\dot{\varphi}(t)=\sum_{j=1}^{s}A_j\varphi(t+\tau_j),\qquad t\in\mathbb{R},$$

with the initial condition $\varphi(0) = I$. The existence of such a matrix follows from Corollary 1, which implies that any solution of the homogeneous equation (7) can be represented as $x(t) = \varphi(t)x(0)$ and an arbitrary solution of the inhomogeneous equation (12) can be represented as $x(t) = \varphi(t)x(0) + \psi(t)$, where $\psi(t)$ is a particular solution of Eq. (12). We use this fact to proceed directly to the proof of the theorem.

**Sufficiency.** Assume that the trivial solution is the unique $2\pi$-periodic solution of the homogeneous equation (7). Then it follows from Proposition 1 and the fact that the solutions of the homogeneous equation (7) can be represented as $x(2\pi) = \varphi(2\pi)x(0)$ that the equation $x = \varphi(2\pi)x$ must have only one solution $x = 0$. Therefore, we have $\det(I - \varphi(2\pi)) \neq 0$. On the other hand, for an arbitrary solution $x(\cdot)$ of the inhomogeneous equation (12) we have $x(2\pi) = \varphi(2\pi)x(0) + \psi(2\pi)$. Since the periodic solution satisfies the condition $x(0) = x(2\pi)$, the problem of determining the periodic solution of the inhomogeneous equation is reduced to solving the equation $(I - \varphi(2\pi))x = \psi(2\pi)$. Since $\det(I - \varphi(2\pi)) \neq 0$, this implies the uniqueness of the $2\pi$-periodic solution of the inhomogeneous equation (12).

**Necessity.** Assume that the inhomogeneous equation (12) has a unique $2\pi$-periodic solution. We argue by contradiction. Assume that the homogeneous equation (7) has at least one $2\pi$-periodic solution in addition to the zero solution. It follows that $\det(I - \varphi(2\pi)) = 0$. In this case, the equation $(I - \varphi(2\pi))x = \psi(2\pi)$ has either no or infinitely many solutions, which contradicts the uniqueness of the $2\pi$-periodic solution of the inhomogeneous equation (12). The proof of the theorem is complete.

When proving the necessity of the conditions of the theorem, we have made the conclusion that if the homogeneous equation (7) has a nonzero periodic solution, then the corresponding inhomogeneous equation (12) can have either infinitely many periodic solutions or none of them. Let us illustrate both of these possible cases by examples.

Consider the simplest one-dimensional ordinary differential equation $\dot{x}(t) = \xi(t)$, for which $A_j \equiv 0$, $j = 1, \ldots, s$, and the corresponding linear homogeneous equation $\dot{x} = 0$ has infinitely many periodic solutions $x(t) = C$, $C \in \mathbb{R}$. If we take the function $\xi(t) \equiv 1$, then the family of solutions of the inhomogeneous equation has the form $x(t) = t + C$, $C \in \mathbb{R}$; i.e., there exist no periodic solutions. On the other hand, if we set $\xi(t) \equiv \cos t$, then the solutions of the inhomogeneous equation become $x(t) = \sin t + C$, $C \in \mathbb{R}$; i.e., all the solutions are periodic.

Now, we use Theorem 3 and Lemma 1 to state a corollary refining Theorem 3.

**Corollary 2.** *Assume that inequality* (14) *holds for some* $\mu \in (0, 1)$. *The inhomogeneous equation* (12) *has a unique* $2\pi$*-periodic solution if and only if conditions* (11) *are satisfied for the* $n \times n$ *matrices* $A_j$, $j = 1, \ldots, s$, *and the deviations* $(\tau_1, \ldots, \tau_s) \in [0, 2\pi) \times \cdots \times [0, 2\pi)$.

## 4. PERIODIC SOLUTION OPERATOR

Consider the linear homogeneous equation (7) in which the deviations $\tau_j \in [0, 2\pi)$, $j \in \{1, \ldots, s\}$, are commensurable and for which conditions (11) are satisfied (and hence, by Corollary 2, the inhomogeneous equation (12) has a unique $2\pi$-periodic solution). Every homogeneous equation (7) of this kind determines the operator $\mathbb{P}$ that takes each $2\pi$-periodic function $\xi(\cdot) \in C^{(r)}(\mathbb{R}; \mathbb{R}^n)$ to the $2\pi$-periodic solution $x(\cdot)$ of the inhomogeneous equation (12). The operator $\mathbb{P}$ thus defined is called the periodic solution operator. Obviously, if $\xi(\cdot) \in C^{(r)}(\mathbb{R}; \mathbb{R}^n)$, then $\mathbb{P}\xi(\cdot) \in C^{(r+1)}(\mathbb{R}; \mathbb{R}^n)$.

For each $r \in \mathbb{N}_0$, we define the spaces

$$C_{2\pi}^{(r)}(\mathbb{R}; \mathbb{R}^n) = \{x(\cdot) \in C^{(r)}(\mathbb{R}; \mathbb{R}^n) : \; x^{(j)}(t) = x^{(j)}(t + 2\pi), \; j = 0, \ldots, r, \; t \in \mathbb{R}\}.$$

Therefore, the operator

$$\mathbb{P} : C_{2\pi}^{(r)}(\mathbb{R}; \mathbb{R}^n) \to C_{2\pi}^{(r+1)}(\mathbb{R}; \mathbb{R}^n), \qquad \mathbb{P}\xi(\cdot) = x(\cdot)$$

is defined for each $r \in \mathbb{N}_0$. Obviously, the operator $\mathbb{P}$ is linear and injective. We neither write the index $r$ on the operator $\mathbb{P}$ nor indicate its dependence on Eq. (7); this will not lead to a misunderstanding. Moreover, the operator $\mathbb{P}$ for $r \in \mathbb{N}$ is the restriction of the similar operator with index $(r - 1)$.

For each $r \in \mathbb{N}_0$, we introduce the spaces

$$C_{2\pi}^{(r),n} = \{x(\cdot) \in C^{(r)}([0, 2\pi]; \mathbb{R}^n) :\ x^{(j)}(0) = x^{(j)}(2\pi),\ j = 0, \dots, r\}.$$

In other words, the space $C_{2\pi}^{(r),n}$ consists of restrictions of the functions in the space $C_{2\pi}^{(r)}(\mathbb{R}; \mathbb{R}^n)$ to the interval $[0, 2\pi]$. The norm in this space is the same as in the space $C^{(r)}([0, 2\pi]; \mathbb{R}^n)$.

Since the mapping $\iota : C_{2\pi}^{(r)}(\mathbb{R}; \mathbb{R}^n) \to C_{2\pi}^{(r),n}$ that takes each function $y(\cdot) \in C_{2\pi}^{(r)}(\mathbb{R}; \mathbb{R}^n)$ to its restriction $\hat{y}(\cdot)$ to the interval $[0, 2\pi]$ is obviously a bijection, we see that the operator $\mathbb{P}$ generates the operator $\hat{\mathbb{P}} : C_{2\pi}^{(r),n} \to C_{2\pi}^{(r+1),n}$ defined by the relation $\hat{\mathbb{P}} = \iota \mathbb{P} \iota^{-1}$. Obviously, the operator $\hat{\mathbb{P}}$ is injective, and the mapping $\iota$ and its inverse are continuous.

Let $\mathbb{J} : C_{2\pi}^{(r+1),n} \to C_{2\pi}^{(r),n}$, $r \in \mathbb{N}_0$, be the operator of natural embedding. In what follows, the operator of periodic solutions is understood as a linear operator $\mathbb{J}\hat{\mathbb{P}} : C_{2\pi}^{(r),n} \to C_{2\pi}^{(r),n}$, $r \in \mathbb{N}_0$. Obviously, the operator $\mathbb{J}\hat{\mathbb{P}}$ is injective as well.

**Proposition 2.** *Assume that inequality* (14) *holds and conditions* (11) *are satisfied for some* $\mu \in (0, 1)$. *Then the operator*

$$\mathbb{J}\hat{\mathbb{P}} : C_{2\pi}^{(0),n} \to C_{2\pi}^{(0),n}$$

*is continuous.*

**Proof.** The proof readily follows from Theorem 2 and Remark 1.

In what follows, in addition to the continuity of the operator $\mathbb{J}\hat{\mathbb{P}}$, we also need an exact estimate of its norm. But it is a rather difficult problem to obtain such an estimate. In fact, it suffices to have estimates for the restriction of the operator in question to the subspace $C_{2\pi}^{(1),n}$. To this end, we set

$$\mathbb{A} = \left\| \left( \sum_{j=1}^{s} A_j \right)^{-1} \right\|_{\mathbb{R}^n}, \qquad \mathbb{D} = \left( \sum_{k=1}^{\infty} \|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}^2 \right)^{1/2}. \tag{15}$$

**Proposition 3.** *Let the assumptions of Proposition* 2 *be satisfied. Then the following estimate holds*:

$$\sup_{\substack{\hat{\xi}(\cdot) \in C_{2\pi}^{(1),n} \\ \|\hat{\xi}(\cdot)\|_{C_{2\pi}^{(0),n}} = 1}} \|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{C_{2\pi}^{(0),n}} \le \sqrt{\mathbb{A}^2 + 2\mathbb{D}^2}. \tag{16}$$

**Proof.** 1. *Construction of the action of the operator* $\mathbb{J}\hat{\mathbb{P}}$ *on the functions* $\hat{\xi}(\cdot)$ *of class* $C_{2\pi}^{(1),n}$ *in explicit form.* We use Fourier series to construct the operator $(\mathbb{J}\hat{\mathbb{P}})^{-1}$. We expand the periodic solution $x(\cdot)$ of Eq. (12) and the function $\xi(\cdot)$ on the right-hand side in this equation in Fourier series,

$$x(t) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_{2k-1} \cos(kt) + \alpha_{2k} \sin(kt)), \qquad \xi(t) = \beta_0 + \sum_{k=1}^{\infty} (\beta_{2k-1} \cos(kt) + \beta_{2k} \sin(kt)).$$

Since

$$\hat{\xi}(t) = \dot{\hat{x}}(t) - \sum_{j=1}^{s} A_j \hat{x}(t + \tau_j),$$

where the hat over a function means its restriction to the interval $[0, 2\pi]$, we replace the functions with their Fourier expansions and match the coefficients of like basis functions to obtain

$$\begin{aligned}
\beta_0 &= -\Big(\sum_{j=1}^{s} A_j\Big)\alpha_0,\\
\beta_{2k-1} &= -\Big(\sum_{j=1}^{s} A_j\cos(k\tau_j)\Big)\alpha_{2k-1} + \Big(kI - \sum_{j=1}^{s} A_j\sin(k\tau_j)\Big)\alpha_{2k},\\
\beta_{2k} &= \Big(-kI + \sum_{j=1}^{s} A_j\sin(k\tau_j)\Big)\alpha_{2k-1} - \Big(\sum_{j=1}^{s} A_j\cos(k\tau_j)\Big)\alpha_{2k}, \qquad k\in\mathbb{N}.
\end{aligned}$$

For each $k\in\mathbb{N}$, the coefficients $\beta_{2k-1}$ and $\beta_{2k}$ satisfy the matrix equation

$$\begin{pmatrix}\beta_{2k-1}\\ \beta_{2k}\end{pmatrix} = \mathbb{A}_k\begin{pmatrix}\alpha_{2k-1}\\ \alpha_{2k}\end{pmatrix},$$

where the $2n\times 2n$ matrix $\mathbb{A}_k$ is defined by formula (8).

It follows from the assumption of Proposition 2 that such matrices are nonsingular, and hence we have

$$\alpha_0 = -\Big(\sum_{j=1}^{s} A_j\Big)^{-1}\beta_0 \quad\text{and}\quad \begin{pmatrix}\alpha_{2k-1}\\ \alpha_{2k}\end{pmatrix} = \mathbb{A}_k^{-1}\begin{pmatrix}\beta_{2k-1}\\ \beta_{2k}\end{pmatrix}, \qquad k\in\mathbb{N}.$$

Thus, the operator $\mathbb{J}\hat{\mathbb{P}}$ has the following explicit form:

$$\begin{aligned}
(\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot))(t) &= \alpha_0 + \sum_{k=1}^{\infty}(\alpha_{2k-1}\cos(kt) + \alpha_{2k}\sin(kt))\\
&= -\Big(\sum_{j=1}^{s} A_j\Big)^{-1}\beta_0 + \sum_{k=1}^{\infty}\{((\overline{A}_k^2 + \underline{A}_k^2)^{-1}\underline{A}_k\beta_{2k-1} - (\overline{A}_k^2 + \underline{A}_k^2)^{-1}\underline{A}_k\beta_{2k})\cos(kt)\\
&\quad + ((\overline{A}_k^2 + \underline{A}_k^2)^{-1}\underline{A}_k\beta_{2k-1} + (\overline{A}_k^2 + \underline{A}_k^2)^{-1}\overline{A}_k\beta_{2k})\sin(kt)\}, \qquad t\in[0,2\pi].
\end{aligned}$$

2. *Majorant for the norm* $\|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{C_{2\pi}^{(0),n}}$, $\hat{\xi}(\cdot)\in C_{2\pi}^{(1),n}$, $\|\hat{\xi}(\cdot)\|_{C_{2\pi}^{(0),n}}\le 1$. To prove the estimate (16), we need to solve the following extremal problem:

$$\|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{C_{2\pi}^{(0),n}} \to \sup_{\hat{\xi}(\cdot)} \tag{17}$$

under the condition

$$\hat{\xi}(\cdot)\in C_{2\pi}^{(1),n}, \qquad \|\hat{\xi}(\cdot)\|_{C_{2\pi}^{(0),n}}\le 1. \tag{18}$$

By the obvious inequality $\alpha_{i,2k-1}\cos(kt)+\alpha_{i,2k}\sin(kt)\le\sqrt{\alpha_{i,2k-1}^2+\alpha_{i,2k}^2}$ (here the $\alpha_{i,m}$ and $\beta_{i,m}$ are the $i$th coordinates of the vectors $\alpha_m$ and $\beta_m$, respectively, $i=1,\dots,n$, $m\in\mathbb{N}_0$), we have the estimate

$$\|\alpha_{2k-1}\cos(kt)+\alpha_{2k}\sin(kt)\|_{\mathbb{R}^n}\le\sqrt{\|\alpha_{2k-1}\|_{\mathbb{R}^n}^2+\|\alpha_{2k}\|_{\mathbb{R}^n}^2}.$$

Further, it is easily seen that

$$\begin{aligned}
\|\alpha_{2k-1}\|_{\mathbb{R}^n}^2+\|\alpha_{2k}\|_{\mathbb{R}^n}^2 &= \left\|\begin{pmatrix}\alpha_{2k-1}\\ \alpha_{2k}\end{pmatrix}\right\|_{\mathbb{R}^{2n}}^2 = \left\|\mathbb{A}_k^{-1}\begin{pmatrix}\beta_{2k-1}\\ \beta_{2k}\end{pmatrix}\right\|_{\mathbb{R}^{2n}}^2 \le \|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}^2\left\|\begin{pmatrix}\beta_{2k-1}\\ \beta_{2k}\end{pmatrix}\right\|_{\mathbb{R}^{2n}}^2\\
&= \|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}^2(\|\beta_{2k-1}\|_{\mathbb{R}^n}^2+\|\beta_{2k}\|_{\mathbb{R}^n}^2).
\end{aligned}$$

With regard to the above-derived inequalities, we obtain the estimate

$$\|\alpha_{2k-1}\cos(kt) + \alpha_{2k}\sin(kt)\|_{\mathbb{R}^n} \le \sqrt{\|\mathbb{A}_k^{-1}\|^2_{\mathbb{R}^{2n}}(\|\beta_{2k-1}\|^2_{\mathbb{R}^n} + \|\beta_{2k}\|^2_{\mathbb{R}^n})}\,.$$

Thus, for each function $\hat{\xi}(\cdot) \in C^{(1),n}_{2\pi}$, $\|\hat{\xi}(\cdot)\|_{C^{(0),n}_{2\pi}} = 1$, the norm $\|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{C^{(0),n}_{2\pi}}$ can be majorized as follows:

$$\|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{C^{(0),n}_{2\pi}} \le \left\|\left(\sum_{j=1}^{s} A_j\right)^{-1}\beta_0\right\|_{\mathbb{R}^n} + \sum_{k=1}^{\infty}\left(\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}\sqrt{\|\beta_{2k-1}\|^2_{\mathbb{R}^n} + \|\beta_{2k}\|^2_{\mathbb{R}^n}}\right).$$

Below, we show that the series on the right-hand side of this inequality converges.

3. *Auxiliary extremal problem required to complete the estimation of the norm* $\|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{C^{(0),n}_{2\pi}}$. It is rather difficult to estimate the right-hand side of the last inequality. Thus, we replace such a problem by a simpler one. To this end, we maximize the value of the right-hand side of this inequality on the wider set of functions, $\|\hat{\xi}(\cdot)\|_{L_2([0,2\pi];\mathbb{R}^n)} \le \sqrt{2\pi}$. Then the extremal problem is stated as follows:

$$\left\|\left(\sum_{j=1}^{s} A_j\right)^{-1}\beta_0\right\|_{\mathbb{R}^n} + \sum_{k=1}^{\infty}\left(\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}\sqrt{\|\beta_{2k-1}\|^2_{\mathbb{R}^n} + \|\beta_{2k}\|^2_{\mathbb{R}^n}}\right) \to \sup_{\beta_k,\ k\in\mathbb{N}_0} \tag{19}$$

under the constraint

$$\|\hat{\xi}(\cdot)\|_{L_2([0,2\pi];\mathbb{R}^n)} \le \sqrt{2\pi}. \tag{20}$$

Obviously, the supremum of the functional in problem (19), (20) is in this case not less than the supremum of the functional in problem (17), (18). By the Parseval identity with respect to the orthogonal basis $\{1, \cos(kt), \sin(kt)\}_{k\in\mathbb{N}}$ of the space $L_2([0,2\pi];\mathbb{R}^n)$, we have

$$\|\hat{\xi}(\cdot)\|^2_{L_2([0,2\pi];\mathbb{R}^n)} = \int_0^{2\pi}\|\hat{\xi}(t)\|^2_{\mathbb{R}^n}\,dt = 2\pi\|\beta_0\|^2_{\mathbb{R}^n} + \pi\sum_{k=1}^{\infty}\|\beta_k\|^2_{\mathbb{R}^n}.$$

Therefore, the extremal problem (19), (20) can be rewritten in equivalent form as follows: Find the supremum (19) under the constraint

$$\|\beta_0\|^2_{\mathbb{R}^n} + \frac{1}{2}\sum_{k=1}^{\infty}\|\beta_k\|^2_{\mathbb{R}^n} \le 1. \tag{21}$$

4. *Completion of the proof of the proposition.* Consider the Hilbert space $l_2$ of numerical sequences and its elements $r_1$ and $r_2$ that we define as

$$r_1 = \left\{\frac{1}{\sqrt{2}}\left\|\left(\sum_{j=1}^{s} A_j\right)^{-1}\right\|_{\mathbb{R}^n}, \|\mathbb{A}_1^{-1}\|_{\mathbb{R}^{2n}}, \|\mathbb{A}_2^{-1}\|_{\mathbb{R}^{2n}}, \dots\right\},$$
$$r_2 = \left\{\sqrt{2}\|\beta_0\|_{\mathbb{R}^n}, \sqrt{\|\beta_1\|^2_{\mathbb{R}^n} + \|\beta_2\|^2_{\mathbb{R}^n}}, \sqrt{\|\beta_3\|^2_{\mathbb{R}^n} + \|\beta_4\|^2_{\mathbb{R}^n}}, \dots\right\}.$$

One can readily see that the sequence $r_1$ belongs to the space $l_2$, because, as was proved in 2, $\|\mathbb{A}_k^{-1}\|^2_{\mathbb{R}^{2n}} = O(1/k^2)$ as $k \to \infty$, and this just implies the inequality $\|r_1\|_{l_2} < +\infty$. The optimization problem (19), (21) has the following form in the new terminology: $(r_1, r_2)_{l_2} \to \sup_{r_2\in l_2}$ under the condition $\|r_2\|^2_{l_2} \le 2$. Applying the Cauchy–Schwarz inequality, we obtain the estimate

$$(r_1, r_2)_{l_2} \le \|r_1\|_{l_2}\|r_2\|_{l_2} \le \sqrt{2}\,\|r_1\|_{l_2}.$$

The norm of the element $r_1$ has the form

$$\|r_1\|_{l_2}^2 = \frac{1}{2}\bigg\|\bigg(\sum_{j=1}^{s} A_j\bigg)^{-1}\bigg\|_{\mathbb{R}^n}^2 + \sum_{k=1}^{\infty}\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}^2.$$

It is well known that the equality in the Cauchy–Schwarz inequality is attained only in the case of collinear vectors. Therefore, if we take $\overline{\beta}_k$, $k \in \mathbb{N}_0$, such that the vector $r_2$ becomes collinear to the vector $r_1$ and the relation $\|r_2\|_{l_2} = \sqrt{2}$ holds, then the original maximization problems is solved. Obviously, there exist $\overline{\beta}_k$, $k \in \mathbb{N}_0$, with this property. In this case, the objective functional (19) at the point of maximum takes the value

$$\bigg\|\bigg(\sum_{j=1}^{s} A_j\bigg)^{-1}\overline{\beta}_0\bigg\|_{\mathbb{R}^n} + \sum_{k=1}^{\infty}(\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}\sqrt{\|\overline{\beta}_{2k-1}\|_{\mathbb{R}^n}^2 + \|\overline{\beta}_{2k}\|_{\mathbb{R}^n}^2})$$
$$= \Bigg(\bigg\|\bigg(\sum_{j=1}^{s} A_j\bigg)^{-1}\bigg\|_{\mathbb{R}^n}^2 + 2\sum_{k=1}^{\infty}\|\mathbb{A}_k^{-1}\|_{\mathbb{R}^{2n}}^2\Bigg)^{1/2} = \sqrt{\mathbb{A}^2 + 2\mathbb{D}^2},$$

which implies the estimate (16). The proof of the proposition is complete.

**Remark 2.** Similar estimates for the norm of the periodic solution operator $\hat{\mathbb{JP}}$ were obtained for the case of scalar linearization in [1]. But in our case, the value of $\mathbb{D}$ in formula (15) is calculated in terms of the norms of the matrices $\mathbb{A}_k^{-1}$, $k \in \mathbb{N}$, while in [1], a similar quantity was calculated in terms of $1/\det A_k$ (see formula (16) in [1]). This can be explained by the fact that, for the scalar case, we have the relation $\|A_k^{-1}\|_{\mathbb{R}^2} = 1/\sqrt{\det A_k}$. (The matrix $A_k$ in [1] is an analog of the matrix $\mathbb{A}_k$.) Therefore, the result stated in this paper is a generalization of the result obtained in [1] for the scalar linearization. Note that there is no relationship between the values of the determinant $\det \mathbb{A}_k$ and the norm $\|\mathbb{A}_k\|_{\mathbb{R}^{2n}}$ except for the well-known inequality $(\det \mathbb{A}_k)^{1/(2n)} \le \|\mathbb{A}_k\|_{\mathbb{R}^{2n}}$ in the nonscalar case.

## 5. EXISTENCE AND UNIQUENESS OF A $2\pi$-PERIODIC SOLUTION OF THE NONLINEAR EQUATION

In this concluding section, we obtain conditions that ensure the existence and uniqueness of periodic solutions of the nonlinear functional-differential equation of point type (1), where $g(\cdot) \in C^{(1)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$ is a $2\pi$-periodic function in the variable $t$. Along with Eq. (1), we also consider Eq. (2) obtained by linearizing Eq. (1). If the function $g(\cdot)$ in Eq. (1) satisfies the Lipschitz condition with a constant $L_g$, then the function $f$ (defined by formula (3)) in Eq. (2) satisfies the Lipschitz condition with some constant $L_f$ as well. Associated with each linearization (2) of Eq. (1) is a linear inhomogeneous system (7). In turn, if the assumptions of Corollary 2 are satisfied for the matrices $A_j$, $j \in \{1,\ldots,s\}$, and the deviations $(\tau_1,\ldots,\tau_s) \in [0,2\pi) \times \cdots \times [0,2\pi)$, then the operator $\hat{\mathbb{JP}}$ is well defined.

We define an operator

$$\mathbb{F} : C_{2\pi}^{(r)}(\mathbb{R};\mathbb{R}^n) \to C_{2\pi}^{(r)}(\mathbb{R};\mathbb{R}^n), \qquad r = 0,1,$$

by the rule

$$\mathbb{F}[x(\cdot)](t) = f(t, x(t+\tau_1),\ldots,x(t+\tau_s)), \qquad t \in \mathbb{R}.$$

The operator $\mathbb{F}$ generates the operator

$$\hat{\mathbb{F}} : C_{2\pi}^{(r),n} \to C_{2\pi}^{(r),n}, \qquad r = 0,1,$$
$$\hat{\mathbb{F}}[\hat{x}(\cdot)](t) = f(t, \hat{x}(t+\tau_1),\ldots,\hat{x}(t+\tau_s)), \qquad t \in [0,2\pi],$$

where, just as above, the hat over a function means its restriction to the interval $[0,2\pi]$. In other words, $\hat{\mathbb{F}} = \iota\mathbb{F}\iota^{-1}$, and in particular, the correspondence $\mathbb{F} \to \hat{\mathbb{F}}$ is bijective, because $\mathbb{F} = \iota^{-1}\hat{\mathbb{F}}\iota$.

**Theorem 4.** *Let the following conditions be satisfied*:

(a) *The function* $g(\cdot) \in C^{(1)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$ *in the nonlinear equation* (1) *is* $2\pi$*-periodic in the variable* $t$ *and satisfies the Lipschitz condition* (5), *and the Lipschitz constant of the function* $f(\cdot)$ *is equal to* $L_f$.

(b) *Inequality* (14) *holds for some* $\mu \in (0,1)$ *with the constant* $M$ *given by relation* (13).

(c) *Conditions* (11) *are satisfied.*

*If the inequality*

$$sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} < 1, \tag{22}$$

*holds, where the numbers* $\mathbb{A}$ *and* $\mathbb{D}$ *are defined by relations* (15), *then Eq.* (1) *has a* $2\pi$*-periodic solution. Such a solution is unique and belongs to the space* $C^{(2)}_{2\pi}(\mathbb{R};\mathbb{R}^n)$.

*Moreover, for any initial function* $\hat{x}^0(\cdot) \in C^{(1),n}_{2\pi}$ *the sequence* $\hat{x}^m(\cdot) = (\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}})^m[\hat{x}^0(\cdot)]$, $m \in \mathbb{N}$, *converges in the norm of the space* $C^{(0),n}_{2\pi}$ *to the same function* $\hat{x}(\cdot) \in C^{(2),n}_{2\pi}$, *and the following estimate of the convergence rate takes place*:

$$\|(\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}})^m[\hat{x}^0(\cdot)] - \hat{x}(\cdot)\|_{C^{(0),n}_{2\pi}} \le \left(sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2}\right)^m \|\hat{x}^0(\cdot) - \hat{x}(\cdot)\|_{C^{(0),n}_{2\pi}}. \tag{23}$$

*The periodic solution* $x(\cdot) \in C^{(2)}_{2\pi}(\mathbb{R};\mathbb{R}^n)$ *of Eq.* (1) *is obtained by the* $2\pi$*-periodic continuation of the function* $\hat{x}(\cdot)$ *to the entire real line* $\mathbb{R}$; *i.e.*, $x(\cdot) = \iota^{-1}\hat{x}(\cdot)$.

**Proof.** In the space $C^{(0),n}_{2\pi}$, we define the operator equation

$$(\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{x}(\cdot)])(t) = \hat{x}(t), \qquad t \in [0, 2\pi]. \tag{24}$$

By Corollary 2, the $2\pi$-periodic continuation of any solution of Eq. (24) to the entire real line determines a periodic solution of Eq. (2) (of Eq. (1), respectively), and vice versa. Since the function $g(\cdot) \in C^{(1)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$ is $2\pi$-periodic, it follows that each solution of Eq. (24) belongs to the space $C^{(2),n}_{2\pi}$.

It follows from the Lipschitz conditions for the function $f(\cdot)$ that the inequality

$$\|\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{C^{(0),n}_{2\pi}} \le sL_f\|\hat{y}(\cdot) - \hat{z}(\cdot)\|_{C^{(0),n}_{2\pi}}$$

holds for any $\hat{y}(\cdot), \hat{x}(\cdot) \in C^{(0),n}_{2\pi}$. By Proposition 3, for any $\hat{y}(\cdot), \hat{z}(\cdot) \in C^{(1),n}_{2\pi}$ we have the chain of inequalities

$$\begin{aligned}\|\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{y}(\cdot)] - \mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{C^{(0),n}_{2\pi}} &= \|\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{C^{(0),n}_{2\pi}} \left\|\mathbb{J}\hat{\mathbb{P}}\left(\frac{\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]}{\|\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{C^{(0),n}_{2\pi}}}\right)\right\|_{C^{(0),n}_{2\pi}} \\ &\le \sqrt{\mathbb{A}^2 + 2\mathbb{D}^2}\|\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{C^{(0),n}_{2\pi}} \\ &\le sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2}\|\hat{y}(\cdot) - \hat{z}(\cdot)\|_{C^{(0),n}_{2\pi}}.\end{aligned} \tag{25}$$

By inequality (22), the estimate (25) means that the operator $\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}}$ is contracting in the subspace $C^{(1),n}_{2\pi}$ of the metric space $C^{(0),n}_{2\pi}$. Since the subspace $C^{(1),n}_{2\pi}$ is invariant under the action of the operator $\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}}$ (because, by assumption, $g \in C^{(1)}(\mathbb{R} \times \mathbb{R}^{n\times s}; \mathbb{R}^n)$) and is complete, it follows that this operator has a unique fixed point $\hat{x}(\cdot)$ in this subspace, and the sequence $(\mathbb{J}\hat{\mathbb{P}}\mathbb{F})^m[\hat{x}^0(\cdot)](t)$, $m \in \mathbb{N}_0$, converges to this point at the rate (23) for any initial function $\hat{x}^0(\cdot) \in C^{(1),n}_{2\pi}$. The proof of the theorem is complete.

We point out that when a linearization is chosen in Theorem 4, the linear part contains only the deviations which are present in the right-hand side of the original functional-differential equation (1). This is essential, because if it turns out that the linearization is chosen so that the function

$f(\cdot)$ has at least one deviation $\overline{\tau}$ that does not coincide with any of the deviations $\tau_1, \dots, \tau_s$, then one can readily verify that the inequality $sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} < 1$ is, as a rule, violated.

**Example.** Consider an example in which the scalar linearization considered in [1] does not permit proving the existence of a unique periodic solution, while the matrix linearization allows one to prove the existence of such a solution.

Consider the problem

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \varepsilon f(t, x_1(t+\tau), x_2(t+\tau)).$$

The function $f(\cdot)$ is $2\pi$-periodic in the time variable with Lipschitz constant $L_f$, and the deviation $\tau$ is chosen to be such that the nonresonance conditions (11) are satisfied. The quantity $\varepsilon > 0$ is a sufficiently small parameter. In this problem, for the matrix linearization it is most natural to take the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

One can readily see that if we take a sufficiently small value of the parameter $\varepsilon > 0$, then the inequality $2\varepsilon L_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} < 1$ is satisfied, which, in turn, guarantees the existence of a unique $2\pi$-periodic solution of such an equation.

Now let us verify whether it is possible to determine the existence of a unique $2\pi$-periodic solution for a given equation by isolating the scalar linear part. In this case, the equation must be written in the form

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = a\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} - a\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \varepsilon f(t, x_1(t+\tau), x_2(t+\tau)),$$

and it is necessary to verify whether there exists a value of the parameter $a \in \mathbb{R}$ for which the inequality $2L_{\tilde{f}}\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} < 1$ holds, where $\mathbb{A} = 1/|a|$, $\mathbb{D} = \sum_{k=1}^{\infty} 1/\det A_k$ (see formula (16) and Theorem 4 in [1]), and $L_{\tilde{f}}$ is the Lipschitz constant of the function

$$\tilde{f}(t, x(t), x(t+\tau)) = \begin{pmatrix} -1-a & 0 \\ 0 & 1-a \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \varepsilon f(t, x_1(t+\tau), x_2(t+\tau)).$$

Obviously,

$$\left\| \begin{pmatrix} -1-a & 0 \\ 0 & 1-a \end{pmatrix} \right\|_{\mathbb{R}^2} = 1 + |a|$$

and the inequalities $L_{\tilde{f}} \ge 1 + |a|$ and $\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} > \mathbb{A} = 1/|a|$ hold, which implies the estimate $2L_{\tilde{f}}\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} > 2(1 + |a|)/|a| > 2$. Therefore, the criterion $2L_{\tilde{f}}\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} < 1$ for the existence of a $2\pi$-periodic solution is not satisfied for the scalar linearization for any values of $\varepsilon$ and $a \in \mathbb{R}$, despite the fact that such a solution exists for sufficiently small $\varepsilon$.

## 6. VARIATIONAL PROBLEM RELATED TO THE PROBLEM OF EXISTENCE OF A $2\pi$-PERIODIC SOLUTION OF THE NONLINEAR EQUATION

The existence and uniqueness of a $2\pi$-periodic solution of the original nonlinear equation (1) was established by studying the properties of the linearization of the right-hand side of Eq. (2). The criterion for the existence and uniqueness of a periodic solution is stated as the strict inequality (22). Therefore, the statement of the following variational problem is natural.

**Variational problem.** Minimize the functional

$$J(A_1, \dots, A_s) = sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} \to \inf_{A_1,\dots,A_s \in \mathbb{M}_n(\mathbb{R})} \tag{26}$$

under the constraints

$$M\sum_{j=1}^{s}\mu^{-|\tau_j|} < \ln\mu^{-1},$$

where $M=\max_{1\le j\le s}\|A_j\|_{\mathbb{R}^n}$, $\sum_{j=1}^{s}A_j\in\mathbb{GL}_n(\mathbb{R})$, and $\mathbb{A}_k\in\mathbb{GL}_{2n}(\mathbb{R})$ for all $k\in\mathbb{N}$. Here $\mathbb{M}_n(\mathbb{R})$ is the ring of real $n\times n$ matrices and $\mathbb{GL}_n(\mathbb{R})$ is the group of invertible real $n\times n$ matrices. The quantities $\mathbb{A}$ and $\mathbb{D}$ are determined by formulas (15), and the matrices $\mathbb{A}_k$, $k\in\mathbb{N}$ are given by formulas (8). Let us state a theorem on the existence and uniqueness of a periodic solution in terms of the variational problem.

**Theorem 5.** *Assume that the function $g(\cdot)\in C^{(1)}(\mathbb{R}\times\mathbb{R}^{n\times s};\mathbb{R}^n)$ in the nonlinear functional-differential equation* (1) *is $2\pi$-periodic in the variable $t$ and satisfies the Lipschitz condition* (5). *If the infimum $J_*$ of the functional in the variational problem* (26) *satisfies the inequality*

$$J_*<1,$$

*then Eq.* (1) *has a $2\pi$-periodic solution. This solution is unique and belongs to the space $C^{(2)}_{2\pi}(\mathbb{R};\mathbb{R}^n)$. Moreover, for any initial function $\hat{x}^0(\cdot)\in C^{(1),n}_{2\pi}$ the sequence $\hat{x}^m(\cdot)=(\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}})^m[\hat{x}^0(\cdot)]$, $m\in\mathbb{N}$, converges in the norm of the space $C^{(0)}_{2\pi}(\mathbb{R};\mathbb{R}^n)$ to one and the same function $\hat{x}(\cdot)\in C^{(2),n}_{2\pi}$ and the rate of convergence satisfies the estimate*

$$\|(\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}})^m[\hat{x}^0(\cdot)]-\hat{x}(\cdot)\|_{C^{(0),n}_{2\pi}}\le J_*^m\|\hat{x}^0(\cdot)-\hat{x}(\cdot)\|_{C^{(0),n}_{2\pi}}. \tag{27}$$

*The periodic solution $x(\cdot)\in C^{(2)}_{2\pi}(\mathbb{R};\mathbb{R}^n)$ of Eq.* (1) *is obtained by the $2\pi$-periodic continuation of the solution $\hat{x}(\cdot)$ to the entire real line $\mathbb{R}$.*

**Proof.** Take a sequence $(A_1(k),\dots,A_s(k))$, $k\in\mathbb{N}$, such that $J(A_1(k),\dots,A_s(k))\to J_*$ as $k\to\infty$. Starting from a sufficiently large $N$, all assumptions of Theorem 4 are satisfied for the linearization of the right-hand side of the original equation (1) with the matrices $A_1(k),\dots,A_s(k)$, $k\ge N$, belonging to this sequence. Therefore, for such an equation, there exists a unique $2\pi$-periodic solution satisfying the following estimates for $k\ge N$:

$$\|(\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}})^m[\hat{x}^0(\cdot)]-\hat{x}(\cdot)\|_{C^{(0),n}_{2\pi}}\le (sL_f\sqrt{\mathbb{A}(k)^2+2\mathbb{D}(k)^2})^m\|\hat{x}^0(\cdot)-\hat{x}(\cdot)\|_{C^{(0),n}_{2\pi}}.$$

If we take the limit as $k\to\infty$ of the right-hand side of this inequality, then it becomes the corresponding inequality (27). The proof of the theorem is complete.

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