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ORDINARY DIFFERENTIAL EQUATIONS

Stability of Solutions of Stochastic Functional-Differential Equations with Locally Lipschitz Coefficients in Hilbert Spaces

M. M. Vas'kovskii* **and I. V. Kachan****

Belarusian State University, Minsk, 220030 Belarus
*e-mail: *vaskovskii@bsu.by, **ilyakachan@gmail.com*

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Abstract— We obtain sufficient conditions for the stability of weak solutions of nonlinear stochastic functional-differential equations in Hilbert spaces with random coefficients satisfying the nonlocal Lipschitz condition.

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Many physical, biological, and economic phenomena can be modeled by stochastic partial differential equations and stochastic delay differential equations [1, 2]. Stochastic partial differential equations can be treated as Itô stochastic differential equations in appropriate Hilbert spaces.

Assume that the following objects are given: a probability space $(\Omega, \mathcal{F}, P)$ with a flow $(\mathcal{F}_t)$, $t \geq -h$, of σ-algebras, $\mathcal{F}_t = \mathcal{F}_0$ for $t \in [-h, 0]$; two separable Hilbert spaces H and U; a nonnegative trace class operator Q_w on the space U; an $\mathcal{F}_t$-coordinated Brownian motion $W(t,\omega)$ ranging in U with covariance operator Q_w; a linear operator A defined on a set $D(A)$ everywhere dense in the space H and generating a C_0-semigroup $S(t)$ on H; functions $f : \mathbb{R}^+ \times \Omega \times C_h \to H$ and $g : \mathbb{R}^+ \times \Omega \times C_h \to L_2(U, H)$, where $\mathbb{R}^+ = [0, \infty)$, $C_h = C([-h, 0], H)$, $C_0 = H$, and $L_2(U, H)$ is the Hilbert space of Hilbert–Schmidt operators acting from the space U to the space H; a continuous $\mathcal{F}_0$-measurable stochastic process $\xi : [-h, 0] \times \Omega \to D(A)$ such that $E(\sup_{t\in[-h,0]} \|\xi(t)\|^p) < \infty$ for some $p > 2$. We assume that the functions $f(t, \omega, \varphi)$ and $g(t, \omega, \varphi)$ are measurable, $\mathcal{F}_t$-coordinated for any fixed $\varphi \in C_h$, and continuous in φ for any fixed $(t, \omega) \in \mathbb{R}^+ \times \Omega$ and satisfy the following two conditions.

1. Local Lipschitz condition. For any $a > 0$, there exists a constant q_a such that, for all $t \in [0, a]$ and any $(\mathcal{F}, \beta(C_h))$-measurable random variables $\varphi, \psi : \Omega \to C_h$ for which $\|\varphi\| \leq a$ and $\|\psi\| \leq a$ almost surely (a.s.), the inequality $\|f(t, \omega, \varphi) - f(t, \omega, \psi)\| + \|g(t, \omega, \varphi) - g(t, \omega, \psi)\| \leq q_a\|\varphi - \psi\|$ holds a.s.

2. Condition of linear order of growth. There exists a continuous function $k\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $t \in \mathbb{R}^+$ and any $(\mathcal{F}, \beta(C_h))$-measurable random variable $\eta : \Omega \to C_h$ for which $E\|\eta\|^p < \infty$, the inequality $\|f(t, \omega, \eta)\| + \|g(t, \omega, \eta)\| \leq k(t)(1 + \|\eta\|)$ is satisfied a.s.

Consider the stochastic differential equation

$$dX(t,\omega) = (AX(t,\omega) + f(t,\omega, X_t(\omega)))\,dt + g(t,\omega, X_t(\omega))\,dW(t), \qquad t > 0, \qquad X \in H, \tag{1}$$

with the initial condition

$$X(0,\omega) = \xi(t,\omega), \qquad t \in [-h, 0], \qquad h > 0. \tag{2}$$

In [3, 4], the Lyapunov functional method was used to prove stability theorems for solutions of Eq. (1) with deterministic globally Lipschitz coefficients f, g and without delay. In [5], a theorem on the exponential stability of solutions of Eq. (1) with generator A of an analytic semigroup and deterministic globally Lipschitz coefficients f and g was proved. In the present paper, we use the Lyapunov functional method to prove the stability theorem for solutions of Eq. (1) with delay, with generator A of a semigroup of class C_0, and with random locally Lipschitz coefficients f and g.

In what follows, we omit the argument ω to simplify the presentation. All integrals in the definitions below are written under the assumption that they exist and are finite.

Definition 1. A continuous $\mathcal{F}_t$-coordinated stochastic process $X(t)$, $t \geq -h$, is called a *weak solution* of Eq. (1) with the initial condition (2) if the following relation holds with probability 1:

$$X(t) = \begin{cases} \xi(t), & t \in [-h, 0], \\ S(t)\xi(0) + \int_0^t S(t-s) f(s, X_s)\, ds + \int_0^t S(t-s) g(s, X_s)\, dW(s), & t \in \mathbb{R}^+. \end{cases}$$

Definition 2. A continuous $\mathcal{F}_t$-coordinated stochastic process $X(t)$, $t \geq -h$, ranging in the set $D(A)$ for almost all $(t, \omega) \in [-h, \infty) \times \Omega$ is called a *strong solution* of Eq. (1) with the initial condition (2) if the following relation holds with probability 1:

$$X(t) = \begin{cases} \xi(t), & t \in [-h, 0], \\ \xi(0) + \int_0^t AX(t)\, dt + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dW(s), & t \in \mathbb{R}^+. \end{cases}$$

Definition 3. A (weak or strong) solution $X(t)$ of Eq. (1) with the initial condition (2) is said to be *unique* if any other solution $Y(t)$ of Eq. (1) with the initial condition (2) coincides a.s. with $X(t)$, i.e., if $P(X(t) = Y(t)$ for all $t \geq -h) = 1$.

Definition 4. A weak solution $X(t)$ of problem (1), (2) is said to be *stable* if there exist constants $\gamma > 0$ and $T > 0$ and a function $\lambda : [T, \infty) \to \mathbb{R}^+$ such that the following conditions are satisfied:

1. $\lim\limits_{t\to\infty} \lambda(t) = \infty$.
2. The function $\log \lambda(t)$ is uniformly continuous in $t \geq T$.
3. There exists a constant $\tau \geq 0$ such that $\limsup\limits_{t\to\infty} \log\log t / \log \lambda(t) \leq \tau$.
4. The inequality $\limsup\limits_{t\to\infty} \log \|X(t)\| / \log \lambda(t) \leq -\gamma$ holds a.s.

For any nonnegative functional $V(t, x)$ continuously differentiable with respect to t and twice continuously differentiable with respect to x, we define mappings LV and QV by the formulas

$$\begin{aligned} LV(t, x, \varphi) &= V_t'(t, x) + \langle V_x'(t, x), Ax + f(t, \varphi)\rangle \\ &\quad + \frac{1}{2}\operatorname{tr}[V_{xx}''(t, x)(g(t, \varphi)Q_w^{1/2})(g(t, \varphi)Q_w^{1/2})^*], \quad t \in \mathbb{R}^+, \quad x \in D(A), \quad \varphi \in C_h, \\ QV(t, x, \varphi) &= \operatorname{tr}[V_{xx}''(t, x) \otimes V_{xx}''(t, x)(g(t, \varphi)Q_w^{1/2})(g(t, \varphi)Q_w^{1/2})^*], \quad t \in \mathbb{R}^+, \quad x \in H, \quad \varphi \in C_h. \end{aligned}$$

Theorem. *Assume that there exists a functional $V(t, x) \in C^{1,2}(\mathbb{R}^+ \times H; \mathbb{R}^+)$, nonnegative continuous functions $\psi_1(t)$ and $\psi_2(t)$, and a continuous nonnegative function $\lambda(t)$ satisfying conditions (1)–(3) in Definition 4 such that the following conditions are satisfied for some constants $r > 0$, $m \geq 0$, and $\mu, \nu, \theta \in \mathbb{R}$ and a nonincreasing positive function $\zeta(t)$:*

1. *$\|x\|^r (\lambda(t))^m \leq V(t, x)$ for all $(t, x) \in \mathbb{R}^+ \times H$.*
2. *$LV(t, x, \varphi) + \zeta(t) QV(t, x, \varphi) \leq \psi_1(t) + \psi_2(t) V(t, x)$ for all $x \in D(A)$, $t \in \mathbb{R}^+$, and $\varphi \in C_h$.*

3. $\limsup_{t\to+\infty} \log(\int_0^t \psi_1(s)\,ds)/\log\lambda(t) \le \nu,\qquad \limsup_{t\to+\infty} \int_0^t \psi_2(s)\,ds/\log\lambda(t) \le \theta,$
$\liminf_{t\to+\infty} \log\zeta(t)/\log\lambda(t) \ge -\mu.$

4. $-m+\theta+\max\{\nu,\mu+\tau\} < 0,$ *where* $\tau = \limsup_{t\to\infty} \log\log t/\log\lambda(t).$

Then any weak solution $X(t)$ *of problem* (1), (2) *is stable.*

Proof. To prove the theorem, we need the following two lemmas.

Lemma 1 [3]. *Assume that the following conditions are satisfied a.s. for any* $T>0$:

(a) $\xi(0)\in D(A)$, $S(t-r)f(r,\varphi)\in D(A)$, *and* $S(t-r)g(r,\varphi)z\in D(A)$ *for any* t, r, $T\ge t>r\ge 0$, $\varphi\in C_h$, *and* $z\in H$.

(b) $\int_0^T\int_0^t \|AS(t-r)f(r,\varphi)\|\,dr\,dt<\infty.$

(c) $\int_0^T\int_0^t \|AS(t-r)g(r,\varphi)\|^2\,dr\,dt<\infty.$

Then a weak solution $X(t)$ *of problem* (1), (2) *is a strong solution.*

By $\rho(A)$ we denote the resolvent set of the operator A, i.e., the set of $l\in\mathbb{C}$ for which the operator $R(l,A)=(lI-A)^{-1}$ is well defined (the resolvent of the operator A). We also write $R(l)=lR(l,A)$ and $\rho_{\mathbb{R}}(A)=\rho(A)\cap\mathbb{R}$.

For problem (1), (2), consider the approximating Cauchy problem

$$\begin{aligned} dX^l(t) &= AX^l(t)+R(l)f(t,X_t^l)\,dt+R(l)g(t,X_t^l)\,dW(t), \qquad t\in\mathbb{R}^+,\\ X^l(t) &= \xi(t),\qquad t\in[-h,0], \end{aligned}\tag{3}$$

where $l\in\rho_{\mathbb{R}}(A)$.

Lemma 2. *For any sufficiently large* $l\in\rho_{\mathbb{R}}(A)$, *the Cauchy problem* (3) *has a unique strong solution* X^l, *and there exists a subsequence* X^{l_n} *such that* $X^{l_n}(t)\to X(t)$ *as* $n\to\infty$ *a.s. uniformly in* $t\in[0,T]$, *where* $X(t)$ *is a weak solution of problem* (1), (2) *and* $T>0$ *is an arbitrary number.*

Proof. By Theorem 3.1 in [6], we have the integral representation

$$R(l,A)x=\int_0^{+\infty} e^{-l\,t}S(t)x\,dt,\qquad x\in H,$$

of the resolvent for all $l\in\rho(A)$ for which this integral exists and is finite. Theorem 2.2 in [6] implies the existence of constants M and $\beta\ge 0$ such that $\|S(t)\|\le Me^{\beta t}$ for all $t\ge 0$. Therefore, for the norm of the operator $R(l,A)$ we have the estimate

$$\|R(l,A)\|=\sup_{\|x\|=1}\|R(l,A)x\|\le\int_0^{+\infty}e^{-lt}\sup_{\|x\|=1}\|S(t)x\|\,dt\le M\int_0^{+\infty}e^{-(l-\beta)t}\,dt=\frac{M}{l-\beta},\qquad l>\beta,$$

which implies that $(\beta,+\infty]\subset\rho(A)$ and, in addition, $\|R(l)\|\le Ml/(l-\beta)\le 2M$ for $l\ge 2\beta$. In what follows, we consider only $l\ge 2M$ and refer to them as "sufficiently large" l.

Let us prove that the Cauchy problem (3) has a unique weak solution for sufficiently large l. Since the functions f and g have a linear order of growth and satisfy the local Lipschitz condition, we see that the following relations are satisfied for any $r>1$:

$$\begin{aligned} E\|R(l)f(t,\varphi)-R(l)f(t,\psi)\|^r &\le (2M)^rE\|f(t,\varphi)-f(t,\psi)\|^r\le(2Mq_a)^rE\|\varphi-\psi\|^r,\\ E\|R(l)g(t,\varphi)-R(l)g(t,\psi)\|^r &\le (2M)^rE\|g(t,\varphi)-g(t,\psi)\|^r\le(2Mq_a)^rE\|\varphi-\psi\|^r,\\ E\|R(l)f(t,\eta)\|^r &\le (2M)^rE(k(t)(1+\|\eta\|))^r\\ &\le (2M)^rE((k(t))^r\cdot 2^{r-1}(1+\|\eta\|^r))=\frac{(4Mk(t))^r}{2}(1+E\|\eta\|^r),\\ E\|R(l)g(t,\eta)\|^r &\le (2M)^rE(k(t)(1+\|\eta\|))^r\\ &\le (2M)^rE((k(t))^r\cdot 2^{r-1}(1+\|\eta\|^r))=\frac{(4Mk(t))^r}{2}(1+E\|\eta\|^r), \end{aligned}$$

where $\varphi, \psi, \eta : \Omega \to H$ are arbitrary $\mathcal{F}_0$-measurable random variables with finite moment of order p such that $\|\varphi\| \le a$ and $\|\zeta\| \le a$ a.s. Thus, the Cauchy problem (3) satisfies the assumptions of Theorem 1 in [7] and hence has a unique weak solution $X^l(t)$, $t \ge -h$.

Let us prove that the weak solution $X^l(t)$, $t \ge -h$, is also a strong solution of the Cauchy problem (3). To this end, it suffices to verify the assumptions of Lemma 1, first fixing a time interval $[0, T]$. By Theorem 2.4.c in [6], the operators A and $S(t)$ [and $(lI - A)$ and $S(t)$] commute with each other. The following relations hold for any $r \in [0, t)$, $x \in H$, and $u \in U$:

$$(lI - A)(S(t-r)R(l)f(r,\varphi)) = S(t-r)(lI - A)l(lI - A)^{-1}f(r,\varphi) = lS(t-r)f(r,\varphi) \in H,$$
$$(lI - A)(S(t-r)R(l)g(r,\varphi)u) = S(t-r)(lI - A)l(lI - A)^{-1}g(r,\varphi)u = lS(t-r)g(r,\varphi)u \in H,$$

which implies the inclusions

$$S(t-r)R(l)f(r,\varphi), S(t-r)R(l)g(r,\varphi)u \in D(lI - A) = D(A).$$

Moreover, $\xi(0) \in D(A)$; i.e., assumption (a) of Lemma 1 is satisfied. Further, we estimate the integral in assumption (b) of Lemma 1,

$$\int_0^T\int_0^t \|AS(t-r)R(l)f(r,X_r^l)\|\,dr\,dt \le \int_0^T\int_0^t \|(A - lI)S(t-r)l(lI - A)^{-1}f(r,X_r^l)\|\,dr\,dt$$
$$+ \int_0^T\int_0^t \|lS(t-r)l(lI - A)^{-1}f(r,X_r^l)\|\,dr\,dt = I_1 + I_2,$$

$$I_1 = \int_0^T\int_0^t \|(A - lI)S(t-r)l(lI - A)^{-1}f(r,X_r^l)\|\,dr\,dt$$
$$= \int_0^T\int_0^t \|S(t-r)(A - lI)l(lI - A)^{-1}f(r,X_r^l)\|\,dr\,dt$$
$$= l\int_0^T\int_0^t \|S(t-r)f(r,X_r^l)\|\,dr\,dt.$$

Since the operators $lI - A$ and $S(t)$ commute, it follows that so do the operators $R(l)$ and $S(t)$. Therefore,

$$I_2 = \int_0^T\int_0^t \|lS(t-r)R(l)f(r,X_r^l)\|\,dr\,dt = \int_0^T\int_0^t \|lR(l)S(t-r)f(r,X_r^l)\|\,dr\,dt$$
$$\le l\int_0^T\int_0^t \|R(l)\|\|S(t-r)f(r,X_r^l)\|\,dr\,dt \le 2Ml\int_0^T\int_0^t \|S(t-r)f(r,X_r^l)\|\,dr\,dt,$$

$$\int_0^T\int_0^t \|AS(t-r)R(l)f(r,X_r^l)\|\,dr\,dt \le (2M+1)l\int_0^T\int_0^t \|S(t-r)\|\|f(r,X_r^l)\|\,dr\,dt$$
$$\le (2M+1)l\int_0^T\int_0^t Me^{\beta(t-r)}k(r)(1 + \|X_r^l\|)\,dr\,dt$$
$$= M(2M+1)l\int_0^T e^{\beta t}\,dt\int_0^t e^{-\beta r}k(r)(1 + \|X_r^l\|)\,dr = I_3.$$

Since the process $X^l(r)$ is a.s. continuous, we have

$$M_T = \sup_{-h\le r\le T} \|X^l(r)\| < \infty, \qquad \|X_r^l\| = \sup_{r-h\le s\le r} \|X_r^l\| \le M_T,$$

where $r \le t \le T$, and since the function $k(r)$ is continuous on the interval $[0,T]$, it obviously follows that the integral I_3 is a.s. finite; i.e., assumption (b) is satisfied. Assumption (c) can be verified in a similar way,

$$\int_0^T\int_0^t \|AS(t-r)R(l)g(r,X_r^l)\|^2\,dr\,dt \le 2(I_1+I_2),$$

$$I_1 = \int_0^T\int_0^t \|(A-lI)S(t-r)l(l\,I-A)^{-1}g(r,X_r^l)\|^2\,dr\,dt \le l^2\int_0^T\int_0^t \|S(t-r)g(r,X_r^l)\|^2\,dr\,dt,$$

$$I_2 = \int_0^T\int_0^t \|lS(t-r)R(l)g(r,X_r^l)\|^2\,dr\,dt \le 4M^2l^2\int_0^T\int_0^t \|S(t-r)g(r,X_r^l)\|^2\,dr\,dt,$$

$$\int_0^T\int_0^t \|AS(t-r)R(l)g(r,X_r^l)\|^2\,dr\,dt \le 2l^2(4M^2+1)\int_0^T\int_0^t \|S(t-r)\|^2\|g(r,X_r^l)\|^2\,dr\,dt$$

$$\le 4l^2(4M^2+1)M^2\int_0^T e^{2\beta t}\,dt\int_0^t e^{-2\beta r}(k(r))^2(1+\|X_r^l\|^2)dr < +\infty.$$

Thus, all assumptions of Lemma 1 are satisfied, and hence the Cauchy problem (3) has a strong solution. This solution is unique, because each strong solution is a weak solution (see Proposition 2.1 in [3]) and the weak solution of problem (3) is unique.

Now let us prove the existence of the desired subsequence X^{l_n}. First, we show that for any $T\in\mathbb{R}^+$ there exists a constant $C(T)>0$ such that the following inequality holds for the weak solution of problem (1), (2):

$$E\Big(\sup_{-h\le s\le T}\|X(s)\|^p\Big) \le C(T).$$

Indeed, using the Hölder and Burkholder inequalities and the fact that the functions f and g have a linear order of growth, we obtain

$$E\Big(\sup_{0\le s\le T}\|X(s)\|^p\Big) \le 3^{p-1}\bigg(E\|S(T)\xi(0)\|^p + C_0(T)E\int_0^T \|S(T-s)f(s,X_s)\|^p\,ds$$

$$+E\bigg(\sup_{0\le s\le T}\bigg\|\int_0^s S(s-r)g(r,X_r)\,dW(r)\bigg\|^p\bigg)\bigg)$$

$$\le \widetilde{C}_1(T)+\widetilde{C}_2(T)E\int_0^T(1+\|X_s\|^p)\,ds+\widetilde{C}_3(T)E\int_0^T(1+\|X_s\|^p)\,ds$$

$$\le C_1(T)+C_2(T)\int_0^T E\Big(\sup_{0\le s\le T}\|X_s\|^p\Big)\,ds = C_1(T)+C_2(T)\int_0^T E\Big(\sup_{-h\le s\le T-h}\|X(s)\|^p\Big)\,ds$$

$$\le C_1(T)+C_2(T)\int_0^T E\Big(\sup_{-h\le s\le T}\|X(s)\|^p\Big)\,ds,$$

$$E\biggl(\sup_{-h\le s\le T}\|X(s)\|^p\biggr)=\max\biggl\{E\biggl(\sup_{-h\le s\le 0}\|X(s)\|^p\biggr),E\biggl(\sup_{0\le s\le T}\|X(s)\|^p\biggr)\biggr\}$$
$$=\max\biggl\{\max_{-h\le s\le 0}E\|\xi(s)\|^p,E\biggl(\sup_{0\le s\le T}\|X(s)\|^p\biggr)\biggr\}\le\max_{-h\le s\le 0}E\|\xi(s)\|^p+E\biggl(\sup_{0\le s\le T}\|X(s)\|^p\biggr)$$
$$\le\overline{C}_1(T)+C_2(T)\int_0^T E\biggl(\sup_{-h\le s\le T}\|X(s)\|^p\biggr)\,ds.$$

By the Gronwall inequality, this implies the inequality

$$E(\sup_{-h\le s\le T}\|X(s)\|^p)\le\overline{C}_1(T)e^{C_2(T)T}\stackrel{\mathrm{def}}{=}C(T),$$

as stated.

Since $\|R(l)\|\le 2M$ for sufficiently large l, we can in a similar way prove the following estimate of the solution $X^l(t)$ of problem (3): there exists a constant $K(T)$ such that

$$E(\sup_{-h\le s\le T}\|X^l(s)\|^p)\le K(T).$$

We write $C_T=\max(C(T),K(T))$.

For any $a,T>0$ and a sufficiently large l, we define the set

$$\Omega_l^{a,T}=\biggl\{\omega\in\Omega:\ \max\biggl(\sup_{-h\le s\le T}\|X(s)\|,\ \sup_{-h\le s\le T}\|X^l(s)\|\biggr)\le a\biggr\}$$

and its characteristic function $\zeta_l^{a,T}=1_{\Omega_l^{a,T}}(\omega)$. Since

$$X(t)-X^l(t)=\int_0^t S(t-s)(I-R(l))f(s,X_s)\,ds+\int_0^t S(t-s)(I-R(l))g(s,X_s)\,dW(s)$$
$$+\int_0^t S(t-s)R(l)(f(s,X_s)-f(s,X_s^l))\,ds$$
$$+\int_0^t S(t-s)R(l)(g(s,X_s)-g(s,X_s^l))\,dW(s),\qquad t\ge 0,$$

we have

$$E\biggl(\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p\zeta_l^{a,T}\biggr)\le 3^{p-1}E\sup_{0\le t\le T}\biggl\|\int_0^t S(t-s)R(l)(f(s,X_s)-f(s,X_s^l))\,ds\biggr\|^p\zeta_l^{a,T}$$
$$+3^{p-1}E\sup_{0\le t\le T}\biggl\|\int_0^t S(t-s)R(l)(g(s,X_s)-g(s,X_s^l))\,dW(s)\biggr\|^p\zeta_l^{a,T}$$
$$+3^{p-1}E\sup_{0\le t\le T}\biggl\|\int_0^t S(t-s)(I-R(l))f(s,X_s)\,ds+\int_0^t S(t-s)(I-R(l))g(s,X_s)\,dW(s)\biggr\|^p\zeta_l^{a,T}.$$

We separately estimate each term on the right-hand side in the resulting inequality. Using the Hölder inequality, we obtain

$$
\begin{aligned}
&3^{p-1}E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)R(l)(f(s,X_s)-f(s,X_s^l))\,ds\right\|^p\zeta_l^{a,T}\\
&\le 3^{p-1}E\sup_{0\le t\le T}\left(\int_0^t \|S(t-s)R(l)(f(s,X_s)-f(s,X_s^l))\|\,ds\right)^p\zeta_l^{a,T}\\
&\le \widetilde{C}_3(T)E\sup_{0\le t\le T}\left(\int_0^t \sup_{0\le\tau\le T}\|S(\tau-s)\|^p\|f(s,X_s)-f(s,X_s^l)\|^p ds\right)\zeta_l^{a,T}\\
&\le C_3(a,T)\int_0^T E\sup_{0\le r\le s}\|X_r-X_r^l\|^p\zeta_l^{a,T}\,ds = C_3(a,T)\int_0^T E\sup_{-h\le r\le s-h}\|X(r)-X^l(r)\|^p\zeta_l^{a,T}\,ds\\
&= C_3(a,T)\int_0^T E\sup_{0\le r\le s-h}\|X(r)-X^l(r)\|^p\zeta_l^{a,T}\,ds \le C_3(a,T)\int_0^T E\sup_{0\le r\le s}\|X(r)-X^l(r)\|^p\zeta_l^{a,T}\,ds.
\end{aligned}
$$

In a similar way, using the Burkholder inequality, we obtain

$$
\begin{aligned}
&3^{p-1}E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)R(l)(g(s,X_s)-g(s,X_s^l))\,dW(s)\right\|^p\zeta_l^{a,T}\\
&\le \widetilde{C}_4(T)E\sup_{0\le t\le T}\left(\int_0^t \|R(l)\|^p\|g(s,X_s)-g(s,X_s^l)\|^p\,ds\right)\zeta_l^{a,T}\\
&\le C_4(a,T)\int_0^T E\sup_{0\le r\le s}\|X_r-X_r^l\|^p\zeta_l^{a,T}\,ds = C_4(a,T)\int_0^T E\sup_{-h\le r\le s-h}\|X(r)-X^l(r)\|^p\zeta_l^{a,T}\,ds\\
&= C_4(a,T)\int_0^T E\sup_{0\le r\le s-h}\|X(r)-X^l(r)\|^p\zeta_l^{a,T}\,ds \le C_4(a,T)\int_0^T E\sup_{0\le r\le s}\|X(r)-X^l(r)\|^p\zeta_l^{a,T}\,ds.
\end{aligned}
$$

Finally, the last term does not exceed

$$
\begin{aligned}
&3^{2p-2}E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)(I-R(l))f(s,X_s)\,ds\right\|^p\zeta_l^{a,T}\\
&\quad+3^{2p-2}E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)(I-R(l))g(s,X_s)\,dW(s)\right\|^p\zeta_l^{a,T} = 3^{2p-2}(I_1+I_2).
\end{aligned}
$$

It remains to estimate the terms I_1 and I_2. By Lemma 3.2 in [6], we have $\|x-R(l)x\| \underset{l\to\infty}{\to} 0$ for any $x\in H$. At the same time,

$$
\begin{aligned}
\|(I-R(l))f(s,X_s)\|^p &\le \|I-R(l)\|^p\|f(s,X_s)\|^p\\
&\le (1+2M)^p\left(2^{p-1}\max_{0\le s\le T}(k(s))\right)^p\left(1+\sup_{-h\le s\le T-h}\|X(s)\|^p\right)=\eta,\\
E\eta &\le 2^{p-1}(1+2M)^p\max_{0\le s\le T}(k(s))^p(1+C(T))<\infty.
\end{aligned}
$$

Therefore, by the Lebesgue dominated convergence theorem, we conclude that

$$E\sup_{0\le t\le T}\|S(t)\|^p\|(I-R(l))f(s,X_s)\|^p \le C_5(T)E\|(I-R(l))f(s,X_s)\|^p \underset{l\to\infty}{\to} 0.$$

In a similar way, we obtain $E\|(I-R(l))g(s,X(s))\|^p \underset{l\to\infty}{\to} 0$. Now, using the Hölder inequality (for the term I_1) and the Burkholder inequality (for the term I_2) and passing to the limit in the integrand of the Lebesgue integral, we obtain

$$\begin{aligned}
I_1 &\le E\sup_{0\le t\le T}\Bigg(\int_0^t \|S(t-s)(I-R(l))f(s,X_s)\|\,ds\Bigg)^p \zeta_l^{a,T}\\
&\le \widetilde{C}_6(T)E\sup_{0\le t\le T}\Bigg(\int_0^t \sup_{0\le\tau\le T}\|S(\tau-s)\|^p\|(I-R(l))f(s,X_s)\|^p ds\Bigg)\zeta_l^{a,T}\\
&\le C_6(T)\int_0^T E\|(I-R(l))f(s,X_s)\|^p ds \underset{l\to\infty}{\to} 0,\\
I_2 &\le C_7(T)E\sup_{0\le t\le T}\Bigg(\int_0^t \|S(t-s)\|^p\|(I-R(l))g(s,X_s)\|^p ds\Bigg)\zeta_l^{a,T}\\
&\le C_8(T)\int_0^T E\|(I-R(l))g(s,X_s)\|^p ds \underset{l\to\infty}{\to} 0.
\end{aligned}$$

Thus, there exist constants $\widetilde{C}(a,T)$ and $\varepsilon(l)>0$ such that

$$E\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p\zeta_l^{a,T} \le \widetilde{C}(a,T)\int_0^T E\sup_{0\le r\le s}\|X(r)-X^l(r)\|^p\zeta_l^{a,T}\,ds+\varepsilon(l),$$

where $\varepsilon(l)\underset{l\to\infty}{\to} 0$. By the Gronwall lemma, this inequality implies that

$$E\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p\zeta_l^{a,T} \le \varepsilon(l)e^{T\widetilde{C}(a,T)} \underset{l\to\infty}{\to} 0 \quad \text{for any } a>0 \text{ and } T\in\mathbb{R}^+.$$

Let us show that this implies the convergence $\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p \underset{l\to\infty}{\to} 0$ in probability for any $T\in\mathbb{R}^+$. We use the Chebyshev inequality to obtain the estimates

$$\begin{aligned}
&P\Bigg(\sup_{0\le t\le T}\|X(t)\|>a\Bigg) \le a^{-p}E\Bigg(\sup_{0\le t\le T}\|X(t)\|\Bigg)^p = a^{-p}E\sup_{0\le t\le T}\|X(t)\|^p \le a^{-p}C_T,\\
&P\Bigg(\sup_{0\le t\le T}\|X^l(t)\|>a\Bigg) \le a^{-p}C_T.
\end{aligned}$$

Therefore, for any $a>0$ and sufficiently large l we have the inequality

$$P(\zeta_l^{a,T}=0) \le P\Bigg(\sup_{0\le t\le T}\|X(t)\|>a\Bigg)+P\Bigg(\sup_{0\le t\le T}\|X^l(t)\|>a\Bigg) \le 2a^{-p}C_T.$$

Take some $\varepsilon_1,\varepsilon_2>0$ and set $a=(4C_T/\varepsilon_2)^{1/p}$. Since

$$E\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p\zeta_l^{a,T} \underset{l\to\infty}{\to} 0,$$

it follows from the Chebyshev inequality that there exists an l_{ε_2} such that the inequality

$$P\Big(\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p\zeta_l^{a,T}>\varepsilon_1\Big)\le\varepsilon_2/2$$

holds for all $l\ge l_{\varepsilon_2}$. Thus, the inequality

$$P\Big(\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p>\varepsilon_1\Big)$$
$$\le P\Big(\Big(\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p\zeta_l^{a,T}>\varepsilon_1\Big)\cap(\zeta_l^{a,T}=1)\Big)+P(\zeta_l^{a,T}=0)\le\varepsilon_2$$

holds for all $l\ge l_{\varepsilon_2}$, and this is the convergence $\sup_{0\le t\le T}\|X(t)-X^l(t)\|^p\underset{l\to\infty}{\to}0$ in probability P.

Since every sequence of random variables converging in probability contains a subsequence converging a.s., it follows that there exists a subsequence $X^{l_n}(t)$ such that

$$\sup_{0\le t\le T}\|X(t)-X^{l_n}(t)\|^p\underset{l\to\infty}{\to}0\quad\text{a.s.};$$

i.e., $X(t)\underset{l\to\infty}{\to}X^{l_n}(t)$ uniformly in $t\in[0,T]$ a.s., and this is precisely the desired subsequence. The proof of the lemma is complete.

Now we continue the proof of the theorem.

We apply the Itô formula to the functional $V(t,x)$ and the (strong) solution $X^l(t)$ of problem (3). We have

$$\begin{aligned}V(t,X^l(t))&=V(0,\xi(0))+\int_0^t L_lV(s,X^l(s),X_s^l)\,ds+\int_0^t\langle V_x'(s,X^l(s)),R(l)g(s,X_s^l)\,dW(s)\rangle\\
&=V(0,\xi(0))+\Big(\int_0^t L_lV(s,X^l(s),X_s^l)\,ds-\int_0^t LV(s,X^l(s),X_s^l)\,ds\Big)\\
&\quad+\int_0^t LV(s,X^l(s),X_s^l)\,ds+\Big(\int_0^t\langle V_x'(s,X^l(s)),R(l)g(s,X_s^l)\,dW(s)\rangle\\
&\quad-\int_0^t\langle V_x'(s,X(s)),g(s,X_s)\,dW(s)\rangle\Big)+\int_0^t\langle V_x'(s,X(s)),g(s,X_s)\,dW(s)\rangle\\
&=V(0,\xi(0))+I_1(t,l)+\int_0^t LV(s,X^l(s),X_s^l)\,ds+I_2(t,l)\\
&\quad+\int_0^t\langle V_x'(s,X(s)),g(s,X_s)\,dW(s)\rangle,\end{aligned}\tag{4}$$

where

$$I_1(t,l)=\int_0^t L_lV(s,X^l(s),X_s^l)\,ds-\int_0^t LV(s,X^l(s),X_s^l)\,ds,$$

$$I_2(t,l)=\int_0^t\langle V_x'(s,X^l(s)),R(l)g(s,X_s^l)\,dW(s)\rangle-\int_0^t\langle V_x'(s,X(s)),g(s,X_s)\,dW(s)\rangle,$$

$$L_l V(t,x,x_t) = V_t'(t,x) + \langle V_x'(t,x), Ax + R(l) f(t,x_t)\rangle + 2^{-1}\,\mathrm{tr}\,[V_{xx}''(t,x)(R(l)g(t,x_t)) \circ Q_w \circ (R(l)g(t,x_t))^*].$$

It follows from the uniform continuity of the function $\log\lambda(t)$ that for each $\varepsilon > 0$ there exist positive integers $N = N(\varepsilon)$ and $k_1 = k_1(\varepsilon)$ such that the inequality

$$|\log\lambda(2^{-N}k) - \log\lambda(t)| \le \varepsilon, \qquad t \in [2^{-N}(k-1), 2^{-N}k],$$

holds for all $k \ge k_1(\varepsilon)$. On the other hand, by the exponential inequality for martingales, the inequality

$$P\biggl(\omega : \sup_{0\le t\le w}\biggl(\int_0^t \langle V_x'(s,X(s)), g(s,X_s)\,dW(s)\rangle - \int_0^t \frac{u}{2} QV(s,X(s),X_s)\,ds\biggr) > v\biggr) \le e^{-uv}$$

holds for any positive constants u, v, and w. In particular, by setting $u = 2\zeta(2^{-N}k)$, $v = \log(2^{-N}(k-1))/\zeta(2^{-N}k)$, and $w = 2^{-N}k$, $k = 2,3,\dots$, and then by applying the Borel–Cantelli lemma, we see that, outside a set $\widetilde{\Omega}$ of probability measure zero ($P(\widetilde{\Omega}) = 0$), i.e., for each $\omega \in \Omega\setminus\widetilde{\Omega}$, there exists a positive integer $k_0(\varepsilon,\omega)$ such that the inequality

$$\int_0^t \langle V_x'(s,X(s)), g(s,X_s)\,dW(s)\rangle \le \frac{\log(2^{-N}(k-1))}{\zeta(2^{-N}k)} + \zeta(2^{-N}k)\int_0^t QV(s,X(s),X_s)\,ds \tag{5}$$

holds for all $t \in [0, 2^{-N}k]$ and $k = k(\omega) \ge k_0(\varepsilon,\omega)$. We replace the last expression in (4) with its estimate (5) and use assumption 2 of the theorem. Then for any $\omega \in \Omega\setminus\widetilde{\Omega}$, $P(\widetilde{\Omega}) = 0$, we obtain

$$\begin{aligned}
V(t,X^l(t)) &\le \frac{\log(2^{-N}(k-1))}{\zeta(2^{-N}k)} + V(0,\xi(0)) + I_1(t,l) + \int_0^t LV(s,X^l(s),X_s^l)\,ds + I_2(t,l)\\
&\quad + \int_0^t \zeta(2^{-N}k)QV(s,X(s),X_s)\,ds - \int_0^t \zeta(2^{-N}k)QV(s,X^l(s),X_s^l)\,ds\\
&\quad + \int_0^t \zeta(2^{-N}k)QV(s,X^l(s),X_s^l)\,ds \le \frac{\log(2^{-N}(k-1))}{\zeta(2^{-N}k)} + V(0,\xi(0))\\
&\quad + \int_0^t LV(s,X^l(s),X_s^l)\,ds + \int_0^t \zeta(s)QV(s,X^l(s),X_s^l)\,ds + I_1(t,l) + I_2(t,l) + I_3(t,l)\\
&\le \frac{\log(2^{-N}(k-1))}{\zeta(2^{-N}k)} + V(0,\xi(0)) + \int_0^t (\psi_1(s) + \psi_2(s)V(s,X^l(s)))\,ds\\
&\quad + I_1(t,l) + I_2(t,l) + I_3(t,l)
\end{aligned}$$

for all $t \in [0, 2^{-N}k]$ and $k \ge \max\{k_0(\varepsilon,\omega), k_1(\varepsilon)\}$. Here

$$I_3(t,l) = \zeta(2^{-N}k)\int_0^t (QV(s,X(s),X_s) - QV(s,X^l(s),X_s^l))\,ds.$$

Therefore, by the Gronwall lemma, the inequality

$$V(t,X^l(t)) \le \exp\Bigl(\int_0^t \psi_2(s)\,ds\Bigr)\Bigl(V(0,\xi(0)) + \frac{\log(2^{-N}(k-1))}{\zeta(2^{-N}k)} + \sup_{t\in[0,2^{-N}k]}\Bigl(|I_1(t,l)| + |I_2(t,l)| + |I_3(t,l)| + \int_0^t \psi_1(s)\,ds\Bigr)\Bigr)$$

holds a.s. for all $t \in [0, 2^{-N}k]$ and $k \ge \max\{k_0(\varepsilon,\omega), k_1(\varepsilon)\}$.

Now let us show that there exists a subsequence $(l_n)_{n=1}^{\infty} \subset \mathbb{R}^+$ such that $I_1(t,l_n)$, $I_2(t,l_n)$, $I_3(t,l_n) \underset{n\to\infty}{\to} 0$ a.s. uniformly in $t \in [0,2^{-N}k]$. Indeed, take a subsequence $X^{l_n}(t)$ satisfying the assumptions of Lemma 2: $X^{l_n}(t) \underset{n\to\infty}{\to} X(t)$ a.s. uniformly in $t \in [0,2^{-N}k]$. There exist subsets $\Omega_k \subset \Omega$ with $P(\Omega_k) = 0$ such that $X^{l_n}(t) \underset{n\to\infty}{\to} X(t)$ a.s. uniformly in $t \in [0,2^{-N}k]$ for any $\omega \in \Omega\setminus\Omega_k$. Therefore, for all $\omega \in \Omega \setminus (\bigcup_{k\ge 2}\Omega_k \cup \widetilde{\Omega})$ we have

$$\begin{aligned}\sup_{t\in[0,2^{-N}k]} |I_1(t,l)| &\le \int_0^{k/2^N} |L_{l_n}V(s,X^{l_n}(s),X_s^{l_n}) - LV(s,X^{l_n}(s),X_s^{l_n})|\,ds \\ &\le \int_0^{k/2^N} |\langle V_x'(s,X^{l_n}(s)),(I-R(l_n))f(s,X_s^{l_n})\rangle|\,ds \\ &\quad + \frac{1}{2}\int_0^{k/2^N} |\operatorname{tr}[V_{xx}''(s,X^{l_n}(s))\{(R(l_n)g(s,X_s^{l_n})\circ Q_w \circ (R(l_n)g(s,X_s^{l_n}))^* \\ &\quad - g(s,X_s^{l_n})\circ Q_w\circ g(s,X_s^{l_n}))^*\}]|\,ds \underset{n\to\infty}{\to} 0\end{aligned}$$

for all $k \ge \max\{k_0(\varepsilon,\omega), k_1(\varepsilon)\}$. In a similar way, we can prove that

$$\sup_{t\in[0,2^{-N}k]} |I_2(t,l)|, \qquad \sup_{t\in[0,2^{-N}k]} |I_3(t,l)| \underset{n\to\infty}{\to} 0.$$

Therefore, letting n to infinity, we see that the inequality

$$V(t,X(t)) \le \Bigl(V(0,\xi(0)) + \frac{\log(2^{-N}k)}{\zeta(2^{-N}k)} + \frac{\log((k-1)/k)}{\zeta(2^{-N}k)} + \int_0^{2^{-N}k} \psi_1(s)\,ds\Bigr)\exp\Bigl(\int_0^t \psi_2(s)\,ds\Bigr)$$

holds a.s. for all $t \in [0,2^{-N}k]$ and $k \ge \max\{k_0(\varepsilon,\omega), k_1(\varepsilon)\}$.

Thus, we use assumption 3 of the theorem and the uniform continuity of the function $\log\lambda(t)$ to conclude that, given an $\varepsilon > 0$, there exists a positive integer $k_2(\varepsilon,\omega)$ such that

$$\begin{aligned}\log V(t,X(t)) &\le \log(V(0,\xi(0)) + \lambda(2^{-N}k)^{(\mu+\tau+2\varepsilon)} + \lambda(2^{-N}k)^{(\mu+\varepsilon)}\log((k-1)/k) \\ &\quad + \lambda(2^{-N}k)^{(\nu+\varepsilon)}) + (\theta+\varepsilon)\log\lambda(t) \\ &\le \log(V(0,\xi(0)) + e^{\varepsilon(\mu+\tau+2\varepsilon)}\lambda(t)^{(\mu+\tau+2\varepsilon)} + e^{\varepsilon(\mu+\varepsilon)}\lambda(t)\log((k-1)/k) \\ &\quad + e^{\varepsilon(\nu+\varepsilon)}\lambda(t)^{(\nu+\varepsilon)}) + (\theta+\varepsilon)\log\lambda(t)\end{aligned}$$

for all $t \in [2^{-N}(k-1), 2^{-N}k]$ and $k \ge \max\{k_0(\varepsilon,\omega), k_1(\varepsilon), k_2(\varepsilon,\omega)\}$, which implies the inequality

$$\limsup_{t\to+\infty} \log V(t,X(t))/\log\lambda(t) \le \max\{\nu+\varepsilon, \mu+\tau+2\varepsilon\} + \theta + \varepsilon.$$

Here we let ε to zero and obtain the inequality

$$\limsup_{t\to+\infty}\frac{\log V(t,X(t))}{\log\lambda(t)}\le\max\{\nu,\mu+\tau\}+\theta.$$

Finally, using assumptions 1 and 4 of the theorem, we have

$$\limsup_{t\to+\infty}\frac{\log\|X(t)\|}{\log\lambda(t)}\le\limsup_{t\to+\infty}\frac{1}{r}\frac{\log(\lambda(t)^{-m}V(t,X(t)))}{\log\lambda(t)}\le-\frac{m-(\max\{\nu,\mu+\tau\}+\theta)}{r}<0\quad\text{a.s.},$$

as desired. The proof of the theorem is complete.

Example. Consider the stochastic differential system (the values of the parameters $\alpha, m > 0$ will be specified below)

$$dX_t(x)=\left(\frac{d^2}{dx^2}X_t(x)+\alpha\sin(X_t(x)+e^{-mt/2}\cos X_t^1)\right)dt+\alpha e^{-mt/2}X_t(x)\,dW_t,\quad t>0,\quad 0<x<\pi,$$

$$dX_t^1=\left(\alpha X_t^1\sin X_t^1+\left(\int\limits_0^\pi X_t(x)^2\,dx\right)^{1/2}\right)dt+\alpha e^{-mt/2}\left(\int\limits_0^\pi X_t(x)^2\,dx\right)^{1/2}dW_t,\quad t>0,$$

as an equation for $\bar{X}_t=(X_t(\cdot),X_t^1)^{\mathrm T}$ in the space $H\times\mathbb{R}$ with the initial condition $\bar{X}_0=(X_0(x),X_0^1)^{\mathrm T}=(x_0(x),x_0^1)^{\mathrm T}$, $0<x<\pi$. Here $H=L_2[0,\pi]$ and $U=\mathbb{R}$. In concise form, this equation becomes

$$d\bar{X}_t=(\bar{A}\bar{X}_t+f(t,\bar{X}_t))\,dt+g(t,\bar{X}_t)\,dW_t,\tag{6}$$

where

$$f(t,\bar{X}_t)=\alpha(\sin(X_t(x)+e^{-mt/2}\cos X_t^1),X_t^1\sin X_t^1+\|X_t(x)\|_H)^{\mathrm T},$$

$$g(t,\bar{X}_t)=\alpha e^{-mt/2}\left(X_t(x),\left(\int\limits_0^\pi X_t(x)^2\,dx\right)^{1/2}\right)^{\mathrm T},\qquad\bar{A}=\begin{pmatrix}A&0\\0&0\end{pmatrix},$$

$A=d^2/dx^2$, and $D(A)=\{u\in C_2[0,\pi]:u(0)=u(\pi)=0\}$. Obviously, the functions f and g satisfy the condition of linear order of growth; the fact that these functions satisfy the local Lipschitz condition is proved at the end of the example to avoid any interruption of our presentation.

Equation (6) written in integral form becomes

$$\bar{X}_t=\int\limits_0^\pi G(t,x,s)\bar{X}_0(s)\,ds+\int\limits_0^t\int\limits_0^\pi G(t-\tau,x,s)f(s,x_\tau^1)\,ds\,d\tau+\int\limits_0^t\int\limits_0^\pi G(t-\tau,x,s)g(s,x_\tau^1)\,ds\,dW(\tau),$$

where $G(t,x,y)=2\pi^{-1}\sum_{n=1}^\infty e^{-n^2t}\sin(nx)\sin(ny)$. The operator $S(t):H\times\mathbb{R}\to H\times\mathbb{R}$ is accordingly determined by the relation

$$S(t)\bar{u}(\cdot)=\left(\int\limits_0^\pi G(t,\cdot,s)u(s)\,ds,u^1\right).$$

Set

$$V(t,\bar{u})=V(t,u)=e^{mt}\|u\|^2,\qquad u\in H.\tag{7}$$

Obviously, $V_t'(t,\bar{u})=mV(t,u)$. By definition, the Fréchet derivative V_u' at a point $u\in H$ is the bounded linear functional such that

$$V(t,u+h)-V(t,u)-V_u'(t,u)h=o(\|h\|)$$

for $h \in H$ with sufficiently small $\|h\|$; i.e.,

$$e^{mt} \int_0^\pi (u(s) + h(s))^2 \, ds - e^{mt} \int_0^\pi (u(s))^2 \, ds = V_u'(t,u)h + o(\|h\|).$$

Since $(u(s)+h(s))^2 - (u(s))^2 = 2u(s)h(s) + h^2(s)$, we have $2e^{mt}\langle u, h\rangle + e^{mt}\|h\|^2 = V_u'(t,u)h + o(\|h\|)$, which implies that $V_u'(t,u)h = 2e^{mt}\langle u, h\rangle$ for any $h \in H$. In a similar way, $V_{uu}''h = 2e^{mt}\langle h, \cdot\rangle$ for any $h \in H$. Note that

$$\begin{aligned}
\langle V_u'(t,u), Au\rangle &= V_u' Au = 2e^{mt} \int_0^\pi u(s)u''(s)\, ds = 2e^{mt} u(s)u'(s)|_{s=0}^{\pi} - 2e^{mt}\int_0^\pi (u'(s))^2\, ds \\
&= -2e^{mt}\|u'\|^2 = -2\|u'\|^2\|u\|^{-2}V(u), \\
\langle V_u'(t,u), f(t,\bar{u})\rangle &\le |\langle V_u'(t,u), f(t,\bar{u})\rangle| \le 2\alpha e^{mt}\int_0^\pi |u(s)\sin(u(s) + e^{-mt/2}\cos u^1)|\, ds \\
&\le 2\alpha e^{mt}\int_0^\pi |u(s)||u(s) + e^{-mt/2}\cos u^1|\, ds \\
&\le 2\alpha e^{mt}\int_0^\pi |u(s)|^2\, ds + 2\alpha e^{mt}\int_0^\pi |u(s)||e^{-mt/2}\cos u^1|\, ds \\
&\le 2\alpha e^{mt}\|u\|^2 + \alpha e^{mt}\int_0^\pi |u(s)|^2\, ds + \alpha e^{mt}\int_0^\pi e^{-mt}|\cos u^1|^2\, ds \le 3\alpha V(t,u) + \alpha\pi.
\end{aligned}$$

One can readily see that

$$\|V_u'(t,u)\| = \sup_{\|h\|=1} 2e^{mt}|\langle u, h\rangle| \le 2e^{mt}\|h\| \sup_{\|h\|=1}\|u\| = 2e^{mt}\|u\|$$

and, in a similar way,

$$\|V_{uu}''(t,u)\| = 2e^{mt}\sup_{\|h\|=1}\sup_{\|u\|=1}|\langle h, u\rangle| \le 2e^{mt}.$$

Moreover, for any bounded linear operator B the trace $\operatorname{tr}(BQ_w) = \sum\langle BQ_w u_k, u_k\rangle$ satisfies the estimate $|\operatorname{tr}(BQ_w)| \le \|B\|\operatorname{tr} Q_w$. (For (u_k) it suffices to take the basis of eigenvectors of the operator Q_w.) In our case, $U = \mathbb{R}$, and hence $Q_w = I$ and $\operatorname{tr} Q_w = 1$. Thus,

$$\begin{aligned}
2^{-1}\operatorname{tr}[V_{uu}''(t,\bar{u})(g(t,\bar{u})Q_w^{1/2})(g(t,\bar{u})Q_w^{1/2})^*] &\le 2^{-1}\|V_{uu}''\|\|g(t,\bar{u})\|^2 \le 2e^{mt}\alpha^2 e^{-mt}\|u\|^2 \le 2\alpha^2 V(t,\bar{u}), \\
|QV(t,\bar{u})| &= |\operatorname{tr}[V_{uu}''(t,\bar{u}) \otimes V_{uu}''(t,\bar{u})(g(t,\bar{u})Q_w^{1/2})(g(t,\bar{u})Q_w^{1/2})^*]| \\
&\le \|V_{uu}''\|^2\|g(t,\bar{u})\|^2 \le 8e^{mt}\alpha^2\|u\|^2 = 8\alpha^2 V(t,\bar{u}).
\end{aligned}$$

For the nonincreasing positive function $\zeta(t)$ we take the function identically equal to 1, set $\lambda_0 = \inf_{u\in D(A)}\|u'\|^2/\|u\|^2$, and estimate the expression

$$\begin{aligned}
LV(t,\bar{u}) + \zeta(t)QV(t,\bar{u}) &\le V_t'(t,\bar{u}) + \langle V_u'(t,\bar{u}), Au\rangle + \langle V_u'(t,\bar{u}), f(t,\bar{u})\rangle \\
&\quad + |2^{-1}\operatorname{tr}[V_{uu}''(t,\bar{u})(g(t,\bar{u})Q_w^{1/2})(g(t,\bar{u})Q_w^{1/2})^*]| + |QV(t,\bar{u})| \\
&\le mV(t,u) - 2\lambda_0 V(t,u) + \alpha\pi + 3\alpha V(t,u) + 2\alpha^2 V(t,u) + 8\alpha^2 V(t,u) \\
&= \alpha\pi + (-2\lambda_0 + m + 3\alpha + 10\alpha^2)V(t,u). \qquad (8)
\end{aligned}$$

Since $u(0)=0$, we have

$$(u(x))^2 = \Big(\int_0^x u'(s)\,ds\Big)^2 \le \int_0^x ds \int_0^x (u'(s))^2\,ds = x\int_0^x (u'(s))^2\,ds \le \pi \int_0^\pi (u'(s))^2\,ds.$$

We integrate this inequality over $x \in [0,\pi]$ and obtain

$$\|u\|^2 = \int_0^\pi (u(x))^2\,dx \le \int_0^\pi \Big(\pi \int_0^\pi (u'(s))^2\,ds\Big)\,dx = \pi^2 \int_0^\pi (u'(s))^2\,ds = \pi^2\|u'\|^2.$$

Thus, for any function $u \in D(A)\setminus\{0\}$ we have the inequality $\|u'\|^2/\|u\|^2 \ge \pi^{-2}$, and hence $\lambda_0 \ge \pi^{-2} > 0$. Therefore, by the choice of a sufficiently small $\alpha > 0$ and a sufficiently large $m > 0$, we ensure that the constant $\beta = -2\lambda_0 + m + 3\alpha + 10\alpha^2$ is positive and $-2\lambda_0 + 3\alpha + 10\alpha^2$ is negative.

Let us show that we can choose functions and constants in the statement of the theorem so that its assumptions 1–4 are satisfied. By the choice (7) of the functional $V(t,\bar u)$, assumption 1 is satisfied if we take $r = 2$ and $\lambda(t) = e^t$. (The constant $m > 0$ was determined above.) Since $\log\log t/\log\lambda(t) = t^{-1}\log\log t \underset{t\to\infty}{\to} 0$, we have $\tau = 0$. By setting $\psi_1(t) \equiv \alpha\pi$, $\psi_2(t) \equiv \beta$, and, according to the above, $\zeta(t) \equiv 1$, we see that assumption 2 is satisfied as well owing to inequality (8). Further, $\log\zeta(t)/\log\lambda(t) = 0$; i.e., $\mu = 0$; $\log(\int_0^t \psi_1(s)\,ds)/\log\lambda(t) = t^{-1}\log\alpha\pi t \underset{t\to\infty}{\to} 0$; i.e., $\nu = 0$; and hence, $\max\{\nu,\mu+\tau\} = 0$. Finally, $\int_0^t \psi_2(s)\,ds/\log\lambda(t) = \beta t/t = \beta$; i.e., $\theta = \beta$. Therefore, assumption 3 is satisfied, and since the number $-m+\theta+\max\{\nu,\mu+\tau\} = -m+\beta = -2\lambda_0+3\alpha+10\alpha^2$ is negative as was shown above, we see that assumption 4 is satisfied as well.

Thus, as was shown in the proof of the theorem, the estimate $\limsup_{t\to+\infty}(t^{-1}\log\|X(t)\|) \le (-2\lambda_0 + 3\alpha + 10\alpha^2)/2$ holds a.s. for the weak solution $X(t)$ of Eq. (6).

Let us show that the function g satisfies the global Lipschitz condition and the function f satisfies the local (but not global) Lipschitz condition. Indeed, the inequality

$$\begin{aligned}\|g(t,\bar x) - g(t,\bar y)\|^2 &= e^{-mt}\alpha^2 \int_0^\pi |x(s)-y(s)|^2\,ds + e^{-mt}\alpha^2(\|x\|-\|y\|)^2 \le 2e^{-mt}\alpha^2\|x-y\|^2\\ &\le 2\alpha^2\|x-y\|^2 \le 2\alpha^2\|x-y\|^2 + 2\alpha^2|x^1-y^1|^2 = 2\alpha^2\|\bar x-\bar y\|^2\end{aligned}$$

holds for the function g, and the relation

$$\begin{aligned}\|f(t,\bar x) - f(t,\bar y)\|^2 = \int_0^\pi &|\sin(x(s) + e^{-mt/2}\cos x^1) - \sin(y(s) + e^{-mt/2}\cos y^1)|^2\,ds\\ &+ |x^1\sin x^1 - y^1\sin y^1 + \|x\| - \|y\||^2\end{aligned}$$

holds for the function f. Since $|\sin x - \sin y| = |\cos\theta_1||x-y| \le |x-y|$ and $|\cos x - \cos y| = |-\sin\theta_2||x-y| \le |x-y|$ for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\|f(t,\bar x) - f(t,\bar y)\|^2 &\le \int_0^\pi |x(s)-y(s) + e^{-mt/2}\cos x^1 - e^{-mt/2}\cos y^1|^2\,ds\\ &\quad + |x^1\sin x^1 - y^1\sin y^1 + \|x\| - \|y\||^2\\ &\le 2\int_0^\pi |x(s)-y(s)|^2\,ds + 2e^{-mt}\int_0^\pi |\cos x^1 - \cos y^1|^2\,ds\\ &\quad + 2|x^1\sin x^1 - y^1\sin y^1|^2 + 2|\|x\| - \|y\||^2\\ &\le 4\|x-y\|^2 + 2\pi|x^1-y^1|^2 + 2|x^1\sin x^1 - y^1\sin y^1|^2.\end{aligned}$$

The function $h(x) = x \sin x$, $x \in \mathbb{R}$, satisfies the local Lipschitz condition (and does not satisfy the global one). Therefore, for any $a > 0$ there exists a constant $q_a > 0$ such that the inequality $|x^1 \sin x^1 - y^1 \sin y^1| \le q_a |x^1 - y^1|$ holds for all $|x^1|, |y^1| < a$. But if $\|\bar{x}\|^2 = \int_0^\pi (x(s))^2\, ds + (x^1)^2 \le a^2$ and $\|\bar{y}\|^2 \le a^2$, then we also have $|x^1| \le a$ and $|y^1| \le a$. Therefore, the inequality

$$\|f(\bar{x}) - f(\bar{y})\|^2 \le 4\|x - y\|^2 + 2(\pi + q_a^2)|x^1 - y^1|^2 \le \max\{4, 2(\pi + q_a^2)\}\|\bar{x} - \bar{y}\|^2$$

holds for any $a > 0$ provided that $\|\bar{x}\|, \|\bar{y}\| \le a$; i.e., the local Lipschitz condition is satisfied.

By Theorem 1 in [7], Eq. (6) with the initial condition $X_0 = \xi \in D(A)$ has a unique weak solution, which is stable by the theorem proved in the present paper.

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