ISSN 0012-2661, Differential Equations, 2018, Vol. 54, No. 9, pp. 1225–1235. © Pleiades Publishing, Ltd., 2018.
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INTEGRAL EQUATIONS

Discretization Methods for Three-Dimensional Singular Integral Equations of Electromagnetism

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Received April 1, 2018

Abstract—Theorems providing the convergence of approximate solutions of linear operator equations to the solution of the original equation are proved. The obtained theorems are used to rigorously mathematically justify the possibility of numerical solution of the 3D singular integral equations of electromagnetism by the Galerkin method and the collocation method.

DOI: 10.1134/S0012266118090100

INTRODUCTION

To solve linear operator equations considered in infinitely-dimensional Hilbert spaces, for example, integral equations, numerically, it is first necessary to discretize the problem, i.e., reduce it to a system of linear algebraic equations (SLAE). Then the following two basic questions arise:

1. How close is the solution obtained by using SLAE to the solution of the original problem?

2. Are the properties of the original operator, which permit using the iteration methods for solving the SLAE, preserved when passing to a finite-dimensional operator?

The answer to the first question does not depend on the solution method but is determined by the method of approximation of the original operator and its properties. Theoretically, it is here important to prove the possibility of obtaining a solution with an arbitrary accuracy in principle. The answer to the second question depends on the properties of the original operator, the discretization method, and the applied iteration method.

In the paper, we introduce a sequence of finite-dimensional spaces belonging to the Hilbert space under study and, in these spaces, write the equations approximating the original operator equation. We prove several theorems ensuring that the approximate solutions converge to the solution of the original problem. Then we consider the 3D singular integral equations describing the problems of interaction of the electromagnetic field with three-dimensional inhomogeneous dielectric structures. Based on the obtained theorems, we mathematically rigorously justify the possibility of numerically solving these equations by using the Galerkin method and the collocation method.

OPERATOR EQUATIONS

Consider the operator equation

$$\hat{A}u = f \tag{1}$$

with a bounded linear operator $\hat{A}$ acting in a Hilbert space H.

First, we describe the methods for solving linear equation (1) based on the Galerkin method and the iteration method of minimal discrepancies. Let $H_N \subset H$ be a finite-dimensional subspace of dimension N with a basis $\{v_n^{(N)}\}$, $n = 1, \dots, N$. We seek an approximate solution u_N of Eq. (1) on the basis of the Galerkin method

$$u_N = \sum_{n=1}^{N} \alpha_n^{(N)} v_n^{(N)}, \tag{2}$$

where the unknown coefficients $\alpha_n^{(N)}$ are determined from the system of N linear algebraic equations

$$\sum_{n=1}^{N} \alpha_n^{(N)} A_{nm}^{(N)} = (f, v_m^{(N)}), \tag{3}$$

$$A_{nm}^{(N)} = (\hat{A} v_n^{(N)}, v_m^{(N)}). \tag{4}$$

Equations (3), (4) are obtained from the requirement that the discrepancy in the approximate solution $h_N = \hat{A} u_N - f$ must be orthogonal to the subspace H_N.

Let $\hat{O}_N$ be the operator of orthogonal projection of the space H onto the subspace H_N; i.e., let $(v - \hat{O}_N v, \omega) = 0$ for any $v \in H$, $\omega \in H_N$. Then Eqs. (3), (4) can be written in the operator form

$$\hat{O}_N \hat{A} u_N = \hat{O}_N f, \qquad u_N \in H_N. \tag{5}$$

Let us state the following conditions.

Condition A. *A sequence of subspaces $\{H_N\}$ is ultimately dense in H; i.e., for any vector $u \in H$ there exist vectors $\tilde{u}_N \in H_N$, $N = 1, 2, \ldots$, such that*

$$\|u - \tilde{u}_N\| = \inf_{\omega \in H_N} \|u - \omega\| \le \varepsilon(u, N),$$

where the $\varepsilon(u, N)$ are the approximation error estimates and $\varepsilon(u, N) \to 0$ as $N \to \infty$.

Condition B. *For the operator $\hat{O}_N$ of orthogonal projection of the space H onto the subspaces H_N and any vector $v_N \in H_N$, the inequality*

$$\|(\hat{O}_N \hat{A} - \hat{A}) v_N\| \le \delta(N) \|v_N\|$$

is satisfied, and $\delta(N) \to 0$ as $N \to \infty$. Moreover, $\|\hat{O}_N f - f\| \to 0$ as $N \to \infty$.

We have the following assertion [1, p. 26; 2, p. 9].

Theorem 1. *Assume that $\hat{A}$ is a bounded linear operator acting in the Hilbert space H and such that for any $v \in H$ the inequality*

$$|(\hat{A} v, v)| \ge p_0 (v, v), \qquad p_0 > 0, \tag{6}$$

holds. Then there exists a unique solution of Eq. (1) *in the space H, the norm of the inverse operator $\hat{A}^{-1}$ satisfies the estimate $\|\hat{A}^{-1}\| \le 1/p_0$, and the iterations*

$$u_{n+1} = u_n - \tau_n h_n, \qquad h_n = (\hat{A} u_n - f), \qquad \tau_n = \frac{(h_n, \hat{A} h_n)}{(\hat{A} h_n, \hat{A} h_n)}, \qquad n = 0, 1, \ldots, \tag{7}$$

converge to the solution of Eq. (1) *for any initial approximation $u_0 \in H$.*

An important special case of inequality (6) is the condition

$$|\operatorname{Im}(\hat{A} v, v)| \ge p_0 (v, v), \qquad p_0 > 0, \tag{8}$$

which is satisfied for many problems of mathematical physics and will be used below.

The iteration procedure (7) is called the iteration method of minimal discrepancies (MMD). There exist many generalizations of the scheme (7) for which the convergence of iterations can be faster but the convergence condition is the same.

Now we prove the following assertion.

Theorem 2. *Assume that for any $v \in H$ inequality* (6) *and condition A or condition B are satisfied. Then there exists a unique solution of the system of linear equations* (3), (4), *which can be obtained by using the iteration MMD. Here the sequence u_N converges to the solution of Eq.* (1); *i.e.*, $\lim_{N\to\infty} \|u - u_N\| = 0$.

Proof. Let $b^{(N)} = \{b_n^{(N)}\}$ be an N-dimensional complex vector in the Euclidean space $\mathbb{E}_N$, and let $A^{(N)} = (A_{nm}^{(N)})$ be the linear operator defined in this space by the expression (4). Then, by (4) and (6), we have

$$\begin{aligned}|(A^{(N)}b^{(N)}, b^{(N)})_{\mathbb{E}_N}| &= \left|\sum_{n,m=1}^{N} A_{nm}^{(N)} b_n^{(N)} (b_m^{(N)})^*\right| \\ &= \left|\left(\hat{A}\left[\sum_{n=1}^{N} b_n^{(N)} \upsilon_n^{(N)}\right], \left[\sum_{m=1}^{N} b_m^{(N)} \upsilon_m^{(N)}\right]\right)_H\right| \ge \rho_0 q_0 (b^{(N)}, b^{(N)})_{\mathbb{E}_N},\end{aligned}$$

where q_0 is the smallest eigenvalue of the Hermitian positive definite matrix $\beta_{nm} = (\upsilon_n^{(N)}, \upsilon_m^{(N)})$, $n, m = 1, \dots, N$.

Thus, by Theorem 1, there exists a unique solution of system (3), (4), and a numerical solution of it can be obtained by MMD iterations. Note that, by this theorem, there exists a unique solution of Eq. (1) in the space H as well.

Further, multiplying (3) by $(\alpha_m^{(N)})^*$ and summing over m with regard to (2) and (4), we obtain

$$(\hat{A}u_N, u_N) = (f, u_N), \tag{9}$$

whence, by inequality (6),

$$\|u_N\| \le p_0^{-1}\|f\|. \tag{10}$$

Let condition A be satisfied. Then the solution u of Eq. (1) can be represented as

$$u = \tilde{u}_N + x_N, \qquad \tilde{u}_N \in H_N, \qquad \lim_{N\to\infty} \|x_N\| = 0. \tag{11}$$

By relations (6), (9), with regard to $\hat{A}u = f$, we obtain

$$\begin{aligned}p_0\|u - u_N\|^2 &\le |(\hat{A}(u - u_N), (u - u_N))| = |(f - \hat{A}u_N, u - u_N)| = |(f - \hat{A}u_N, u)| \\ &= |(f - \hat{A}u_N, \tilde{u}_N) + (f - \hat{A}u_N, x_N)|.\end{aligned} \tag{12}$$

Further, since $\tilde{u}_N \in H_N$, we have $(f - \hat{A}u_N, \tilde{u}_N) = 0$. Therefore, inequalities (10) and (12) imply the estimate

$$\|u - u_N\|^2 \le \frac{1}{p_0}\left[\|f\| + \frac{\|\hat{A}\|}{p_0}\|f\|\right]\|x_N\|,$$

whence, with regard to representation (11), we conclude that $\|u - u_N\| \to 0$ as $N \to \infty$.

Let condition B be satisfied. By Eqs. (5) and (1), we have

$$\hat{A}(u_N - u) + (\hat{O}_N\hat{A} - \hat{A})u_N = \hat{O}_N f - f. \tag{13}$$

Applying the operator $\hat{A}^{-1}$ to both parts of (13) and taking into account the estimate (10) and condition B, we obtain

$$\|u - u_N\| \le \frac{1}{p_0}\left[\frac{1}{p_0}\delta(N)\|f\| + \|\hat{O}_N f - f\|\right],$$

which implies that $\|u - u_N\| \to 0$ as $N \to \infty$. The proof of the theorem is complete.

Consider another method for solving operator equation (1). Assume that there is a sequence of finite-dimensional Hilbert spaces (H_N), $H_N \subset H$, where N is the dimension of the space H_N, and linear operators $\hat{P}_N$ projecting the space H onto the space H_N are defined such that $\hat{P}_N(H) = H_N$ and $\hat{P}_N^2 = P_N$. Note that, in general, $\hat{P}_N$ need not be orthogonal projection operators. Obviously, the operators $\hat{P}_N$ do not change elements of H_N. An example of such an operator is the following one: $\hat{P}_N$ projects the function space H into the space of piecewise constant functions whose values are determined by the elements $u \in H$ at isolated nodal points (the collocation method).

Consider the following two equations: the original equation (1) in the space H and the equation

$$\hat{P}_N \hat{A} u_N = \hat{P}_N f, \qquad u_N \in H_N, \tag{14}$$

in the subspace H_N. Equation (1) is said to be exact, and Eq. (14) is said to be approximate. Note that, by introducing a basis in the subspace H_N, one can reduce Eq. (14) to an equivalent SLAE.

We write

$$\hat{A}_1 = \frac{\hat{A} + \hat{A}^*}{2}, \qquad \hat{A}_2 = \frac{\hat{A} - \hat{A}^*}{2i}. \tag{15}$$

Then $\hat{A} = \hat{A}_1 + i\hat{A}_2$, and the operators $\hat{A}_1$ and $\hat{A}_2$ are obviously self-adjoint in the space H.

We introduce the following conditions.

Condition C. *For any vector $v_N \in H_N$, the inequality*

$$\|(\hat{P}_N \hat{A} - \hat{A}) v_N\| \le \delta(N) \|v_N\|$$

holds, and $\delta(N) \to 0$ as $N \to \infty$. Moreover, $\|\hat{P}_N f - f\| \to 0$ as $N \to \infty$.

Condition D. *The operators $\hat{P}_N \hat{A}_1$ and $\hat{P}_N \hat{A}_2$ are self-adjoint in the space H_N, and for any vector $v_N \in H_N$, the inequality*

$$\|(\hat{P}_N \hat{A}_2 - \hat{A}_2) v_N\| \le \delta(N) \|v_N\| \tag{16}$$

holds, where $\delta(N) \to 0$ as $N \to \infty$. Moreover, if u is a solution of Eq. (1), then

$$\lim_{N\to\infty} \|u - \hat{P}_N u\| = 0. \tag{17}$$

Let us prove the following assertion.

Theorem 3. *Assume that either inequality (6) and condition C or inequality (8) and condition D are satisfied. Then there exists a value N_0 such that for $N \ge N_0$ the solution of the approximate equation (14) exists, is unique, and can be obtained by using MMD. In this case, the sequence of approximate solutions converges to the exact solution; i.e., $\lim_{N\to\infty} \|u - u_N\| = 0$.*

Proof. First, consider condition C. We use the obvious inequality $|a| \ge |b| - |a-b|$, set $N \ge N_0$, where N_0 is determined by the condition $\delta(N_0) < p_0$, and obtain the following chain of inequalities for any $v_N \in H_N$:

$$\begin{aligned}|(\hat{P}_N \hat{A} v_N, v_N)| &= |(\hat{A} v_N, v_N) + (\hat{P}_N \hat{A} v_N - \hat{A} v_N, v_N)| \\ &\ge |(\hat{A} v_N, v_N)| - |(\hat{P}_N \hat{A} v_N - \hat{A} v_N, v_N)| \ge (p_0 - \delta(N)) \|v_N\|^2.\end{aligned}$$

Thus, by Theorem 1, the solution of Eq. (14) exists, is unique, and can be obtained by MMD, and by Theorem 1, the norm of the solution u_N satisfies the estimate

$$\|u_N\| \le \frac{\|\hat{P}_N f\|}{p_0 - \delta(N)} \le \frac{\|f\|}{p_0 - \delta(N)} + \frac{\|\hat{P}_N f - f\|}{p_0 - \delta(N)}. \tag{18}$$

By Eqs. (1) and (14), we have

$$\hat{A}(u_N - u) + (\hat{P}_N\hat{A} - \hat{A})u_N = \hat{P}_N f - f.$$

Applying the operator $\hat{A}^{-1}$ to both sides of this relation and taking into account condition C, we obtain the inequality

$$\|u_N - u\| \le \frac{\delta(N)}{p_0}\|u_N\| + \frac{1}{p_0}\|\hat{P}_N f - f\|,$$

whence, with regard to the estimate (18) and condition C, we conclude that $\|u - u_N\| \to 0$ for $N \to \infty$.

Let condition D and inequality (8) be satisfied. By setting $N \ge N_0$ with regard to the fact that the inner products $(\hat{P}_N\hat{A}_1 v_N, v_N)$ and $(\hat{P}_N\hat{A}_2 v_N, v_N)$ are real numbers, for any $v_N \in H_N$ we obtain the chain of inequalities

$$\begin{aligned}|(\hat{P}_N\hat{A}v_N, v_N)| &\ge |(\hat{P}_N\hat{A}_2 v_N, v_N)| = |(\hat{A}_2 v_N, v_N) + (\hat{P}_N\hat{A}_2 v_N - \hat{A}_2 v_N, v_N)| \\ &\ge |(\hat{A}_2 v_N, v_N)| - |(\hat{P}_N\hat{A}_2 v_N - \hat{A}_2 v_N, v_N)| \ge (p_0 - \delta(N))\|v_N\|^2. \qquad (19)\end{aligned}$$

Thus, just as in the case considered above, there exists a unique solution of Eq. (14), which can be obtained by MMD, and the norm of the solution u_N satisfies the estimate (18).

Let u be a solution of Eq. (1), and let u_N be a solution of Eq. (14). Obviously,

$$\|u - u_N\| \le \|u - \hat{P}_N u\| + \|\hat{P}_N u - u_N\|. \qquad (20)$$

Further, with regard to inequality (19), we obtain

$$\begin{aligned}(p_0 - \delta(N))\|\hat{P}_N u - u_N\|^2 &\le |(\hat{P}_N\hat{A}\hat{P}_N u - \hat{P}_N\hat{A}u_N, \hat{P}_N u - u_N)| \\ &= |(\hat{P}_N\hat{A}\hat{P}_N u - \hat{P}_N\hat{A}u + \hat{P}_N\hat{A}u - \hat{P}_N\hat{A}u_N, \hat{P}_N u - u_N)|.\end{aligned}$$

Then, since $\hat{P}_N\hat{A}u = \hat{P}_N\hat{A}u_N = \hat{P}_N f$, we obtain the estimate

$$\|\hat{P}_N u - u_N\| \le \frac{1}{p_0 - \delta(N)}\|\hat{P}_N\hat{A}(\hat{P}_N u - u)\|. \qquad (21)$$

Now from relations (17), (20), (21) with regard to the boundedness of the operator $\hat{A}$, we find that $\|u - u_N\| \to 0$ as $N \to \infty$. The proof of the theorem is complete.

INTEGRAL EQUATIONS

The results obtained in the preceding section are of general character as applied to linear operator equations. Their main objective is to prove the possibility of solving the equations under certain conditions in principle. But in the specific applications of the results obtained above, it is extremely important to take into account the special properties of the equations. In what follows, we consider the integral equations describing the problem of scattering of electromagnetic fields on three-dimensional inhomogeneous anisotropic dielectric structures [3].

Consider the three-dimensional singular integral equation for the electric field in a bounded domain of inhomogeneity Q:

$$\begin{aligned}\vec{E}(x) + \frac{1}{3}(\hat{\varepsilon}_r(x) - \hat{I})\vec{E}(x) - \text{p.v.}\int\limits_Q ((\hat{\varepsilon}_r(y) - \hat{I})\vec{E}(y), \operatorname{grad})\operatorname{grad} G(R)\,dy \\ - k_0^2\int\limits_Q (\hat{\varepsilon}_r(y) - \hat{I})\vec{E}(y)G(R)\,dy = \vec{E}^0(x), \qquad x \in Q, \qquad (22)\end{aligned}$$

where $G(R) = \exp(ik_0R)/(4\pi R)$, $R = |x - y|$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $\hat{\varepsilon}_r = \hat{\varepsilon}/\varepsilon_0$, and $k_0 = \omega(\varepsilon_0\mu_0)^{1/2}$. We also use the equivalent integro-differential equation

$$\vec{E}(x) - k_0^2 \int\limits_Q (\hat{\varepsilon}_r - 1)\vec{E}(y)G(R)\,dy - \operatorname{grad} \div \int\limits_Q (\hat{\varepsilon}_r - 1)\vec{E}(y)G(R)\,dy = \vec{E}^0(x). \tag{23}$$

First, consider the case in which the Galerkin method is used to solve integral equations. If we multiply Eqs. (22) or (23) by the tensor function $(\hat{\varepsilon} - \varepsilon_0\hat{I})^*$, then the obtained equivalent equations satisfy conditions of the form (6) and (8) [1, p. 94]. The finite-dimensional subspaces H_N of the Hilbert space $\vec{L}_2(Q)$ satisfying condition A can be constructed by various methods. For example, these can be spaces of piecewise constant or piecewise linear functions. One can also consider spline functions. This means that, by Theorem 2, to solve the equations numerically, one can use SLAE (3), (4), and $u_N \to u$ as $N \to \infty$. Thus, the Galerkin method can be used to solve the corresponding integral equations numerically, and MMD iterations can be used to solve the SLAE obtained by discretization.

To apply the Galerkin method, it is necessary to calculate the inner products of the form

$$(\hat{A}\vec{U}, \vec{V})_{\vec{L}_2}, \tag{24}$$

where $\vec{U}$ and $\vec{V}$ are known vector functions and $\hat{A}$ is the integral operator of the equations. The inner product in the Hilbert space $\vec{L}_2(Q)$ is defined as an appropriate integral. Therefore, to calculate the product (24), one can use not the three-dimensional singular integral equation (22) but the equivalent equation (23) and introduce the differentiation operation grad div into the integrand in the inner product. Indeed, summing over repeated indices, we obtain

$$\left(\operatorname{grad}_x \operatorname{div}_x \int\limits_Q \vec{U}(y)\frac{1}{4\pi R}\,dy, \vec{V}(x)\right)_{\vec{L}_2(Q)} = \int\limits_Q\int\limits_Q \frac{(3\alpha_n\alpha_m - \delta_{nm})}{4\pi R^3} U_m(y)V_n^*(x)\,dx\,dy, \tag{25}$$

where the numbers α_1, α_2, and α_3 are determined by formulas (6) in [3].

The integral (25) exists in the usual sense. The inner products of other integral operators in (23) are calculated in the standard way.

Now consider how the collocation method is used to solve the three-dimensional singular integral equation (22) numerically. Note an important fact. The kernels of the integral operator of Eq. (22) depend only on the difference of Cartesian coordinates of the points x and y. Therefore, in the discretization, it is expedient to take this fact into account and obtain a SLAE matrix that has appropriate symmetry properties.

To this end, in rectangular Cartesian coordinates we introduce finitely many points, i.e., a grid such that the domain Q completely lies in the rectangular parallelepiped Π with sides N_lh_l, $l = 1, 2, 3$, where h_l are the grid steps in the Cartesian coordinates. The parallelepiped Π is divided by this grid into cells (elementary parallelepipeds) $\Pi(p)$, $p = (p_1, p_2, p_3)$, $p_l = 0, \dots, N_l - 1$. We define the domain $\tilde{Q}$ as the union N_Q of cells whose centers lie inside the domain Q. Obviously, $N_Q \le N_1N_2N_3$. Then the choice of the grid is determined by the shape of the domain Q and the requirement that the desired solution, the external field, and the medium parameters vary inside the cells weakly. The nodal points at which the values of the desired functions are determined are the centers $x(p)$ of the cells with coordinates

$$x_1(p) = p_1h_1 + h_1/2, \qquad x_2(p) = p_2h_2 + h_2/2, \qquad x_3(p) = p_3h_3 + h_3/2.$$

Using the results obtained in [3], we can the represent integro-differential equation (23), which is equivalent to the three-dimensional singular equation (22), as the following three-dimensional singular integral equation:

$$\vec{E}(x) + \hat{\gamma}(\hat{\varepsilon}_r(x) - \hat{I})\vec{E}(x) - \lim_{\delta \to 0} \int\limits_{Q \setminus \Pi(\delta)} ((\hat{\varepsilon}_r(y) - \hat{I})\vec{E}(y), \operatorname{grad}) \operatorname{grad} G(R)\, dy$$
$$- k_0^2 \int\limits_{Q} (\hat{\varepsilon}_r(y) - \hat{I})\vec{E}(y) G(R)\, dy = \vec{E}^0(x), \qquad x \in Q. \tag{26}$$

Here $\Pi(\delta)$ is a parallelepiped of diameter δ centered at the point $x = y$ and the tensor $\hat{\gamma}$ is determined by the shape of the parallelepiped.

Then the integral equation (26) can be approximated by the following SLAE of dimension N_Q for the values of the unknown field $u(p)$ at the nodal points of the domain Q:

$$u(p) - \sum_{y(q) \in Q} B(p-q) v(q) u(q) = u_0(p), \qquad x(p) \in Q, \tag{27}$$

where $u(p) \equiv u(x(p))$, $u_0(p) \equiv u_0(x(p))$, $u(q) \equiv u(x(q))$. Since the nodal points are at the cell center, it follows that the accuracy of the integral operator approximation is of order h^2, $h = \sqrt{h_1^2 + h_2^2 + h_3^2}$.

In system (27), $u(p)$ and $u_0(p)$ are vectors, and $B(p-q)$ and $v(q)$ are 3×3 matrices whose entries are determined by the formulas

$$v_{mk}(q) = \varepsilon_{mk}(y(q)) - \delta_{mk},$$
$$B_{nm}(p-q) = \int\limits_{\Pi_q} G(R) \left[\left(\frac{3}{R^2} - \frac{3ik_0}{R} - k_0^2 \right) \alpha_n \alpha_m + \left(k_0^2 + \frac{ik_0}{R} - \frac{1}{R^2} \right) \delta_{nm} \right] dy, \qquad p \neq q.$$

Here

$$R = \sqrt{(y_1 - x_1(p))^2 + (y_2 - x_2(p))^2 + (y_3 - x_3(p))^2},$$

and the numbers α_n are given by the relations

$$\alpha_n = \frac{x_n(p) - y_n}{|x(p) - y|}, \qquad n = 1, 2, 3.$$

To determine $B_{nm}(0,0,0)$, consider an auxiliary relation. It was shown in [4] that if $\vec{J} = \text{const}$ in the domain V, then

$$\operatorname{grad} \operatorname{div}_x \int\limits_V \frac{1}{4\pi R} \vec{J}\, dy = -\hat{\gamma}\vec{J},$$

where $R = |x - y|$ and the tensor $\hat{\gamma}$ is determined by the shape of the domain boundary. If the domain V is a parallelepiped, the Cartesian axes are parallel to its edges, and the point x is at its center, then the tensor $\hat{\gamma}$ has the diagonal form $\gamma_{nm} = \gamma_n \delta_{nm}$. Here $\gamma_n = (1/4\pi)\Omega_n$, and Ω_n is the double solid angle with vertex at the parallelepiped center spanned by the face perpendicular to the x_n-axis, $n = 1, 2, 3$. Obviously, $\gamma_1 + \gamma_2 + \gamma_3 = 1$.

By appropriate transformations, one obtains

$$\gamma_1 = f(h_1, h_2, h_3), \qquad \gamma_2 = f(h_2, h_1, h_3), \qquad \gamma_3 = f(h_3, h_1, h_2),$$
$$f(h_1, h_2, h_3) = 1 - \frac{2}{\pi} \left\{ \arcsin \left[\frac{h_1 h_3}{\sqrt{h_1^2 + h_2^2}\sqrt{h_2^2 + h_3^2}} \right] + \arcsin \left[\frac{h_1 h_2}{\sqrt{h_1^2 + h_3^2}\sqrt{h_2^2 + h_3^2}} \right] \right\}.$$

Note that if the domain V is a cube, then, just as in the case of a ball, we have $\gamma_1 = \gamma_2 = \gamma_3 = 1/3$.

With regard to the preceding, we obtain

$$B_{nn}(0,0,0) = -\gamma_n + \int\limits_{\Pi_p} \left\{ k_0^2 G(R)(1-\alpha_n^2) + \frac{k_0^2}{4\pi R}[3\Phi(R)\alpha_n^2 - \Phi(R)] \right\} dy, \quad B_{nm}(0) = 0, \quad n \neq m,$$

$$\Phi(R) = 1 + (1 - ik_0R)\frac{\exp(ik_0R) - 1 - ik_0R}{(k_0R)^2}.$$

Note that $\Phi(R)$ is an everywhere differentiable function of coordinates. Also note that the use of multi-index numbering for the unknown vector and the right-hand side of the equations allows one to represent the symmetry properties of the obtained SLAE matrix in the most transparent form.

The above argument implies symmetry relations for the elements of the array $\{B(p-q)\}$,

$$B_{nm} = B_{mn}, \qquad B_{nm}(p-q) = \operatorname{sgn}(p_n - q_n)\operatorname{sgn}(p_m - q_m)B_{nm}(\tilde{p}), \tag{28}$$

where $\tilde{p} = (|p_1 - q_1|, |p_2 - q_2|, |p_3 - q_3|)$ and $\operatorname{sgn}(a) = \begin{cases} 1, & a \geq 0, \\ -1, & a < 0. \end{cases}$

Obviously, it follows from (28) that the number of elements of the array $\{B(p-q)\}$, which must be calculated and stored in the computer memory, is proportional to $N = N_1N_2N_3$.

The quantities $B(p)$ are calculated either analytically if the cell size is small or by numerical integration formulas. In both cases, the computational expenditures for determining the array $\{B(p)\}$ are a small part of the total amount of computations necessary to solve the SLAE.

If the integral equations are solved by the Galerkin method on a rectangular grid with equal basis functions inside each cell (elementary parallelepiped), then we have the same estimate for the number of distinct entries of the SLAE matrix.

Now we show that all conditions of Theorem 3 (condition D and an inequality of the form (8)) are satisfied for the collocation method applied to Eq. (26). Consider the subspaces H_N as the spaces of piecewise constant vector functions in the domain Q. Obviously, $H_N \subset \vec{L}_2(Q)$. We write Eq. (26) in symbolic form

$$\vec{E}(x) - \frac{1}{\varepsilon_0}\hat{S}((\hat{\varepsilon} - \varepsilon_0\hat{I})\vec{E})(x) = \vec{E}^0(x), \qquad x \in Q. \tag{29}$$

To simplify the calculations, we assume that $\operatorname{Im}\varepsilon_0 = 0$, but this restriction can be removed.

We multiply Eq. (29) by the tensor function $\hat{\delta}^* = (\hat{\varepsilon} - \varepsilon_0\hat{I})^*$ and obtain

$$(\hat{A}\vec{E})(x) \equiv \hat{\delta}^*(x)\vec{E}(x) - \frac{\hat{\delta}^*(x)}{\varepsilon_0}\hat{S}(\hat{\delta}\vec{E})(x) = \hat{\delta}^*(x)\vec{E}^0(x), \qquad x \in Q. \tag{30}$$

We assume that there is damping in the medium, but it can be arbitrarily small. This means that the Hermitian tensor function $\hat{\delta}_2(x) = (\hat{\varepsilon}(x) - \hat{\varepsilon}^*(x))/(2i)$ is positive definite in the domain Q. It is shown in the monograph [1, p. 94] that in this case an inequality of the form (8) holds for the operator $\hat{A}$ in Eq. (30). We also note that Eqs. (29) and (30) are equivalent.

Let A, B, and C be linear operators in a Hilbert space. Then $(ABC)^* = C^*B^*A^*$. Therefore, Eq. (30) implies that

$$\hat{A} = \hat{\delta}^* - \frac{\hat{\delta}^*}{\varepsilon_0}\hat{S}\hat{\delta}, \qquad \hat{A}^* = \hat{\delta} - \frac{\hat{\delta}^*}{\varepsilon_0}\hat{S}^*\hat{\delta}. \tag{31}$$

We use (31) to show that the operators $\hat{P}_N\hat{A}_1$ and $\hat{P}_N\hat{A}_2$ are self-adjoint in the space H_N and inequality (16) holds for any vector $v_N \in H_N$. Here the $\hat{P}_N$ are linear operators projecting the space $\vec{L}_2(Q)$ onto the subspace H_N, and the self-adjoint operators $\hat{A}_1$ and $\hat{A}_2$ are determined by (15).

It follows from (31) and (15) that

$$\hat{A}_2 = \frac{1}{2i}\left[(\hat{\delta}^* - \hat{\delta}) - \frac{\hat{\delta}^*}{\varepsilon_0}(\hat{S} - \hat{S}^*)\hat{\delta}\right]. \tag{32}$$

For the integral operator $\hat{S}_2 = (\hat{S} - \hat{S}^*)/(2i)$, by Eqs. (23) and (29), we have the representation

$$(\hat{S}_2\vec{v})(x) = \int\limits_Q (\vec{v}(y), \operatorname{grad}_x) \operatorname{grad}_x G_2(R)\, dy + k_0^2 \int\limits_Q \vec{v}(y) G_2(R)\, dy$$
$$\equiv \int\limits_Q \hat{G}(x-y)\vec{v}(y)\, dy; \qquad G_2(R) = \sin(k_9 R)/(4\pi R), \qquad R = |x-y|. \tag{33}$$

Obviously, this representation implies that the tensor function $\hat{G}(x-y)$ is a differentiable function of the points x and y.

We define the domain Q as the union of M elementary parallelepipeds of the grid introduced above. We also assume that the tensor dielectric permeability function is constant in the interior of each cell. In all Hilbert spaces H_N, $N > M$, the domain Q does not vary, and the tensor dielectric permeability function in Q remains unchanged. For example, this can be attained by decreasing each step of the grid by a factor of two when passing to the successive subspace.

With regard to the above restrictions, from (32) and (33) we obtain

$$(\hat{P}_N\hat{A}_2 - \hat{A}_2) = -\frac{\hat{\delta}^*}{\varepsilon_0}(\hat{P}_N\hat{S}_2 - \hat{S}_2)\hat{\delta},$$

which implies that

$$\|(\hat{P}_N\hat{A}_2 - \hat{A}_2)v_N\| \le \frac{1}{\varepsilon_0}\max_{x\in Q}\|\hat{\delta}(x)\|^2\|(\hat{P}_N\hat{S}_2 - \hat{S}_2)\|\|v_N\|. \tag{34}$$

By the representation (33), we obtain the estimate

$$\|(\hat{P}_N\hat{S}_2 - \hat{S}_2)\| \le \left(\sum_{n=1}^{N_Q}\int\limits_{Q_n}\|\hat{G}(x_n - y) - \hat{G}(x-y)\|^2\, dx\, dy\right)^{1/2}, \tag{35}$$

where N_Q is the number of elementary parallelepipeds in the domain Q and n is the sequential number of the parallelepiped.

Since the tensor function $\hat{G}(x-y)$ is a smooth function of the points x and y, we have $\|(\hat{P}_N\hat{S}_2 - \hat{S}_2)\| \to 0$ as $N \to \infty$. (This corresponds to the fact that the grid steps tend to zero.) Thus, the estimates (34) and (35) imply inequality (16).

Now let us show that the operators $\hat{P}_N\hat{A}_1$ and $\hat{P}_N\hat{A}_2$ are self-adjoint in the space H_N. The operator $\hat{A}_2$ has the form (32), and the operator $\hat{A}_1$ can be represented as

$$\hat{A}_1 = \frac{1}{2}\left[(\hat{\delta}^* + \hat{\delta}) - \frac{\hat{\delta}^*}{\varepsilon_0}(\hat{S} + \hat{S}^*)\hat{\delta}\right].$$

First, consider the operator $\hat{A}_2$. Obviously, the operators $\hat{P}_N(\hat{\delta}^* - \hat{\delta})/(2i)$ are self-adjoint operators in the space H_N. Now consider the integral operator in (32). By the above assumptions about the tensor dielectric permeability function, we have $\hat{P}_N\hat{\delta}^*U = \hat{\delta}^*\hat{P}_N U$, where U is an arbitrary function. Therefore, we have the obvious relations

$$(\hat{P}_N\hat{\delta}^*\hat{S}_2\hat{\delta}\vec{u}, \vec{v}\,) = (\hat{P}_N\hat{S}_2\hat{\delta}\vec{u}, \hat{\delta}\vec{v}\,), \qquad (\vec{u}, \hat{P}_N\hat{\delta}^*\hat{S}_2\hat{\delta}\vec{v}\,) = (\hat{\delta}\vec{u}, \hat{P}_N\hat{S}_2\hat{\delta}\vec{v}\,),$$

where $\vec{u}, \vec{v} \in H_N$. We write $\vec{U} = \hat{\delta}\vec{u}$ and $\vec{V} = \hat{\delta}\vec{v}$. By the above assumptions, it is obvious that $\vec{U}, \vec{V} \in H_N$.

We write the integral operator $\hat{S}_2$ componentwise in Cartesian coordinates,

$$(\hat{P}_N \hat{S}_2)\vec{U})_n(x_p) = \sum_{q=1}^{N_Q} \sum_{m=1}^{3} \int_{Q_q} G_{nm}(x_p - y) U_m(q)\, dy, \tag{36}$$

where $n = 1, 2, 3$ and $p = 1, \ldots, N_Q$; $U_m(q)$ is the value of the mth component of the vector function $\vec{U}$ in the qth elementary parallelepiped, and x_p is the center of the cell Q_p. The specific form of the functions G_{nm} can be obtained from the representation (33).

With regard to the representation (36), we have

$$\begin{aligned} (\hat{P}_N \hat{S}_2 \vec{U}, \vec{V})_{\vec{L}_2} &= \Delta Q \sum_{q=1}^{N_Q} \sum_{p=1}^{N_Q} \sum_{n=1}^{3} \sum_{m=1}^{3} U_m(q) V_n^*(p) \int_{Q_q} G_{nm}(x_p - y)\, dy, \\ (\vec{U}, \hat{P}_N \hat{S}_2 \vec{V})_{\vec{L}_2} &= \Delta Q \sum_{q=1}^{N_Q} \sum_{p=1}^{N_Q} \sum_{n=1}^{3} \sum_{m=1}^{3} U_n(p) V_m^*(q) \int_{Q_q} G_{nm}^*(x_p - y)\, dy, \end{aligned} \tag{37}$$

where ΔQ is the volume of the elementary parallelepiped. Since the integral operator $\hat{S}_2$ is self-adjoint in the space $\vec{L}_2$, we have

$$G_{nm}^*(x - y) = G_{mn}(y - x). \tag{38}$$

In the second relation in (37), we rearrange the indices as $n \to m$, $m \to n$, $p \to q$, $q \to p$. Then, with regard to (38), we obtain

$$(\vec{U}, \hat{P}_N \hat{S}_2 \vec{V})_{\vec{L}_2} = \Delta Q \sum_{q=1}^{N_Q} \sum_{p=1}^{N_Q} \sum_{n=1}^{3} \sum_{m=1}^{3} U_m(q) V_n^*(p) \int_{Q_p} G_{nm}(y - x_q)\, dy. \tag{39}$$

Now the first relation in (37) and the expression (39) imply that

$$(\hat{P}_N \hat{S}_2 \vec{U}, \vec{V})_{\vec{L}_2} - (\vec{U}, \hat{P}_N \hat{S}_2 \vec{V})_{\vec{L}_2} = \Delta Q \sum_{q=1}^{N_Q} \sum_{p=1}^{N_Q} \sum_{n=1}^{3} \sum_{m=1}^{3} U_m(q) V_n^*(p) G_{nmpq}, \tag{40}$$

where

$$G_{nmpq} = \left[\int_{Q_q} G_{nm}(x_p - y)\, dy - \int_{Q_p} G_{nm}(y - x_q)\, dy \right]. \tag{41}$$

We change variables in the integrals (41). In the first integral, $y = x_q + z$, i.e., $y_l = x_{ql} + z_l$, $l = 1, 2, 3$, and in the second integral, $y = x_p + z$. Then we obtain

$$G_{nmpq} = \left[\int_{\Pi_0} G_{nm}(x_p - x_q - z) dz - \int_{\Pi_0} G_{nm}(x_p - x_q + z)\, dz \right], \tag{42}$$

where Π_0 is an elementary parallelepiped centered at the origin. We introduce the notation $d = x_p - x_q$ to write the second integral in (42) in the form

$$\int_{\Pi_0} G_{nm}(d + z)\, dz = \int_{-h_1/2}^{h_1/2} \int_{-h_2/2}^{h_2/2} \int_{-h_3/2}^{h_3/2} G_{nm}(d_1 + z_1, d_2 + z_2, d_3 + z_3)\, dz_1\, dz_2\, dz_3. \tag{43}$$

Changing the variables $z_1 \to -z_1$, $z_2 \to -z_2$, $z_3 \to -z_3$ in the second integral in (43), we obtain

$$\int_{\Pi_0} G_{nm}(x_p - x_q + z)\,dz = \int_{\Pi_0} G_{nm}(x_p - x_q - z)\,dz. \tag{44}$$

It follows from (42) and (44) that all elements in the array G_{nmpq} are zero. Thus, (40) implies that the operator $\hat{P}_N \hat{A}_2$ is self-adjoint in the space H_N. Precisely in the same way, it is proved that the operator $\hat{P}_N \hat{A}_1$ is also self-adjoint in the space H_N.

Under the above restrictions, the solution of the integral equation is smooth inside the elementary parallelepipeds. Therefore, it is obvious that (17) holds.

Thus, all requirements of Theorem 3 (inequality (8) and condition D) are satisfied, and hence, the considered three-dimensional singular integral equation can be solved by the collocation method. Further, as $N \to \infty$ (the grid steps tend to zero), the solutions of SLAE converge to the exact solution of the integral equation, and the SLAE can be solved by the iteration MMD.

The above proof is sound under certain restrictions on the shape of the domain Q and the medium parameters. But numerical studies show that one can efficiently solve specific problems without these restrictions. In particular, the problems of wave scattering in media without decay are solved successfully.

CONCLUSION

The theorems obtained in the paper contain conditions under which approximate solutions of linear operator equations converge to the exact solution. This guarantees that the solution can be obtained with any prescribed accuracy. The results can be used to justify numerical solutions of many integral equations of mathematical physics, in particular, the integral equations of acoustics.

ACKNOWLEDGMENTS

This work was supported by the Ministry of Education and Science of the Russian Federation (project no. 2.1361.2017/4.6).

REFERENCES

1. Samokhin, A.B., *Integral'nye uravneniya i iteratsionnye metody v elektromagnitnom rasseyanii* (Integral Equations and Iteration Methods in Electromagnetic Scattering), Moscow: Radio i Svyaz', 1998.
2. Samokhin, A.B., *Integral Equations and Iteration Methods in Electromagnetic Scattering*, Utrecht: VSP, 2001.
3. Samokhin, A.B., Volume singular integral equations for problems of scattering on three-dimensional dielectric structures, *Differ. Equations*, 2014, vol. 50, no. 9, pp. 1201–1216.
4. Yaghijian, A.D., Electric Green's functions in the source region, *Proc. IEEE*, 1980, vol. 68, pp. 248–263.