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ORDINARY DIFFERENTIAL EQUATIONS

On a Nonlinear Eigenvalue Problem Related to the Theory of Propagation of Electromagnetic Waves

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Abstract—The eigenvalue problem is studied for a quasilinear second-order ordinary differential equation on a closed interval with Dirichlet's boundary conditions (the corresponding linear problem has an infinite number of negative and no positive eigenvalues). An additional (local) condition imposed at one of the endpoints of the closed interval is used to determine discrete eigenvalues. The existence of an infinite number of (isolated) positive and negative eigenvalues is proved; their asymptotics is specified; a condition for the eigenfunctions to be periodic is established; the period is calculated; and an explicit formula for eigenfunction zeroes is provided. Several comparison theorems are obtained. It is shown that the nonlinear problem cannot be studied comprehensively with perturbation theory methods.

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1. PROBLEM STATEMENT AND INTRODUCTORY REMARKS

The theory of propagation of electromagnetic waves in dielectric waveguides filled with a linear medium is currently a fully fledged scientific discipline (see, for example, [1–7]). Tikhonov and Samarskii [1] laid a foundation to linear waveguide theory. If the permittivity of a medium that guides waves depends on electromagnetic field (the nonlinear medium) then we arrive at nonlinear eigenvalue problems [8–12], with no theory developed for them. However, the method of integral dispersion equations [10–12] proves effective for a certain class of such nonlinear problems.

Let us consider a planar waveguide $\Sigma := \{(x, y, z) : 0 \le x \le h, (y, z) \in \mathbb{R}^2\}$ with perfectly conductive walls that a monochromatic TE-polarized wave $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$ travels through, where ω is the circular frequency,

$$\mathbf{E} = (0, \mathrm{E}_y(x), 0)^{\mathrm{TB}} e^{i\gamma z}; \quad \text{and} \quad \mathbf{H} = (\mathrm{H}_x(x), 0, \mathrm{H}_z(x))^{\mathrm{TB}} e^{i\gamma z} \tag{1.1}$$

are the complex amplitudes [8], and γ is a real unknown parameter. The medium permittivity ε in a layer has the form $\varepsilon_l + \alpha \sum_{i=1}^{q} \alpha_i |\mathbf{E}|^{2i}$, where ε_l is a real constant, $\alpha > 0$, $\alpha_q = 1$, and $\alpha_i \ge 0$ $(i = 1, \ldots, q-1)$.

The tangential components of electric field $\mathbf{E}$ (in this case, it is field's yth component) are known [13, p. 85] to turn zero on perfectly conductive walls. It is natural to take that the zth component of magnetic field $\mathbf{H}$ has a fixed (given) value on one of the boundaries, for example, at $x = 0$. The field (1.1) satisfies Maxwell's equations

$$\operatorname{rot} \mathbf{H} = -i\omega\varepsilon\mathbf{E}, \qquad \operatorname{rot} \mathbf{E} = i\omega\mu\mathbf{H}, \tag{1.2}$$

where $\mu > 0$ is the (constant) permeability of free space.

Values $\gamma = \widehat{\gamma}$ such that there exists a nontrivial field (1.1) that meets the above requirements are referred to as waveguide's *propagation constants*. Having the full set of propagation constants (or the essential properties thereof) known is important when designing waveguide systems.

To the best of our knowledge, the solvability of the thus-stated problem has not been established even at $q = 2$. Results on the solvability at $q = 1$ for an open waveguide were first obtained in [11, 14]; similar results for a closed waveguide follow from the results in the present paper (Section 2.2); see [15] for some results at $q = 2$ for an open waveguide.

The above nonlinearity at $q = 1, 2, 3$ is being actively studied, in particular, in nonlinear optics [9, 16, 17] and Schrödinger-equation theory [18]. Of particular significance in physics is also the case of $q > 3$, as the above nonlinearity results from expanding the polarization vector in powers of the field [16, p. 22]. Truncating this expansion to a finite number of terms, one obtains a nonlinearity in the form of a polynomial ($q = 1, 2, 3$ represent the simplest of possible situations). The mathematical theory of propagation of polarized waves in circular cylindrical waveguides filled with a nonlinear medium is far from being completed even for a nonlinearity of the form $|\mathbf{E}|^2$ [19, 20].

Substituting the expressions (1.1) in Eqs. (1.2), allowing for the conditions for the tangential components of electric field to vanish at the waveguide boundaries, and denoting $\omega^2\mu\varepsilon_l - \gamma^2$ by $-\lambda$ and E_y by y, we arrive at the following eigenvalue problem:

$$-y'' = -\lambda y + \alpha\sum_{i=1}^{q}\alpha_i y^{2i+1}, \qquad y \equiv y(x), \qquad x \in [0, h], \tag{1.3}$$

with Dirichlet's boundary conditions

$$y(0) = y(h) = 0 \tag{1.4}$$

and the additional condition

$$y'(0) = p, \tag{1.5}$$

where p is a real constant (without loss of generality $p > 0$), q is a natural number, $\alpha > 0$, $\alpha_q = 1$ and $\alpha_i \geq 0$ ($i = 1, \ldots, q-1$) are real constants, λ is an unknown real spectral parameter. We assume that

$$y(x) \in C^2[0, h]. \tag{1.6}$$

Problem (1.3)–(1.6) will be denoted as $P = P(\alpha, q, p)$.

Definition. A value $\lambda = \widehat{\lambda}$ such that there exists a function $y \equiv y(x; \widehat{\lambda})$ that satisfies problem P is called an *eigenvalue*, while the function $y(x; \widehat{\lambda})$ is referred to as an *eigenfunction* of this problem.

Although of importance in waveguide theory are only those values λ that satisfy the condition $\lambda > -\omega^2\mu\varepsilon_l$, problem P will be studied for all real values λ. In particular, this will make it possible to view qualitative distinctions in the spectral properties of an operator defined by problem P for negative and positive values λ.

At $\alpha = 0$ we obtain a linear problem, which will be denoted as P_0. This problem is known to have an infinite number of simple (multiplicity 1) negative eigenvalues $\lambda = \widetilde{\lambda}_n^-$, where $\widetilde{\lambda}_n^- = -\pi^2 h^{-2}(n-1)^2$, $n = 2, 3, \ldots$ Given the above numbering, the eigenfunction that corresponds to the eigenvalue $\widetilde{\lambda}_n^-$ has precisely n zeroes within the closed interval $[0, h]$.

There are no positive eigenvalues for problem P_0. Those eigenvalues $\widetilde{\lambda}_n^-$ that satisfy the condition $\omega^2\mu\varepsilon_l + \widetilde{\lambda}_n^- > 0$ correspond to the linear case of wave propagation.

Note that many problems in nonlinear waveguide theory remain out of reach for the available methods [21–25]. For example, the known methods based on seeking the minima of certain functionals [21, 22] cannot be applied to studying problem P, because, among others, in the monograph [21, p. 289], the additional condition (natural for the applicability of variational methods) has the form $\|y\| = \text{const}$, while in the work [22], such a condition is not used at all. As shown below (Theorem 2.4), neither the theory of ramification of solutions nor perturbation theory can be used for the comprehensive study of problem P either. Special mention should be made of the work [26], where, based on the theory set forth in the monograph [21], when solving the nonlinear problem of waveguide theory, Kurseeva and Smirnov proved the existence of an infinite number of eigenvalues under the additional condition $\|y\| = \text{const}$, with different values of const corresponding to different eigenvalues. In addition, it can be inferred from the work [26] that the theory from the monograph [21] is inapplicable in the case where, at least, one of the waveguide boundaries is open or the nonlinearity is not a power function.

2. MAIN RESULTS

2.1. Dispersion Equation

Let a function $T \equiv T(\lambda)$ have the form

$$T(\lambda) = \int\limits_{-\infty}^{+\infty} \frac{ds}{w(s;\lambda)}, \tag{2.1}$$

where

$$w(s;\lambda) = s^2 - \lambda + \alpha \sum_{i=1}^{q} \alpha_i \tau^i, \tag{2.2}$$

and a function $\tau \equiv \tau(s;\lambda)$ be implicitly defined by the relation

$$F(s,\tau;\lambda,p) = 0, \tag{2.3}$$

where $F(s,\tau;\lambda,p) = \alpha \sum_{i=1}^{q} \alpha_i (i+1)^{-1} \tau^{i+1} + (s^2 - \lambda)\tau - p^2$.

The following is an important result.

Theorem 2.1 (on spectral equivalence). *A number $\widehat{\lambda}$ is an eigenvalue of problem P if and only if there exists an integer $n = \widehat{n} \geq 2$ such that $\lambda = \widehat{\lambda}$ is a solution of the equation*

$$(n-1)T(\lambda) = h \tag{2.4}$$

for $n = \widehat{n}$; the eigenfunction $y(x;\widehat{\lambda})$ has $\widehat{n}$ (simple) zeroes x_i, where

$$x_i = (i-1)T(\widehat{\lambda}) = (i-1)\frac{h}{\widehat{n}-1}, \qquad i = 1, \ldots, n. \tag{2.5}$$

Relation (2.4) is a family (but not a system) of equations for different n. In order to find all eigenvalues, it is necessary to solve for λ each of the equations in (2.4). Equation (2.4) is called the dispersion equation [10, 11].

Let us separately formulate the following assertion.

Assertion 2.1. *A function $T \equiv T(\lambda)$ possesses the following properties.*

1. $T(\lambda) \in C(-\infty, +\infty)$.

2. *Let $\delta > 0$ be an arbitrary fixed number; then for all $\lambda < -\delta$ the following estimate holds true:*

$$\frac{\pi}{\sqrt{-\lambda + \alpha\alpha_*}} < T(\lambda) < \frac{\pi}{\sqrt{-\lambda}}, \tag{2.6}$$

where α_* is a constant that only depends on δ.

3. *For large $\lambda > 0$, we have the asymptotic formula*

$$T = \left(1 + \frac{1}{q}\right)\frac{\ln\lambda}{\lambda^{1/2}} + \frac{1}{\lambda^{1/2}} \ln\left(\frac{2^{2+2/q}}{p^2}\left(\frac{q+1}{\alpha}\right)^{1/q}\right) + O(\lambda^{-r-1/2}), \tag{2.7}$$

where $0 < r \leq 1/q$.

Of certain interest is the following theorem.

Theorem 2.2 (on periodicity). *If an eigenfunction $y(x;\widehat{\lambda})$ has more than one zero for $x \in (0,h)$, it is periodic with the period $2T(\widehat{\lambda})$.*

2.2. Existence of Eigenvalues and Comparison Theorems

An eigenvalue λ that is a solution of Eq. (2.4) will be denoted, depending on the sign, as $\widehat{\lambda}_n^{\pm}$ (previously, by $\widetilde{\lambda}_n^-$ we denoted the decreasingly ordered eigenvalues of problem P_0); in this case, we take the sequences $\widehat{\lambda}_n^-$ and $\widehat{\lambda}_n^+$ to be ordered decreasingly and increasingly, respectively. If for a certain $n = k$, Eq. (2.4) has several like-sign roots (if multiple, then with the multiplicity taken into account) then all such negative roots will be denoted as $\widehat{\lambda}_k^-$, with the positive ones denoted as $\widehat{\lambda}_k^+$. When it does not lead to confusion, the indices may be dropped. The multiplicity of an eigenvalue is understood to be its multiplicity as a root to dispersion equation (2.4).

Let $\delta > 0$ be an arbitrary fixed number and $n_0^- \geq 2$ be some integer number. The solvability of problem P for $\lambda < -\delta$ is established by the following assertion.

Theorem 2.3. *Problem P has an infinite number of negative eigenvalues $\widehat{\lambda}_n^-$, where $n = n_0^-, n_0^- + 1, \ldots,$ with an accumulation point at infinity. Besides, the following assertions hold true.*

1. *The asymptotics $\widehat{\lambda}_{n+1}^- = O(n^2)$ takes place.*
2. $\lim\limits_{\alpha \to +0} \widehat{\lambda}_n^- = \widetilde{\lambda}_n^-$.
3. *For any sufficiently large n, eigenvalue $\widehat{\lambda}_n^-$ is simple* (*of multiplicity* 1).
4. *For any $n \geq n_0^-$ there exists at least one pair $(\widehat{\lambda}_n^-, y(x;\widehat{\lambda}_n^-))$ such that the eigenfunction $y(x;\widehat{\lambda}_n^-)$ has n simple zeroes* (*for any sufficiently large n this pair is unique*).
5. *The following inequalities hold for any sufficiently large n :*

$$\widetilde{\lambda}_n^- \leq \widehat{\lambda}_n^- \leq \widetilde{\lambda}_{n-1}^-. \tag{2.8}$$

6. *The following asymptotic formula is valid for sufficiently large $|\widehat{\lambda}|$:*

$$\max_{x\in[0,h]} |y(x;\widehat{\lambda})| = O(|\widehat{\lambda}|^{-1/2}). \tag{2.9}$$

The number n_0^- is defined as an integer such that $(n_0^- - 1) \max\limits_{\lambda \in (-\infty,-\delta)} T(\lambda) \geq h$, and

$$(n_0^- - 2) \max_{\lambda \in (-\infty,-\delta)} T(\lambda) < h.$$

Let $n_0^+ \geq 2$ be some integer number. The solvability of problem P for $\lambda > 0$ is established by the following.

Theorem 2.4. *Problem P has an infinite number of positive eigenvalues $\widehat{\lambda}_n^+$, where $n = n_0^+, n_0^+ + 1, \ldots,$ with an accumulation point at infinity. In addition, the following assertions are true.*

1. *For large positive λ and an arbitrary fixed $\Delta > 0$, the following asymptotic inequality is valid*:

$$(1-\Delta)\lambda_n \leq \widehat{\lambda}_n^+ \leq (1+\Delta)\lambda_n, \tag{2.10}$$

where $\lambda_n = g^{-1}(hq/(n-1)(q+1))$, g^{-1} is a function inverse with respect to the function $g(t) = t^{-1/2}\ln t$.

2. *For sufficiently large* $\widehat{\lambda}$, *the following asymptotic formula holds*:

$$\max_{x\in(0,h)} |y(x;\widehat{\lambda})| = O(\widehat{\lambda}^{1/(2q)}). \tag{2.11}$$

The number n_0^+ is defined similar to n_0^-.

Theorem 2.4 demonstrates that results on the positive eigenvalues of problem P cannot be obtained using perturbation theory.

For negative λ, assertion 5 in Theorem 2.3 is essentially a comparison theorem for the eigenvalues of the nonlinear and counterpart linear problems.

As problem P_0 has no positive eigenvalues, results analogous to assertions 2 and 5 in Theorem 2.3 cannot be obtained for $\lambda > 0$. However, the formula (2.7) allows one to obtain essentially much more interesting results, namely, comparison theorems for the eigenvalues of two nonlinear problems. We failed to trace any results of the kind in scientific literature.

The following theorem holds true for the eigenvalues of two different problems of the type $P = P(\alpha, q, p)$.

Theorem 2.5 (comparison). *Let* $\{\widehat{\lambda}_n^+\}_{n=2}^{\infty}$ *and* $\{\widehat{\theta}_n^+\}_{n=2}^{\infty}$ *be the sequences of* (*increasingly ordered positive*) *eigenvalues of problems* $P_1 = P(\alpha, q_1, p_1)$ *and* $P_2 = P(\beta, q_2, p_2)$, *respectively. If either condition* $q_2 > q_1$ *or conditions* $q_1 = q_2 = q$ *and* $p_2\beta > p_1\alpha$ *are fulfilled*, *then for any sufficiently large index* n *we have the inequality*

$$\widehat{\lambda}_n^+ > \widehat{\theta}_n^+. \tag{2.12}$$

To conclude this section, we note that of interest is the case $y'(0) = p$, where $p \equiv p(\lambda)$. If $\lim_{\lambda\to+\infty} p(\lambda) = 0$, then the degree of p tending to zero affects the behavior of function T as $\lambda \to +\infty$. If $\lim_{\lambda\to\infty} T \neq 0$ then there exists only a finite number of eigenvalues. New interesting results arise when using Robin's boundary conditions (including dependent on λ). In this case, there may exist sign-constant periodic solutions [14].

3. PROOFS

3.1. Proof of Theorem 2.1

Equation (1.3) has the first integral

$$y'^2 - \lambda y^2 + \alpha \sum_{i=1}^{q} \frac{\alpha_i}{i+1} y^{2i+2} \equiv C, \tag{3.1}$$

where C is an arbitrary constant. Identity (3.1) and the conditions (1.4) at point $x = 0$ and Eq. (1.5) imply the relation

$$C = p^2. \tag{3.2}$$

It follows from identity (3.1) at point $x = h$ and the equality (3.2) that $y'^2(h) = p^2$. The sign of number $y'(h)$ is determined by the number of zeroes that eigenfunction $y(x)$ has on the closed interval $(0,h)$, viz., if the number of zeroes is k then $y'(h) = (-1)^{k+1}p$, as every root of function $y = y(x)$, as follows from relations (3.1) and (3.2), has the multiplicity of one.

Let us introduce new variables

$$\tau(x) = y^2(x), \qquad \eta(x) = y'(x)/y(x). \tag{3.3}$$

Equation (1.3) can be written in the form of a system as

$$\tau' = 2\tau\eta, \qquad \eta' = -\eta^2 + \lambda - \alpha \sum_{i=1}^{q} \alpha_i \tau^i. \tag{3.4}$$

Allowing for relation (3.2), we obtain the first integral of system (3.4) in the form (2.3), where $s = \eta$.

The second equation in system (3.4) can be written in the form

$$\eta' = -w(\eta;\lambda), \tag{3.5}$$

where the function $w \equiv w(\eta;\lambda)$ is defined by relation (2.2), while the function $\tau \equiv \tau(\eta;\lambda)$ is implicitly defined by Eq. (2.3); in this case, it can be easily seen that any function $\tau(\eta;\lambda)$ is positive and exists for any real η and λ.

If $\lambda < 0$ then, apparently, $w > 0$. However, it is not clear whether function w retains its sign in the case of $\lambda > 0$. Suppose there exists λ for which the right-hand side in Eq. (3.5) turns zero. Hence, we find $\eta^2 - \lambda = -\alpha\sum_{i=1}^{q}\alpha_i\tau^i$. Substituting this expression in the relation $F(\eta,\tau;\lambda,p) = 0$, we arrive at

$$-\alpha\sum_{i=1}^{q}\frac{i\alpha_i}{i+1}\tau^{i+1} = p^2.$$

For $\alpha > 0$, $\alpha_q = 1$, $\alpha_i \geq 0$ $(i = 1,\ldots,q-1)$ and $\tau \geq 0$ the latter relationship is fulfilled for no λ. Thus, $\eta' < 0$ for all $\lambda \in (-\infty,+\infty)$.

As $\eta' < 0$, function $\eta(x)$ monotonically decreases on any segment contained within the interval $(0,h)$, where this function is defined. Relations (3.3) imply that function $\eta(x)$ is discontinuous at those, and only those, points x, where function $y(x)$ vanishes. Let function $y(x)$ have $n \geq 2$ zeroes $0 = x_1 < x_2 < \cdots < x_{n-1} < x_n = h$. In this case, function $\eta(x)$ has precisely n discontinuity points $x_1,\ldots,x_n \in [0,h]$, as $y'(x_i) \neq 0$ for any $i = 1,\ldots,n$ by virtue of relations (3.1) and (3.2). Hence, all the discontinuity points of function $\eta(x)$ will be of the second kind.

By virtue of function η monotonically decreasing, we arrive at the relations

$$\lim_{x\to x_1+0}\eta(x) = +\infty, \qquad \lim_{x\to x_i\pm 0}\eta(x) = \pm\infty \quad (i = 2,\ldots,n-1), \qquad \lim_{x\to x_n-0}\eta(x) = -\infty. \tag{3.6}$$

On each of the intervals $(x_1,x_2), (x_2,x_3),\ldots,(x_{n-1},x_n)$, we solve Eq. (3.5). Applying the method described in the work [10], we arrive at the equalities

$$x_i - \int\limits_{\eta(x_i+0)}^{\eta(x_{i+1}-0)}\frac{ds}{w(s;\lambda)} - x_{i+1} = 0, \qquad i = 1,\ldots,n-1. \tag{3.7}$$

Taking relations (3.6) and the definition (2.1) into account, we obtain from relations (3.7) that

$$0 < x_{i+1} - x_i = T(\lambda), \qquad i = 1,\ldots,n-1. \tag{3.8}$$

The convergence of the improper integrals follows from relations (3.7) and (3.8). The formula (2.5) follows from relation (3.8). Summing the equalities (3.8) for all i, we obtain Eq. (2.4).

So, it has been proved that if $\widehat{\lambda}$ is an eigenvalue of problem P then $\lambda = \widehat{\lambda}$ satisfies Eq. (2.4) for a certain n. Let us prove that any solution of Eq. (2.4) is an eigenvalue.

Let $\lambda = \widehat{\lambda}$ be a solution of Eq. (2.4) for some $n = \widehat{n}$. Since the right-hand side of Eq. (1.3) is smooth in y, the solution of any Cauchy problem for this equation exists and is unique. In this case, the existence of a unique solution $y \equiv y(x;\widehat{\lambda})$ of the Cauchy problem for Eq. (1.3) with the initial conditions $y(0) = 0$, $y'(0) = p$, defined on the closed interval $[0,h]$, follows from identity (3.1).

We use the solution $y \equiv y(x;\widehat{\lambda})$ of the above-indicated Cauchy problem to construct functions $\tau = y^2$ and $\eta = y'/y$. It is evident that $\tau(0) = 0$ and $\lim_{x\to 0+0}\eta(x) = +\infty$. At this step, we do not assert that the condition $\lim_{x\to h-0}\eta(x;\widehat{\lambda}) = -\infty$ is fulfilled. For the sake of definiteness, we assume that $\eta(h) = y'(h)/y(h) = a > -\infty$. Using the thus found τ and η, we construct an expression that is similar to Eq. (2.4) by defining it with the relation

$$\int\limits_{a}^{+\infty}\frac{ds}{w(s;\widehat{\lambda})} + (\widehat{n}-2)\int\limits_{-\infty}^{+\infty}\frac{ds}{w(s;\widehat{\lambda})} = h. \tag{3.9}$$

At the same time, $\lambda = \widehat{\lambda}$ satisfies Eq. (2.4). Note that the integrand terms in Eqs. (3.9) and (2.4) coincide. Subtracting Eq. (2.4) from Eq. (3.9), we arrive at

$$\int\limits_{a}^{+\infty} \frac{ds}{w(s;\widehat{\lambda})} - \int\limits_{-\infty}^{+\infty} \frac{ds}{w(s;\widehat{\lambda})} = 0. \tag{3.10}$$

By virtue of the self-apparent estimates

$$\int\limits_{-\infty}^{+\infty} \frac{ds}{w(s;\widehat{\lambda})} > \int\limits_{a}^{+\infty} \frac{ds}{w(s;\widehat{\lambda})} > 0,$$

relation (3.10) is satisfied only if $a = -\infty$, but the latter implies that $\widehat{\lambda}$ is an eigenvalue. This established the theorem.

3.2. Proof of Assertion 2.1

It can be readily seen that the positive function $\tau \equiv \tau(s;\lambda)$ implicitly defined by Eq. (2.3) exists for all real λ and s. It is also clear that function $\tau(s;\lambda)$ continuously depends on both arguments. As function w defined by the equality (2.2) is always positive (see the proof of Theorem 2.1) and continuous for all real λ and s, function $T(\lambda)$ belongs to space $C(-\infty,+\infty)$.

Let $\lambda < 0$. Let us write Eq. (2.3) in the form

$$(s^2 - \lambda)\tau = p^2 - \alpha \sum_{i=1}^{q} \frac{\alpha_i}{i+1}\tau^{i+1}. \tag{3.11}$$

Relation (3.11) apparently implies that function $\tau \equiv \tau(s;\lambda)$ remains bounded for any real s and all $\lambda < 0$. Now, let $\lambda < -\delta$, where $\delta > 0$ is an arbitrary fixed number. Then relation (3.11) entails the inequality

$$0 < \tau(s;\lambda) < \frac{p^2}{s^2 - \lambda} < \tau_* = \frac{p^2}{\delta}. \tag{3.12}$$

Let us denote $\alpha_* := \sum_{i=1}^{q} \alpha_i \tau_*^i$. Using inequalities (3.12), we obtain the estimates

$$\int\limits_{-\infty}^{+\infty} \frac{ds}{s^2 - \lambda + \alpha\alpha_*} < T(\lambda) < \int\limits_{-\infty}^{+\infty} \frac{ds}{s^2 - \lambda}.$$

By calculating these integrals, we arrive at inequality (2.6).

Let $\lambda > 0$. Relation (2.3) implies that for any fixed value s, function $\tau \equiv \tau(s;\lambda) > 0$ increases without limit together with λ. For this reason, the formula (2.3) is inconvenient for further analysis. However, variables τ and s can be "normed" in such a way that function $\tau(s;\lambda)$ remains bounded for all values λ. The "normed" variables have the form $\tau = \lambda^{1/q}\bar{\tau}$ and $s = \lambda^{1/2}\bar{s}$. In terms of these variables, relation (2.3) takes on the form

$$(\bar{s}^2 - 1)\bar{\tau} = \frac{p^2}{\lambda^{1+1/q}} - \alpha \sum_{i=1}^{q} \frac{\alpha_i}{(i+1)\lambda^{1-i/q}}\bar{\tau}^{i+1}. \tag{3.13}$$

Equation (3.13) implicitly defines a function $\bar{\tau} \equiv \bar{\tau}(\bar{s};\lambda)$ that is positive and bounded for all real s and all $\lambda > 0$. Let us also take note that $\lim_{\bar{s}\to\infty} \bar{\tau}(\bar{s};\lambda) = 0$ regardless of λ. The boundedness of function $\bar{\tau}(\bar{s};\lambda)$ makes it possible to switch from the integral over the entire axis in Eq. (2.4) to an integral over a finite interval.

Changing variables in the integral (2.1), we obtain

$$T(\lambda) = 2\int\limits_0^{+\infty}\left(s^2-\lambda+\alpha\sum_{i=1}^{q}\alpha_i\tau^i\right)^{-1}ds = \frac{2}{\lambda^{1/2}}\int\limits_0^{+\infty}\left(\bar{s}^2-1+\alpha\sum_{i=1}^{q}\frac{\alpha_i}{\lambda^{1-i/q}}\bar{\tau}^i\right)^{-1}d\bar{s}. \tag{3.14}$$

Now, using relation (3.13), let us change variable $\bar{s}$ for $\bar{\tau}$. As in the integral (3.14), variable $\bar{s}$ varies from 0 to $+\infty$, then $\bar{\tau}$ varies from τ_+ to 0, respectively, where τ_+ is the unique positive root of Eq. (3.13) for $\bar{s}=0$. Thus, expressing $\bar{s}$ from relation (3.13), we choose the sign "+."

Using relation (3.13), we obtain

$$\bar{s}^2 = \frac{1}{\bar{\tau}}\left(\frac{p^2}{\lambda^{1+1/q}}+\bar{\tau}-\alpha\sum_{i=1}^{q}\frac{\alpha_i}{(i+1)\lambda^{1-i/q}}\bar{\tau}^{i+1}\right), \tag{3.15}$$

and, hence, calculate the differential

$$d\bar{s} = -\frac{1}{2\sqrt{\bar{\tau}^3}}\left(\frac{p^2}{\lambda^{1+1/q}}+\alpha\sum_{i=1}^{q}\frac{i\alpha_i}{(i+1)\lambda^{1-i/q}}\bar{\tau}^{i+1}\right)\left(\frac{p^2}{\lambda^{1+1/q}}+\bar{\tau}-\alpha\sum_{i=1}^{q}\frac{\alpha_i}{(i+1)\lambda^{1-i/q}}\bar{\tau}^{i+1}\right)^{-1/2}d\bar{\tau}. \tag{3.16}$$

Now we use the formulae (3.15), (3.16) and the relation $\lim\limits_{\bar{s}\to\infty}\bar{\tau}(\bar{s};\lambda)=0$ to transform relation (3.14) to the form

$$T(\lambda) = \frac{1}{\lambda^{1/2}}\int\limits_0^{\tau_+}\frac{1}{\sqrt{\bar{\tau}}}\left(\frac{p^2}{\lambda^{1+1/q}}+\bar{\tau}-\alpha\sum_{i=1}^{q}\frac{\alpha_i}{(i+1)\lambda^{1-i/q}}\bar{\tau}^{i+1}\right)^{-1/2}d\bar{\tau}. \tag{3.17}$$

The equation

$$\frac{p^2}{\lambda^{1+1/q}}+\bar{\tau}-\alpha\sum_{i=1}^{q}\frac{\alpha_i}{(i+1)\lambda^{1-i/q}}\bar{\tau}^{i+1} = 0 \tag{3.18}$$

is nothing else but the first integral (3.13) for $\bar{s}=0$. Equation (3.18) has no fewer than two real roots, including a unique positive root and at least one negative root. We are interested in the greatest negative root, which we will denote by τ_-, and the unique positive root, which we previously denoted as τ_+.

As $\lambda\to+\infty$, we obtain from Eq. (3.18) that $\bar{\tau}(1-\alpha\bar{\tau}^q/(q+1))=0$. The latter equation has at least two real roots $\tau_-^0=0$ and $\tau_+^0=((q+1)/\alpha)^{1/q}$. It can be demonstrated that $\lim_{\lambda\to+\infty}\tau_-=\tau_-^0$ and $\lim_{\lambda\to+\infty}\tau_+=\tau_+^0$.

It is evident that the expression in "larger" radicand in relation (3.17) turns to zero at points $\bar{\tau}=\tau_-$ and $\bar{\tau}=\tau_+$. Then this radicand can be written as

$$\frac{p^2}{\lambda^{1+1/q}}+\bar{\tau}-\alpha\sum_{i=1}^{q}\frac{\alpha_i}{(i+1)\lambda^{1-i/q}}\bar{\tau}^{i+1} = (\bar{\tau}-\tau_-)f_1(\bar{\tau}), \tag{3.19}$$

where $f_1(\tau_+)=0$ and $\lim_{\lambda\to+\infty}f_1(\bar{\tau})=1-\alpha\bar{\tau}^q/(q+1)$.

Taking the above notation into account, we write relation (3.17) in the form

$$T = \frac{1}{\lambda^{1/2}}\int\limits_0^{\tau_+}\frac{f(\bar{\tau})\,d\bar{\tau}}{\sqrt{\bar{\tau}(\bar{\tau}-\tau_-)}}, \tag{3.20}$$

where $f(\bar{\tau})=1/\sqrt{f_1(\bar{\tau})}$.

In what follows, we will need asymptotic expressions for τ_- and τ_+ for large values of λ. The quantities τ_-^0 and τ_+^0 are the first approximations to τ_- and τ_+, respectively. Using Eq. (3.18) and the derived approximations, we calculate

$$\tau_- = -\frac{p^2}{\lambda^{1+1/q}} + O(\lambda^{-3-1/q}) \quad \text{and} \quad \tau_+ = \left(\frac{q+1}{\alpha}\right)^{1/q} + O(\lambda^{-1/q}). \tag{3.21}$$

From the representation (3.20), we find

$$T = \frac{1}{\lambda^{1/2}} \int_0^{\tau_+} \frac{f(\bar\tau)\, d\bar\tau}{\sqrt{\bar\tau(\bar\tau - \tau_-)}} = \frac{1}{\lambda^{1/2}} \int_0^{\tau_+} \frac{f(\bar\tau) - f(\tau_-)}{\sqrt{\bar\tau(\bar\tau - \tau_-)}}\, d\bar\tau + \frac{f(\tau_-)}{\lambda^{1/2}} \int_0^{\tau_+} \frac{d\bar\tau}{\sqrt{\bar\tau(\bar\tau - \tau_-)}}. \tag{3.22}$$

The first term on the right-hand side of relation (3.22) is estimated as follows:

$$\int_0^{\tau_+} \frac{f(\bar\tau) - f(\tau_-)}{\sqrt{\bar\tau(\bar\tau - \tau_-)}}\, d\bar\tau = \lim_{\lambda\to+\infty} \int_0^{\tau_+} \frac{f(\bar\tau) - f(\tau_-)}{\sqrt{\bar\tau(\bar\tau - \tau_-)}}\, d\bar\tau + O(\lambda^{-r_1}),$$

where $r_1 \ge \delta_q > 0$. Hence, we calculate

$$\begin{aligned}\lim_{\lambda\to+\infty} \int_0^{\tau_+} \frac{f(\bar\tau) - f(\tau_-)}{\sqrt{\bar\tau(\bar\tau - \tau_-)}}\, d\bar\tau &= \int_0^{((q+1)/\alpha)^{1/q}} \left(\left(1 - \frac{\alpha}{q+1}\bar\tau^q\right)^{-1/2} - 1\right)\frac{d\bar\tau}{\bar\tau} \\ &= \int_0^{((q+1)/\alpha)^{1/q}} \bar\tau^{q-1}\left(1 - \frac{\alpha}{q+1}\bar\tau^q\right)^{-1/2}\left(1 + \sqrt{1 - \frac{\alpha}{q+1}\bar\tau^q}\right)^{-1} d\bar\tau = \frac{2\ln 2}{q}.\end{aligned}$$

The second term in relation (3.22) can be calculated precisely as

$$\int_0^{\tau_+} \frac{d\bar\tau}{\sqrt{\bar\tau(\bar\tau - \tau_-)}} = 2\ln(\sqrt{\tau_+} + \sqrt{\tau_+ - \tau_-}\,) - 2\ln\sqrt{-\tau_-}.$$

Using the asymptotic formulae (3.21), we find that

$$\ln(\sqrt{\tau_+} + \sqrt{\tau_+ - \tau_-}\,) = \ln 2 + \frac{1}{2q}\ln\frac{q+1}{\alpha} + O(\lambda^{-1/q}), \qquad \ln\sqrt{-\tau_-} = \frac{1}{2}\ln\frac{p^2}{\lambda^{1+1/q}} + O(\lambda^{-2}).$$

From relation (3.19), we obtain

$$f_1(\tau_-) = 1 - \alpha\sum_{i=1}^{q} \frac{\alpha_i \tau_-^i}{\lambda^{1-i/q}}.$$

The last equality and the formula (3.21) imply

$$f(\tau_-) = \left(1 - \alpha\sum_{i=1}^{q} \frac{\alpha_i \tau_-^i}{\lambda^{1-i/q}}\right)^{-1/2} = 1 + O(\lambda^{-2}).$$

By combining the results, we arrive at the asymptotic formula (2.7), where $r = \min\{r_1, 1/q\}$. This establishes the assertion.

3.3. Proof of Theorem 2.2

Let $0 = x_1 < x_2 < \cdots < x_{n-1} < x_n = h$ be the zeroes of eigenfunction $y(x;\widehat{\gamma})$ and $n > 3$. Let us write Eq. (1.3) in the form of the system

$$y' = z, \quad z' = \lambda y - \alpha \sum_{i=1}^{q} \alpha_i y^{2i+1}.$$

It is evident that $(y(x_1), z(x_1)) = (0, p)$. Analyzing the behavior of the derivative y' at points x_i yields the equalities $(y(x_2), z(x_2)) = (0, -p)$ and $(y(x_3), z(x_3)) = (0, p)$. Thus, the relation $(y(x_1), z(x_1)) = (y(x_3), z(x_3))$ holds true. Hence, the trajectory (y, z) is closed, and, therefore, the solution $y(x;\widehat{\gamma})$ is a periodic function [27] with the period that equals the double distance between neighboring zeroes, i.e., $2T$. The theorem is thus proved.

A different method for proving the theorem has been given in the work [10].

3.4. Proof of Theorem 2.3

Inequalities (2.6) apparently imply the relation $\lim\limits_{\lambda\to-\infty} T(\lambda) = 0$. Hence, there exists such an integer $n_0^- \ge 2$ that Eq. (2.4) has, at least, one root $\widehat{\lambda}_n^- < 0$ for every $n = n_0^-, n_0^- + 1, \ldots$ Thus, problem P has an infinite number of negative eigenvalues $\widehat{\lambda}_n^-$, where, evidently, $\lim_{n\to+\infty} \widehat{\lambda}_n^- = -\infty$.

Further, Eq. (2.4) along with inequalities (2.6) provide the double-sided inequalities for eigenvalues

$$\widetilde{\lambda}_n^- < \widehat{\lambda}_n^- < \widetilde{\lambda}_n^- + \alpha\alpha_*, \tag{3.23}$$

where $n = 2, 3, \ldots$ Inequalities (2.6) and (3.23) show that for $\alpha \to +0$, Eq. (2.4) becomes the dispersion equation of the linear problem, while $\widehat{\lambda}_n^- \to \widetilde{\lambda}_n^-$.

It follows from the definition (2.1) that $T(\lambda)$ is a differentiable function of λ for $\lambda \in (-\infty, -\delta]$. Calculating the derivative $dT/d\lambda$, we obtain

$$\frac{dT}{d\lambda} = -\int_{-\infty}^{+\infty} \left(-1 + \alpha \frac{\partial\tau}{\partial\lambda} \sum_{i=1}^{q} i\alpha_i \tau^{i-1}\right) \left(s^2 - \lambda + \alpha \sum_{i=1}^{q} \alpha_i \tau^i\right)^{-2} ds.$$

From Eq. (3.11) we find $\partial\tau/\partial\lambda = \tau(s^2 - \lambda + \alpha \sum_{i=1}^{q} \alpha_i \tau^i)^{-1}$ and, substituting the latter in the expression for $dT/d\lambda$, we arrive at the relation

$$\frac{dT}{d\lambda} = \int_{-\infty}^{+\infty} \left(s^2 - \lambda - \alpha \sum_{i=1}^{q} \alpha_i (i-1)\tau^i\right) \left(s^2 - \lambda + \alpha \sum_{i=1}^{q} \alpha_i \tau^i\right)^{-3} ds.$$

By virtue of inequalities (3.12), the quantity $\tau(s;\lambda)$ is bounded for all s and all $\lambda < 0$. Thus, for all $\lambda < -\alpha \sum_{i=1}^{q} \alpha_i (i-1)\tau_*^i$ the inequality $dT/d\lambda > 0$ holds true. Hence, it follows that any sufficiently large eigenvalue of problem P is simple.

Assertion 4 in the theorem follows from the preceding assertions therein.

From inequalities (3.23), it is clear that there exists an integer $n_0 \ge n_0^-$ such that for all $n \ge n_0$ we have inequality (2.8).

Relations (3.1) and (3.2) imply that $p^2 \ge y'^2$. Evidently, for every eigenfunction $y(x;\widehat{\lambda})$ there exists at least one point $x = \bar{x} \in (0, h)$ where function $y(x;\widehat{\lambda})$ attains its maximum. As at this point we have $y'(\bar{x};\widehat{\lambda}) = 0$, then by putting $y' = 0$ in identity (3.1), we obtain

$$p^2 = |\widehat{\lambda}| y^2 + \alpha \sum_{i=1}^{q} \frac{\alpha_i}{i+1} y^{2i+2}, \tag{3.24}$$

where $y = y(\bar{x};\widehat{\lambda})$. Hence, $p^2 \ge |\widehat{\lambda}| y^2$, but then $\max_{x\in[0,h]} |y(x;\widehat{\lambda})| \to 0$ for $\widehat{\lambda} \to -\infty$.

A more subtle result can be obtained in the following manner. In relation (3.24), we perform the change $y^2 = |\widehat{\lambda}|^{1/q}z$; then we have

$$p^2|\widehat{\lambda}|^{-1-1/q} - \alpha\sum_{i=1}^{q}\alpha_i(i+1)^{-1}z^{i+1}|\widehat{\lambda}|^{-1+i/q} = z.$$

The maximum that we are interested in is the only positive root $\bar{z}$ of this equation; by directing $|\widehat{\lambda}|$ to infinity, we obtain the relation $\alpha z^{q+1}/(q+1) = -z$. Hence it follows that $z = 0$ is the first approximation to $\bar{z}$. Calculating the second approximation, we obtain the asymptotic estimate $\bar{z} = O(|\widehat{\lambda}|^{-1-1/q})$, which, in an evident way, entails the formula (2.9). The theorem is proved.

3.5. Proof of Theorem 2.4

The asymptotic formula (2.7) apparently implies that $\lim_{\lambda\to+\infty} T(\lambda) = 0$. This proves the existence of such an integer $n_0^+ \geq 2$ that Eq. (2.4) has at least one root $\widehat{\lambda}_n^+ > 0$ for every $n = n_0^+, n_0^+ + 1, \ldots$ Thus, problem P has an infinite number of positive eigenvalues $\widehat{\lambda}_n^+$ and, evidently, $\lim_{n\to+\infty}\widehat{\lambda}_n^+ = +\infty$. The asymptotic inequality (2.10) follows from the formula (2.7).

The asymptotic behavior of $\max_{x\in[0,h]}|y(x;\widehat{\lambda}\,)|$ for large $\widehat{\lambda} > 0$ can be established as follows. Apparently, for any eigenfunction $y(x;\widehat{\lambda}\,)$ there exists, at least, one point $x = \bar{x} \in (0,h)$ where function $y(x;\widehat{\lambda}\,)$ has a maximum. Since $y'(\bar{x};\widehat{\lambda}\,) = 0$, by putting $y' = 0$ in identity (3.1), we obtain

$$p^2 - \alpha\sum_{i=1}^{q}\alpha_i(i+1)^{-1}y^{2i+2} = -\widehat{\lambda}y^2,$$

where $y = y(\bar{x};\widehat{\lambda}\,)$. By performing the change $y^2 = \widehat{\lambda}^{1/q}z$ in the last relation, we have

$$\frac{p^2}{\widehat{\lambda}^{1+1/q}} - \alpha\sum_{i=1}^{q}\frac{\alpha_i}{i+1}\frac{z^{i+1}}{\widehat{\lambda}^{1-i/q}} = -z.$$

The maximum that we are interested in is the only positive root $\bar{z}$ of this equation; directing $\widehat{\lambda}$ to infinity, we obtain the relation $\alpha z^{q+1}/(q+1) = z$. Hence it follows that $z = ((q+1)/\alpha)^{1/q}$ is the first approximation to $\bar{z}$. Calculating the second approximation, we obtain an asymptotic estimate $\bar{z} = O(1)$ that, in an evident manner, implies the formula (2.11). This established the theorem.

3.6. Proof of Theorem 2.5

Integrals $T \equiv T_1(\lambda)$ and $T \equiv T_2(\theta)$ correspond to problems P_1 and P_2, respectively. The asymptotic expansions for these integrals, according to the formula (2.7), have the form

$$T_1(\lambda) = \left(1 + \frac{1}{q_1}\right)\frac{\ln\lambda}{\lambda^{1/2}} + \frac{1}{\lambda^{1/2}}\ln\left(\frac{2^{2+2/q_1}}{p_1^2}\left(\frac{q_1+1}{\alpha}\right)^{1/q_1}\right) + O(\lambda^{-r_1-1/2}),$$
$$T_2(\theta) = \left(1 + \frac{1}{q_2}\right)\frac{\ln\theta}{\theta^{1/2}} + \frac{1}{\theta^{1/2}}\ln\left(\frac{2^{2+2/q_2}}{p_2^2}\left(\frac{q_2+1}{\beta}\right)^{1/q_2}\right) + O(\theta^{-r_2-1/2}),$$

where $r_1, r_2 > 0$ and $r_1 \leq 1/q_1$, $r_2 \leq 1/q_2$.

Then for the difference $T_1(\lambda) - T_2(\lambda)$ we have the asymptotic relation

$$T_1(\lambda) - T_2(\lambda) = \left(\frac{1}{q_1} - \frac{1}{q_2}\right)\frac{\ln\lambda}{\lambda^{1/2}} + \frac{1}{\lambda^{1/2}}\ln\left(\frac{2^{2/q_1}p_2^2}{2^{2/q_2}p_1^2}\left(\frac{q_1+1}{\alpha}\right)^{1/q_1}\left(\frac{\beta}{q_2+1}\right)^{1/q_2}\right) + O(\lambda^{-r-1/2}), \quad (3.25)$$

where $r \geq \min\{r_1, r_2\}$.

If $q_2 > q_1$ then relation (3.25) implies that for sufficiently large values λ we have the inequality $T_1(\lambda) > T_2(\lambda)$, which entails inequality (2.12).

Now let $q_1 = q_2 = q$. Then, from Eq.(3.25) we obtain

$$T_1(\lambda) - T_2(\lambda) = \frac{1}{\lambda^{1/2}} \ln\left(\frac{p_2^2}{p_1^2}\left(\frac{\beta}{\alpha}\right)^{1/q}\right) + O(\lambda^{-r-1/2}), \tag{3.26}$$

where $r \geq \min\{r_1, r_2\}$. If $p_2^2 p_1^{-2}(\beta/\alpha)^{1/q} > 1$ then relation (3.26) implies that for sufficiently large values λ we have the inequality $T_1(\lambda) > T_2(\lambda)$, which entails inequality (2.12). The theorem is proved.

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