ISSN 0012-2661, Differential Equations, 2018, Vol. 54, No. 7, pp. 911–918. © Pleiades Publishing, Ltd., 2018.
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NUMERICAL METHODS

Parametrization of the Solution of the Kepler Problem and New Adaptive Numerical Methods Based on This Parametrization

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Received February 5, 2018

Abstract—We propose a one-parameter family of adaptive numerical methods for solving the Kepler problem. The methods preserve the global properties of the exact solution of the problem and approximate the time dependence of the phase variables with the second or fourth approximation order. The variable time increment is determined automatically from the properties of the solution.

DOI: 10.1134/S001226611807008X

1. INTRODUCTION

The solution of the Kepler problem describes the evolution of the state of two gravitationally interacting mass points in the plane of their motion. The global properties of the solution determine the geometry of the problem phase space and the most important physical properties. They include the symplecticity of the mapping of the initial state into the current state and the conservation of the phase volume, the standard integrals of motion (total linear momentum, total angular momentum, and total energy), and the components of the Laplace–Runge–Lenz vector [1, pp. 40–45; 2, pp. 7–8, 170–175].

Widely used numerical methods for solving the Cauchy problem for systems of ordinary differential equations distort the most important properties of the exact solution of the Kepler problem. This distortion is most noticeable on large time intervals. For example, the Verlet method [3] and the variational method [4] are symplectic, conserve the total momentum and the total angular momentum, but do not conserve the total energy and the Laplace–Runge–Lenz vector. The discrete gradient method [2, pp. 162–163; 5; 6] conserves the total momentum, the total angular momentum, and the total energy but is not symplectic. Finally, the well-known explicit Runge–Kutta method $(1/6, 1/3, 1/3, 1/6)$ of the fourth approximation order [7, p. 144 of the Russian translation] is not symplectic and does not conserve any of the first integrals of the Kepler problem except for the total linear momentum. As a result, the methods listed above give approximate orbits of motion qualitatively differing from the exact orbit.

In the case of strongly elongated orbits, the components of the solution of the Kepler problem rapidly change on time intervals small compared with the orbit period. There arises a peculiar boundary layer in a neighborhood of the minimum distance to the force center. In these cases, methods with constant integration step turn out to be inefficient [8]. Therefore, a procedure for choosing the increment value depending on the properties of the solution is needed. It is well known that the standard step choice procedures like the Runge rule significantly distort the solution of the Cauchy problem for Hamiltonian systems [2, p. 255]. Therefore, the development of adaptive numerical methods that are of a sufficiently high approximation order and, in the framework of exact arithmetics, conserve the global properties of exact solutions of the problem is an important problem of computational mathematics.

In the recent years, increased attention has been paid to the development of numerical methods for solving the Kepler problem [9–13]. In particular, new methods based on the exact linearization of the problem by the Levi-Civita transformation have been created [9, 12].

In the present paper, we state the Kepler problem, list the global properties of its solution, and describe a time increment choice procedure based on a specific parametrization of the solution. The main result is a new one-parameter family of one-step adaptive numerical methods conserving all above-listed global properties of the solution of the Kepler problem and approximating the time dependence of the phase variables with the second or the fourth approximation order. This family differs from the previously proposed family [12, 13] in the integration step choice procedure. There are reasons to assume that the new family of methods will be more efficient in the analysis of strongly elongated orbits than the methods used in [13].

2. KEPLER PROBLEM

2.1. Statement of the Problem and Global Properties of Its Solutions

The problem on the motion of gravitationally interacting mass points of masses m_1 and m_2 in the absence of an external force field can be reduced to the problem on the two-dimensional motion of a single virtual point of unit mass in a central field [2, p. 7],

$$dv_x/dt = -\partial H(v_x, v_y, x, y)/\partial x = -\gamma M r^{-3} x, \tag{1}$$
$$dv_y/dt = -\partial H(v_x, v_y, x, y)/\partial y = -\gamma M r^{-3} y, \tag{2}$$
$$dx/dt = \partial H(v_x, v_y, x, y)/\partial v_x = v_x, \tag{3}$$
$$dy/dt = \partial H(v_x, v_y, x, y)/\partial v_y = v_y, \tag{4}$$
$$v_x(0) = v_{x,0}, \qquad v_y(0) = v_{y,0}, \qquad x(0) = x_0, \qquad y(0) = y_0. \tag{5}$$

A fixed attracting force center is located at the origin $(0, 0)$. The unknown functions $x = x(t)$, $y = y(t)$ and $v_x = v_x(t)$, $v_y = v_y(t)$ determine the Cartesian coordinates and the velocity components of the virtual point in the plane of motion at time $t > 0$. The numbers x_0, y_0, $v_{x,0}$, and $v_{y,0}$ define the known initial state of the dynamical system. The Hamiltonian $H = H(v_x, v_y, x, y)$ has the form $H(v_x, v_y, x, y) = 0.5v^2 - \gamma M r^{-1}$, where $v = (v_x^2 + v_y^2)^{1/2}$, $r = (x^2 + y^2)^{1/2}$, $v_0 = (v_{x,0}^2 + v_{y,0}^2)^{1/2}$, $r_0 = (x_0^2 + y_0^2)^{1/2}$, $M = m_1 + m_2$, and $\gamma > 0$ is the gravitational constant.

The motion is two-dimensional if the condition $l_{z,0} = x_0 v_{y,0} - y_0 v_{x,0} \neq 0$ is satisfied. Otherwise, the motion is one-dimensional and occurs along the line passing through the initial position of the virtual point and the origin. In what follows, we assume that $l_{z,0} \neq 0$.

The solutions $(v_x(t), v_y(t), x(t), y(t))$, $t > 0$, of the Kepler problem have several global properties of important geometric and physical meaning [2, p. 7].

1. Any solution realizes a symplectic mapping $\phi(t)$ of the initial state into the current state,

$$(v_{x,0}, v_{y,0}, x_0, y_0) \xrightarrow{\phi(t)} (v_x(t), v_y(t), x(t), y(t)), \qquad \Phi(t)^{\mathrm{T}} J \Phi(t) \equiv J,$$

where

$$\Phi(t) = \frac{\partial(v_x(t), v_y(t), x(t), y(t))}{\partial(v_{x,0}, v_{y,0}, x_0, y_0)}$$

is the Jacobian matrix of the mapping $\phi(t)$ and

$$J = \begin{bmatrix} 0 & E \\ -E & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. The angular momentum l_z, the total energy H, the components of the Laplace–Runge–Lenz vector $e_{\rm LRL}$, and the phase volume $v_{\rm ph}$ are conserved on any solution of problem (1)–(5),

$$\begin{aligned} l_z(t) &= x(t)v_y(t) - y(t)v_x(t) \equiv l_z(0) = l_{z,0}, &(6)\\ H(t) &= 0.5v^2(t) - \gamma M r(t)^{-1} \equiv 0.5v_0^2 - \gamma M r_0^{-1} = -H^*, &(7)\\ e_{{\rm LRL},x}(t) &= v_y^2(t)x(t) - v_x(t)v_y(t)y(t) - \gamma M r^{-1}(t)x(t)\\ &= v_y(t)l_z - \gamma M r^{-1}(t)x(t) \equiv e_{{\rm LRL},x}(0) = e_{{\rm LRL},x,0}, &(8)\\ e_{{\rm LRL},y}(t) &= v_x^2(t)y(t) - v_x(t)v_y(t)x(t) - \gamma M r^{-1}(t)y(t)\\ &= -v_x(t)l_z - \gamma M r^{-1}(t)y(t) \equiv e_{{\rm LRL},y}(0) = e_{{\rm LRL},y,0}, &(9)\\ v_{\rm ph}(t) &\equiv v_{\rm ph}(0) = v_{{\rm ph},0}. \end{aligned}$$

The first integrals (6)–(9) are not functionally independent. They are related by the identity

$$2Hl_z^2 = e_{{\rm LRL},x}^2 + e_{{\rm LRL},y}^2 - \gamma^2 M^2.$$

2.2. Orbit and the Velocity Hodograph

The trajectory of the dynamical system lies in the intersection of the manifolds

$$l_z(v_x, v_y, x, y) = l_{z,0}, \qquad e_{{\rm LRL},x}(v_x, v_y, x, y) = e_{{\rm LRL},x,0}, \qquad e_{{\rm LRL},y}(v_x, v_y, x, y) = e_{{\rm LRL},y,0}$$

in the phase space. The projection of the trajectory onto the (x, y)-plane is the orbit $O(x, y) = 0$, and its projection onto the (v_x, v_y)-plane is the velocity hodograph $H_v(v_x, v_y) = 0$. The equations of the orbit and the velocity hodograph can be derived from the conservation laws (6)–(9). Indeed, identities (6)–(9) imply that

$$\begin{aligned} r &= (\gamma M)(0.5v^2 + H^*)^{-1}, &(10)\\ x &= -(\gamma M)^{-1}(e_{{\rm LRL},x,0} - l_{z,0}v_y)r = -(e_{{\rm LRL},x,0} - l_{z,0}v_y)(0.5v^2 + H^*)^{-1}, &(11)\\ y &= -(\gamma M)^{-1}(e_{{\rm LRL},y,0} + l_{z,0}v_x)r = -(e_{{\rm LRL},y,0} + l_{z,0}v_x)(0.5v^2 + H^*)^{-1}. &(12) \end{aligned}$$

The sum of squared relations (11) and (12) gives the equation of the velocity hodograph,

$$H_v(v_x, v_y) = (e_{{\rm LRL},x,0} - l_{z,0}v_y)^2 + (e_{{\rm LRL},y,0} + l_{z,0}v_x)^2 - (\gamma M)^2 = 0. \qquad (13)$$

The hodograph is either the circle of radius $(\gamma M l_{z,0}^{-1})^2$ centered at the point

$$(-e_{{\rm LRL},y,0}l_{z,0}^{-1}, e_{{\rm LRL},x,0}l_{z,0}^{-1})$$

for $H^* \geqslant 0$ or the arc $v^2 \geqslant -2H^*$ of this circle for $H^* < 0$.

We eliminate the variables v_x and v_y from (10) by using (11) and (12) and obtain the orbit equation

$$\begin{aligned} O(x, y) = {} & (\gamma^2 M^2 - e_{{\rm LRL},x,0}^2)x^2 + (\gamma^2 M^2 - e_{{\rm LRL},y,0}^2)y^2\\ & - 2e_{{\rm LRL},x,0}e_{{\rm LRL},y,0}xy + 2l_{z,0}^2 e_{{\rm LRL},x,0}x + 2l_{z,0}^2 e_{{\rm LRL},y,0}y - l_{z,0}^4 = 0. \qquad (14) \end{aligned}$$

For $H^* > 0$, the orbit is an ellipse with eccentricity

$$\varepsilon = (e_{{\rm LRL},x,0}^2 + e_{{\rm LRL},y,0}^2)^{1/2}(\gamma M)^{-1}, \qquad \varepsilon \in (0, 1).$$

For $H^* < 0$ (respectively, $H^* = 0$), the orbit is a hyperbola with $\varepsilon > 1$ (respectively, a parabola with $\varepsilon = 1$).

2.3. Reduction of the Equations of Motion and Parametrization of the Solution

The substitution of the expressions for r, x, and y from (10)–(12) into Eqs. (1) and (2) gives nonlinear differential equations determining the dependence of the phase variables v_x, v_y on t,

$$dv_x/dt = -\gamma M r^{-3} x = (\gamma M)^{-2}(e_{\mathrm{LRL},x,0} - l_{z,0}v_y)(0.5(v_x^2 + v_y^2) + H^*)^2,$$
$$dv_y/dt = -\gamma M r^{-3} y = (\gamma M)^{-2}(e_{\mathrm{LRL},y,0} + l_{z,0}v_x)(0.5(v_x^2 + v_y^2) + H^*)^2.$$

For the exact linearization of these equations, we parametrize the solutions of the Cauchy problem. Instead of t, we introduce a new independent variable ω by setting

$$dt = r^2(t(\omega))d\omega, \qquad t(0) = 0 \tag{15}$$

and obtain the system of linear Hamiltonian equations

$$dv_x/d\omega = -l_{z,0}v_y + e_{\mathrm{LRL},x,0} = -\partial W/\partial v_y, \tag{16}$$
$$dv_y/d\omega = l_{z,0}v_x + e_{\mathrm{LRL},y,0} = \partial W/\partial v_x \tag{17}$$

with the Hamiltonian $W(v_x, v_y) = (2l_{z,0})^{-1}H_v(v_x, v_y)$.

It follows from (10) and (15) that

$$dt/d\omega = (\gamma M)^2(0.5(v_x^2(\omega) + v_y^2(\omega)) + H^*)^{-2}, \tag{18}$$

or

$$t(\omega) = (\gamma M)^2 \int_0^{\omega} (0.5v^2(x) + H^*)^{-2}\, dx. \tag{19}$$

Note that the parametrization (15) differs from the parametrization $dt = 2r(t(\theta))d\theta$, $t(0) = 0$ considered in [12, 13].

2.4. Parametric Representation of the Exact Solution of the Problem

The exact solution of the Kepler problem can be obtained as a combination of elementary functions by various methods [1, pp. 31–37; 13; 14, pp. 43–54]. In the present paper, we use the parametrization (15), $v_x = v_x(\omega)$, $v_y = v_y(\omega)$, $t = t(\omega)$, $x = x(v_x(\omega), v_y(\omega))$, $y = y(v_x(\omega), v_y(\omega))$.

For system (16), (17), the exact solution of the Cauchy problem $v_x(0) = v_{x,0}$, $v_y(0) = v_{y,0}$ has the form

$$\begin{aligned} v_x(\omega) &= v_{x,0} - (v_{x,0} + e_{\mathrm{LRL},y,0}l_{z,0}^{-1})(1 - \cos\eta) - \operatorname{sgn}(l_{z,0})(v_{y,0} - e_{\mathrm{LRL},x,0}l_{z,0}^{-1})\sin\eta \\ &= v_{x,0} - \gamma M r_0^{-1} l_{z,0}^{-1}[x_0 \operatorname{sgn}(l_{z,0})\sin\eta - y_0(1 - \cos\eta)], \end{aligned} \tag{20}$$

$$\begin{aligned} v_y(\omega) &= v_{y,0} - (v_{y,0} - e_{\mathrm{LRL},x,0}l_{z,0}^{-1})(1 - \cos\eta) + \operatorname{sgn}(l_{z,0})(v_{x,0} + e_{\mathrm{LRL},y,0}l_{z,0}^{-1})\sin\eta \\ &= v_{y,0} - \gamma M r_0^{-1} l_{z,0}^{-1}[y_0 \operatorname{sgn}(l_{z,0})\sin\eta + x_0(1 - \cos\eta)], \end{aligned} \tag{21}$$

where $\eta = |l_{z,0}|\omega$.

The substitution of the solution (20), (21) into (19) permits one to find a monotone increasing dependence of the variable t on the parameter ω. An analysis shows that if $H^* > 0$ $(0 < \varepsilon < 1)$, then the desired dependence exists for all $\omega \in [0, +\infty)$. For $H^* < 0$ $(\varepsilon > 1)$, it exists on the half-interval $[0, \omega_\infty)$. In the first case, it is convenient to write it as

$$\begin{aligned} t(\omega) &= T_t m + t'(\omega'), \qquad \omega = 2\pi|l_{z,0}|^{-1}m + \omega', \\ \omega' &\in [0, 2\pi|l_{z,0}|^{-1}], \qquad 0 < t' < T_t = t'(2\pi|l_{z,0}|^{-1}), \end{aligned} \tag{22}$$

where m is the number of periods $T_\omega = 2\pi|l_{z,0}|^{-1}$ of the solution (20), (21) on the interval $[0, \omega]$, $m = 0, 1, \ldots$, and T_t is the period of the solution in the variable t. In the second case, we write

$$t(\omega) = t'(\omega'), \qquad \omega' \in [0, \omega_\infty), \tag{23}$$

where ω_∞ is the minimum value of ω for which $0.5v^2(\omega) + H^* = 0$, $H^* < 0$.

One can determine the dependence $t'(\omega')$ for $H^*>0$ and $H^*<0$ as a combination of elementary functions using reference literature, say, [15, pp. 163–164]. Since the formulas are very cumbersome, we do not write out the dependence $t'(\omega')$ in final form here.

Thus, for $H^*>0$ the solution of the problem is periodic with period $T_\omega = 2\pi|l_{z,0}|^{-1}$. The dependence (22) of time t' on the parameter ω' is defined for $\omega' \in [0, 2\pi|l_{z,0}|^{-1}]$. The time period of the solution is calculated by the formula

$$T_t = t'(2\pi|l_{z,0}|^{-1}) = 2\pi|l_{z,0}|^3(\gamma M)^{-2}(1-\varepsilon^2)^{-3/2} = 2\pi\gamma M(2H^*)^{-3/2}. \qquad (24)$$

For $H^*<0$ ($\varepsilon>1$), the motion is infinite, and the dependence (23) is defined on the half-interval $[0,\omega_\infty)$, $0<\omega_\infty<T_\omega$. One has $t\to+\infty$ and $v\to(2H^*)^{1/2}$ as $\omega'\to\omega_\infty-0$ (see (19)).

Formulas (10)–(12) and (20)–(24) give a parametric representation of the exact solution of problem (1)–(5) as a combination of elementary functions. This solution permits developing an exact numerical method for solving the Kepler problem. It is also used to test the family of numerical methods proposed below.

3. FAMILY OF ADAPTIVE NUMERICAL METHODS FOR SOLVING THE KEPLER PROBLEM

3.1. Construction of Numerical Methods

Based on the conservation laws (6)–(9), the parametrization (15), and two-stage symmetrically symplectic Runge–Kutta methods [16, 17], we obtain a new family

$$(v_{x,i}, v_{y,i}, x_i, y_i, t_i) \mapsto (v_{x,i+1}, v_{y,i+1}, x_{i+1}, y_{i+1}, t_{i+1})$$

of adaptive numerical methods conserving the global properties of the exact solution of the Kepler problem. The requirement $0.5(v_{x,i+1}^2+v_{y,i+1}^2)-\gamma M r_{i+1}^{-1} = -H^*$ of total energy conservation at any step of the numerical method implies the following expression for r_{i+1} via $v_{x,i+1}$ and $v_{y,i+1}$:

$$r_{i+1} = \gamma M(0.5(v_{x,i+1}^2+v_{y,i+1}^2)+H^*)^{-1}, \qquad i=0,1,\ldots \qquad (25)$$

The requirement of conservation of the components of the Laplace–Runge–Lenz vector, $e_{\mathrm{LRL},x,i+1} = e_{\mathrm{LRL},x,i}$, $e_{\mathrm{LRL},y,i+1} = e_{\mathrm{LRL},y,i}$, and the angular momentum, $l_{z,i+1} = l_{z,i}$, $i=0,1,\ldots$, allows one to express the coordinates via the velocity in the approximate solution as

$$\begin{aligned} x_{i+1} &= -(\gamma M)^{-1}(e_{\mathrm{LRL},x,i} - l_{z,i}v_{y,i+1})r_{i+1} \\ &= -(e_{\mathrm{LRL},x,i} - l_{z,i}v_{y,i+1})(0.5(v_{x,i+1}^2+v_{y,i+1}^2)+H^*)^{-1} \\ &= -(e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i+1})(0.5(v_{x,i+1}^2+v_{y,i+1}^2)+H^*)^{-1}, \qquad (26) \\ y_{i+1} &= -(\gamma M)^{-1}(e_{\mathrm{LRL},y,i} + l_{z,i}v_{x,i+1})r_{i+1} \\ &= -(e_{\mathrm{LRL},y,i} + l_{z,i}v_{x,i+1})(0.5(v_{x,i+1}^2+v_{y,i+1}^2)+H^*)^{-1} \\ &= -(e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i+1})(0.5(v_{x,i+1}^2+v_{y,i+1}^2)+H^*)^{-1}. \qquad (27) \end{aligned}$$

Squaring relations (26) and (27) and summing the results, one obtains

$$H_v(v_{x,i+1}, v_{y,i+1}) = (e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i+1})^2 + (e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i+1})^2 - (\gamma M)^2 = 0, \qquad (28)$$

which means that the point $(v_{x,i+1}, v_{y,i+1})$ in the approximate solution belongs to the exact velocity hodograph (see Eq. (13)). Eliminating the variables $v_{x,i+1}$ and $v_{y,i+1}$ from (25) with the use of relations (26) and (27), one obtains

$$\begin{aligned} O(x_{i+1}, y_{i+1}) &= (\gamma^2M^2 - e_{\mathrm{LRL},x,0}^2)x_{i+1}^2 + (\gamma^2M^2 - e_{\mathrm{LRL},y,0}^2)y_{i+1}^2 \\ &\quad - 2e_{\mathrm{LRL},x,0}e_{\mathrm{LRL},y,0}x_{i+1}y_{i+1} + 2l_{z,0}^2e_{\mathrm{LRL},x,0}x_{i+1} + 2l_{z,0}^2e_{\mathrm{LRL},y,0}y_{i+1} - l_{z,0}^4 = 0, \end{aligned}$$

which means that the point (x_{i+1}, y_{i+1}) in the approximate solution belongs to the exact orbit (14).

Thus, the conditions that the first integrals are conserved in the approximate solution at any step of the proposed methods imply that the coordinates depend on the velocities according to (25)–(27) and the exact velocity hodograph (13) and the exact orbit (14) are conserved.

We associate the points $(v_{x,i+1}, v_{y,i+1})$ of the velocity hodograph and accordingly the points (x_{i+1}, y_{i+1}) of the orbit to the time t_{i+1}; i.e., we approximate Eqs. (16)–(18) so as to satisfy condition (28). This can be done with the use of symmetrically symplectic Runge–Kutta methods [16, 17]. It is well known that these methods conserve the quadratic first integrals of the equations to be approximated [2, p. 97]. Recall that Eqs. (16), (17) have a first integral H_v that is a second-order polynomial. Therefore, the numerical methods chosen to solve the Cauchy problem for Eqs. (16), (17) automatically conserve the velocity hodograph.

We restrict ourselves to two-stage methods for approximating Eqs. (16)–(18). Obviously, symplectic Runge–Kutta methods with more stages can be used to derive methods of higher approximation order.

All two-stage symmetrically symplectic Runge–Kutta methods are contained in the one-parameter family with a free parameter s, $-1 < s < 0$, and are described by the Butcher table shown in the table below (see [7, p. 140 of the Russian translation]).

The methods are of least of the second approximation order. For $s = 0$, the two-stage method degenerates and becomes a one-stage symmetrically symplectic method, the midpoint method. For $s = -3^{-1/2}$, the two-stage method is of the fourth approximation order.

The one-parameter family of two-stage symmetrically symplectic methods gives the system of equations

$$k_{vx,1} = e_{\mathrm{LRL},x,0} - l_{z,0}(v_{y,i} + 0.5\Delta\omega(0.5k_{vy,1} + (0.5 + s)k_{vy,2})), \tag{29}$$
$$k_{vx,2} = e_{\mathrm{LRL},x,0} - l_{z,0}(v_{y,i} + 0.5\Delta\omega((0.5 - s)k_{vy,1} + 0.5k_{vy,2})), \tag{30}$$
$$k_{vy,1} = e_{\mathrm{LRL},y,0} + l_{z,0}(v_{x,i} + 0.5\Delta\omega(0.5k_{vx,1} + (0.5 + s)k_{vx,2})), \tag{31}$$
$$k_{vy,2} = e_{\mathrm{LRL},y,0} + l_{z,0}(v_{x,i} + 0.5\Delta\omega((0.5 - s)k_{vx,1} + 0.5k_{vx,2})) \tag{32}$$

for the unknowns $k_{vx,1}$, $k_{vx,2}$, $k_{vy,1}$, and $k_{vy,2}$ and the formulas (see Eq. (18))

$$\begin{aligned} k_{t,1} &= (\gamma M)^2(0.5((v_{x,i} + 0.5\Delta\omega(0.5k_{vx,1} + (0.5 + s)k_{vx,2}))^2 \\ &\quad + (v_{y,i} + 0.5\Delta\omega(0.5k_{vy,1} + (0.5 + s)k_{vy,2}))^2) + H^*)^{-2}, \\ k_{t,2} &= (\gamma M)^2(0.5((v_{x,i} + 0.5\Delta\omega((0.5 - s)k_{vx,1} + 0.5k_{vx,2}))^2 \\ &\quad + (v_{y,i} + 0.5\Delta\omega((0.5 - s)k_{vy,1} + 0.5k_{vy,2}))^2) + H^*)^{-2} \end{aligned}$$

for calculating $k_{t,1}$ and $k_{t,2}$. The solution of Eqs. (29)–(32) has the form

$$\begin{aligned} k_{vx,1} &= ((e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})(1 - 0.25s(s + 1)(\Delta\omega)^2 l_{z,0}^2) \\ &\quad + 0.5(e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})(\Delta\omega)l_{z,0}(-(s + 1) + 0.25s^3(\Delta\omega)^2 l_{z,0}^2))\Delta^{-1}, \\ k_{vx,2} &= ((e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})(1 - 0.25s(s - 1)(\Delta\omega)^2 l_{z,0}^2) \\ &\quad + 0.5(e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})(\Delta\omega)l_{z,0}((s - 1) - 0.25s^3(\Delta\omega)^2 l_{z,0}^2))\Delta^{-1}, \\ k_{vy,1} &= ((e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})(1 - 0.25s(s + 1)(\Delta\omega)^2 l_{z,0}^2) \\ &\quad - 0.5(e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})(\Delta\omega)l_{z,0}(-(s + 1) + 0.25s^3(\Delta\omega)^2 l_{z,0}^2))\Delta^{-1}, \\ k_{vy,2} &= ((e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})(1 - 0.25s(s - 1)(\Delta\omega)^2 l_{z,0}^2) \\ &\quad - 0.5(e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})(\Delta\omega)l_{z,0}((s - 1) - 0.25s^3(\Delta\omega)^2 l_{z,0}^2))\Delta^{-1}, \end{aligned}$$

where $\Delta = 1 + 0.25(1 - 2s^2)(\Delta\omega)^2 l_{z,0}^2 + (1/16)s^4(\Delta\omega)^4 l_{z,0}^4$.

Table

$0.5(1+s)$	0.25	$0.25+0.5s$
$0.5(1-s)$	$0.25-0.5s$	0.25
	0.5	0.5

The velocity values and the time increment at the $(i+1)$st step are determined by the formulas

$$\begin{aligned} v_{x,i+1} &= v_{x,i} + 0.5\Delta\omega(k_{vx,1} + k_{vx,2}) \\ &= v_{x,i} + 0.5\Delta\omega[(e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})(1 - 0.25s^2(\Delta\omega)^2 l_{z,0}^2) - 0.5(e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})(\Delta\omega)l_{z,0}]\Delta^{-1} \\ &= v_{x,i} + 0.5\Delta\omega[(e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})q_1 - 0.5(e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})q_2], \qquad (33) \\ v_{y,i+1} &= v_{y,i} + 0.5\Delta\omega(k_{vy,1} + k_{vy,2}) \\ &= v_{y,i} + 0.5\Delta\omega[(e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})(1 - 0.25s^2(\Delta\omega)^2 l_{z,0}^2) + 0.5(e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})(\Delta\omega)l_{z,0}]\Delta^{-1} \\ &= v_{y,i} + 0.5\Delta\omega[(e_{\mathrm{LRL},y,0} + l_{z,0}v_{x,i})q_1 + 0.5(e_{\mathrm{LRL},x,0} - l_{z,0}v_{y,i})q_2], \qquad (34) \\ t_{i+1} &= t_i + 0.5\Delta\omega(k_{t,1} + k_{t,2}), \qquad (35) \end{aligned}$$

where $q_1 = (1 - 0.25s^2(\Delta\omega)^2 l_{z,0}^2)\Delta^{-1}$, $q_2 = (\Delta\omega)l_{z,0}\Delta^{-1}$.

To calculate x_{i+1} and y_{i+1}, we use (25)–(27).

4. MAIN RESULT

We have constructed a new one-parameter family of adaptive numerical methods with computational formulas (33)–(35), (26), and (27) for the Kepler problem (1)–(5) starting from the conservation laws for the angular momentum, the total energy, and the Laplace–Runge–Lenz vector. The construction is based on the parametrization (15) and two-stage symmetrically symplectic Runge–Kutta methods [16, 17]. The above-described construction of the methods implies the following assertions.

Proposition 1. *The methods are adaptive. The subsequent increment in time* t_i *on the nonuniform grid* $\Omega_t = \{t_i : t_i = t_{i-1} + \tau_i,\ i = 1, 2, \dots\}$ *is chosen automatically based on the properties of the solution of the problem. In a neighborhood of the force center, where the force and the velocity vary most rapidly, the time increment is significantly less than on the other part of the trajectory of motion. The increment scale is determined by the parameter* $\Delta\omega$, *which is a small fraction of either the solution period* T_ω $(H^* > 0)$ *or of the value* ω_∞ $(H^* < 0)$.

Proposition 2. *The methods are conservative. For any admissible value of the parameter* $\Delta\omega$, *in the framework of exact arithmetics, the methods conserve the angular momentum, the Laplace–Runge–Lenz vector, the total energy, the orbit, and the velocity hodograph.*

Proposition 3. *For small values of the parameter* $\Delta\omega$ $(\Delta\omega \to 0)$ *and* $s \in (-1, 0]$, *the methods for approximating the time dependence of the phase variables are at least of the second approximation order. For* $s = -3^{-1/2}$, *the method is of the fourth approximation order.*

Proposition 4. *The methods realize a symplectic mapping*

$$(v_{x,i}, v_{y,i}, x_i, y_i) \mapsto (v_{x,i+1}, v_{y,i+1}, x_{i+1}, y_{i+1}).$$

Using formulas (33), (34), (26), *and* (27) *to calculate the matrix elements*

$$F_{i+1} = \partial(v_{x,i+1}, v_{y,i+1}, x_{i+1}, y_{i+1})/\partial(v_{x,i}, v_{y,i}, x_i, y_i),$$

one can readily see that all six nontrivial elements of the skew-symmetric symplectic defect matrix $\Delta S_{i+1} = F_{i+1}^{\mathrm{T}} J F_{i+1} - J$ *are zero.*

The results of testing the new methods for solving the Kepler problem by using the exact solution showed their high efficiency in the analysis of elliptic strongly elongated orbits compared with the numerical methods listed in the introduction.

ACKNOWLEDGMENTS

This work was supported in part by Lomonosov Moscow State University (research project "Mathematical modeling in natural sciences and computational methods") and by the Federal Research Center "Computer Science and Control" of Russian Academy of Sciences (project no. 0065-2014-0031).

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