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ORDINARY DIFFERENTIAL EQUATIONS

Functions Determined by the Lyapunov Exponents of Families of Linear Differential Systems Continuously Depending on the Parameter Uniformly on the Half-Line

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Abstract—For families of n-dimensional linear differential systems ($n \ge 2$) whose dependence on a parameter ranging in a metric space is continuous in the sense of the uniform topology on the half-line, we obtain a complete description of the ith Lyapunov exponent as a function of the parameter for each $i = 1, \ldots, n$. As a corollary, we give a complete description of the Lebesgue sets and (in the case of a complete separable parameter space) the range of an individual Lyapunov exponent of such a family.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

Given an $n \in \mathbb{N}$, let $\mathcal{M}^n$ be the space of linear systems

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}^+ \equiv [0, +\infty), \tag{1}$$

with continuous bounded matrix functions A on the half-line $\mathbb{R}^+$ (which we identify with the corresponding systems) with the operations of addition and multiplication by a real number naturally defined for functions. Let us fix some norm $|\cdot|$ on the space $\mathbb{R}^n$ and introduce two topologies on $\mathcal{M}^n$ most frequently used in the theory of Lyapunov exponents, namely, the *uniform topology* defined by the norm

$$\|A\| = \sup_{t \in \mathbb{R}^+} |A(t)|, \qquad A \in \mathcal{M}^n,$$

and the *compact-open topology* defined by the metric [1, p. 533]

$$\varrho_C(A, B) = \sup_{t \in \mathbb{R}^+} \min\{|A(t) - B(t)|, 1/t\}, \qquad A, B \in \mathcal{M}^n,$$

where $|A(t)| = \sup_{|x|=1} |A(t)x|$. Since all norms on $\mathbb{R}^n$ are equivalent, it follows that the resulting topological spaces, which will be denoted by $\mathcal{M}^n_U$ and $\mathcal{M}^n_C$, respectively, are independent of the choice of the norm on $\mathbb{R}^n$.

Definition 1. The *characteristic exponent* of a function $f : [t_0, +\infty) \to \mathbb{R}^m$ (where $t_0 \ge 0$ and $m \in \mathbb{N}$) is the (finite or infinite) number

$$\lambda[f] = \varlimsup_{t \to +\infty} \ln |f(t)|^{1/t}$$

(where we assume that $\ln 0 = -\infty$).

Definition 2. The *Lyapunov exponents* of system (1) are the numbers [2]

$$\lambda_i(A) = \inf_{L\in G_i(S(A))} \sup_{x\in L} \lambda[x], \qquad i = 1, \ldots, n,$$

where $S(A)$ is the solution space of system (1) and $G_i(V)$ is the set of i-dimensional subspaces of a vector space V.

In our notation, in contrast to [2], the Lyapunov exponents are numbered in nondescending order. The above definition of Lyapunov exponents of system (1) is equivalent to their classical definition [2].

Let M be a metric space. Consider a family

$$\dot{x} = A(t,\mu)x, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}^+, \tag{2}$$

of linear differential systems depending on the parameter $\mu \in M$ such that for each μ system (2) belongs to the space $\mathcal{M}^n$ (that is, has continuous coefficients bounded on the half-line). Taking an $i \in \{1,\ldots,n\}$ and assigning the ith Lyapunov exponent of system (2) to each $\mu \in M$, we obtain a function $\Lambda_i^A : M \to \mathbb{R}$, which is called the ith Lyapunov exponent of the family (2).

It is well known and can readily be proved (e.g., see [3, Lemma 4]) that the continuity of the mapping $A : \mathbb{R}^+ \times M \to \operatorname{End}\mathbb{R}^n$ specifying the family (2) is equivalent to the continuity of the mapping of M into $\mathcal{M}_C^n$ defined by the rule $\mu \mapsto A(\cdot\,,\mu)$. Simplest examples show (e.g., see [3, the example before Definition 5]) that, starting already from $n = 1$, the functions Λ_i^A, $i = 1,\ldots,n$, for a continuous bounded mapping $A : \mathbb{R}^+ \times [0,1] \to \operatorname{End}\mathbb{R}^n$ analytic in the first argument for each value of the second one may prove to be everywhere discontinuous. Millionshchikov [4] suggested to use the Baire classification of discontinuous functions to describe the dependence of these and other characteristics of the asymptotic behavior of solutions of linear differential systems on their coefficients. Recall the following definition [5, Sec. 39.2].

Definition 3. Let M be a metric space. The *Baire classes with finite indices* are defined by induction as follows. The zeroth Baire class is the set of continuous functions $M \to \mathbb{R}$. If the classes with numbers less than $k \in \mathbb{N}$ have already been defined, then the kth Baire class is the set of functions $M \to \mathbb{R}$ representable as the pointwise limit of a sequence of functions of the $(k-1)$st class.

Millionshchikov [6] established that, for each $i = 1,\ldots,n$ and every continuous mapping A (not necessarily bounded with respect to t for fixed μ), the function Λ_i^A can be represented as the limit of a decreasing sequence of functions of the first Baire class. For the case in which the space M is complete, he proved that the set of points of upper semicontinuity of the function Λ_i^A contains a dense G_δ set (that is, an intersection of countably many open sets). A complete description of the n-tuples $(\Lambda_1^A,\ldots,\Lambda_n^A)$ for families (2) continuous in $\mathcal{M}_C^n$ was obtained in [7] for any space M, and a complete description of the n-tuples $(M_1,\ldots,M_n)$, where M_i, $i = 1,\ldots,n$, is the set of points of upper semicontinuity (or lower semicontinuity) of the function Λ_i^A for the same families and a complete space M was obtained in [8].

Now we require that the mapping $\mu \mapsto A(\cdot,\mu)$ corresponding to the family (2) be continuous in the uniform topology, that is, that the condition

$$\lim_{\nu\to\mu} \|A(\cdot,\nu) - A(\cdot,\mu)\| = 0, \qquad \mu \in M, \tag{3}$$

be satisfied. The collection of such families (identified with the mappings A defining them) will be denoted by $\mathcal{A}^n(M)$.

It is well known that the function Λ_1^A is continuous for any space M and any family $A \in \mathcal{A}^1(M)$. Indeed, in this case the (unique) Lyapunov exponent of the family (2) is given by the formula

$$\Lambda_1^A(\mu) = \varlimsup_{t\to+\infty} \frac{1}{t}\int_0^t A(\tau,\mu)\,d\tau, \qquad \mu \in M,$$

whence we find that $|\Lambda_1^A(\nu) - \Lambda_1^A(\mu)| \leq \|A(\cdot,\nu) - A(\cdot,\mu)\|$ for all $\mu, \nu \in M$ and hence, by condition (3), the mapping Λ_1^A is continuous for $A \in \mathcal{A}^1(M)$.

The dependence of the Lyapunov exponents Λ_k^A, $k = 1, \ldots, n$, on the parameter is much more complicated for families $A \in \mathcal{A}^n(M)$ with $n \geq 2$.

Perron [9] (see also [10, Sec. 1.4]) constructed an example of an analytic mapping $A \in \mathcal{A}^2([0,1])$ such that the function Λ_2^A is not upper semicontinuous. He also proposed the first nontrivial sufficient conditions under which system (1) is a point of continuity of all Lyapunov exponents on the space $\mathcal{M}_U^n$ simultaneously. (At the same time, there exist no such points in the space $\mathcal{M}_C^n$ even for $n = 1$.) It follows from the Perron–Bylov–Vinograd theorem [11, Th. 15.2.1] that the continuity of all Lyapunov exponents on the space $\mathcal{M}_U^n$ holds at every point of the set of systems with integral separation, which is open and dense in $\mathcal{M}_U^n$ [12]. However, an example of a family $A \in \mathcal{A}^n([0,1])$ such that the function Λ_i^A is everywhere discontinuous (and hence does not belong to the first Baire class [5, Sec. 38.4]) was constructed in [13] for any $n \geq 2$ and $i \in \{1, \ldots, n\}$, and an example of a space M and a family $A \in \mathcal{A}^n(M)$ for which the characteristic function of the set $\{\mu \in M : \Lambda_n(\mu) \leq 0\}$ does not belong to the second Baire class was given in [14]. A complete description of the set of upper (or lower) semicontinuity points of an individual Lyapunov exponent for families in $\mathcal{A}^n(M)$, where M is a complete space, can be found in [15].

The aim of the present paper is to describe the set $\{\Lambda_i^A : A \in \mathcal{A}^n(M)\}$ for every metric space M and any $n \geq 2$ and $i \in \{1, \ldots, n\}$.

2. MAIN RESULT

Definition 4 (cf. [16]). A function $f : M \to \mathbb{R}$ is called an *upper-limit function* if there exists a sequence of continuous functions $f_k : M \to \mathbb{R}$, $k \in \mathbb{N}$, such that

$$f(\mu) = \overline{\lim_{k\to\infty}} f_k(\mu), \qquad \mu \in M. \tag{4}$$

Remark 1. The property of a function indicated in Definition 4 is equivalent to each of the following conditions.

1. The function f can be represented as the pointwise limit of a decreasing sequence of functions of the first Baire class.

2. The preimage of every ray $[r, +\infty)$ ($r \in \mathbb{R}$) under the mapping f is a G_δ set. The functions satisfying this condition form the class $(^*, G_\delta)$ [5, Sec. 37.1].

The equivalence of conditions 1 and 2 was proved in the monograph [5, Sec. 37.1], and the equivalence of condition 2 and Definition 4 was demonstrated in [7, Remark 3].

The following theorem, which gives a complete description of the ith Lyapunov exponent, $i = 1, \ldots, n$, of a family (2) satisfying condition (3), is the main result of this paper.

Theorem. *Let a metric space M, a function $f : M \to \mathbb{R}$, and numbers $n \geq 2$ and $i \in \{1, \ldots, n\}$ be given. A necessary and sufficient condition that there exists a family $A \in \mathcal{A}^n(M)$ such that*

$$\lambda_i(A(\cdot,\mu)) = f(\mu), \qquad \mu \in M, \tag{5}$$

is that f is upper-limit and has a continuous minorant and a continuous majorant. Moreover, if f is bounded, then the mapping specifying the family can be chosen to be bounded.

Remark 2. In the case of $n = 1 = i$, there exists a family (2) with the desired properties if and only if f is continuous. The necessity of this condition was shown in the introduction, and to prove the sufficiency, we set $A(t,\mu) = f(\mu)$ for all $t \in \mathbb{R}^+$ and $\mu \in M$.

For comparison, let us present the result in [7] for an individual Lyapunov exponent: given a metric space M, a function $f : M \to \mathbb{R}$, and numbers $n \geq 1$ and $i \in \{1, \ldots, n\}$, there exists a family (2) with a continuous mapping A satisfying Eq. (5) if and only if f is upper-limit and has an upper semicontinuous minorant. One can readily show that, generally speaking, these conditions on f are broader than those indicated in the statement of the theorem. Indeed, let $M = [0,1]$, and let the function $f : [0,1] \to \mathbb{R}^+$ be given by the formula

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,1] \cap \mathbb{Q}, \\ 1/x + 1/(1-x) & \text{for the other } x. \end{cases}$$

Then the preimage of the ray $[r, +\infty)$ under f coincides with the closed interval $[0,1]$ for each $r \le 0$ and with the set $\{x \in [0,1] \setminus \mathbb{Q} : x(1-x) \le r^{-1}\}$ for $r > 0$. It is easily seen that the latter is a countable intersection of open sets; therefore, it follows from Remark 1 that f is an upper-limit function. Since f is bounded below, we see that it satisfies Eq. (5) with some continuous family (2) for any $n \ge 1$ and $i \in \{1, \ldots, n\}$. On the other hand, the function f is defined on a compact set and is not bounded above; accordingly, it does not have a continuous majorant and hence does not satisfy Eq. (5) for any family in $\mathcal{A}^n(M)$.

For any $n \ge 2$ and $i \in \{1, \ldots, n\}$, this theorem enables us to describe the set $\{\Lambda_i^A(M) : A \in \mathcal{A}^n(M)\} \equiv \mathfrak{R}_i^n(M)$ of ranges of the ith Lyapunov exponent of families in $\mathcal{A}^n(M)$ for the case of a complete separable space M. In what follows, we denote the sets of nonempty Suslin subsets [5, Sec. 32], bounded subsets, and at most countable subsets of the real line by $\mathfrak{S}$, $\mathfrak{B}$, and $\mathfrak{C}$, respectively, and $\mathfrak{P}(M)$ stands for the set of subsets of the real line whose cardinality does not exceed the cardinality of M.

Corollary 1. *Let a metric space M and numbers $n \ge 2$ and $i \in \{1, \ldots, n\}$ be given. Then the following assertions hold.*

1. *If the space M is compact, then $\mathfrak{R}_i^n(M) = \mathfrak{S} \cap \mathfrak{B} \cap \mathfrak{P}(M)$.*

2. *If the space M is a union of a compact set and a countable set and is not compact, then $\mathfrak{R}_i^n(M)$ consists of all sets of the form $S \cup C$, where $S \in \mathfrak{S} \cap \mathfrak{B} \cap \mathfrak{P}(M)$ and $C \in \mathfrak{C}$.*

3. *If the space M is complete and separable and is not a union of a compact set and an at most countable set, then $\mathfrak{R}_i^n(M) = \mathfrak{S}$.*

The structure of the Lebesgue sets [5, Sec. 37.1] of the Lyapunov exponents of families in $\mathcal{A}^n(M)$ is described in the following assertions.

Corollary 2. *Let a metric space M and numbers $n \ge 2$, $i \in \{1, \ldots, n\}$, and $r \in \mathbb{R}$ be given. Then the following assertions hold.*

1. *The collection of sets $\{\mu \in M : \Lambda_i^A(\mu) \ge r\}$, $A \in \mathcal{A}^n(M)$, consists of all G_δ subsets of M.*

2. *The collection of sets $\{\mu \in M : \Lambda_i^A(\mu) > r\}$, $A \in \mathcal{A}^n(M)$, consists of all $G_{\delta\sigma}$ subsets of M (that is, countable unions of G_δ subsets).*

Corollary 3. *Let $f : [0,1] \to \mathbb{R}$ be a bounded upper-limit function. Then for any $n \ge 2$ and $i \in \{1, \ldots, n\}$ there exists a bounded mapping $A : \mathbb{R}^+ \times [0,1] \to \operatorname{End} \mathbb{R}^n$ that is infinitely differentiable in the first argument and analytic in the second argument and satisfies condition* (3) *and Eq.* (5) *for $M = [0,1]$.*

3. PROOFS

To prove the above statements, we need some notation and three lemmas.

Set $\check{\beta} = -e^{3\pi/2}/(e^{3\pi/2} - 1) \in (-\sqrt{2}, -1)$ and define functions φ, Φ, and Ψ by the formulas

$$\varphi(\beta, \theta) = e^\theta(\beta - \sin\theta), \qquad \beta \in [\check{\beta}, 0], \qquad \theta \in [-\pi, 0], \tag{6}$$

$$\Phi(\beta) = \max_{-\pi \le \theta \le 0} \varphi(\beta, \theta), \qquad \beta \in [\check{\beta}, 0], \tag{7}$$

$$\Psi(\beta) = \max_{0 \le \eta \le 2\pi} (\tilde{\Phi}(\beta) e^{-\eta} + \sin\eta), \qquad \beta \in [\check{\beta}, 0], \tag{8}$$

where

$$\tilde{\Phi}(\beta) = \begin{cases} \Phi(\beta), & \beta \in [-1, 0], \\ \Phi(\beta) e^{2\pi}, & \beta \in [\check{\beta}, -1). \end{cases}$$

Lemma 1. *The function* Φ *is continuous and strictly increasing and satisfies the relations*

$$\Phi(\beta) = \varphi(\beta, \theta_\beta), \quad \textit{where} \ \ \theta_\beta = -\frac{\pi}{4} + \arcsin\frac{\beta}{\sqrt{2}}, \quad \beta \in [\check{\beta}, 0], \tag{9}$$

$$\operatorname{sgn}\Phi(\beta) = \operatorname{sgn}(\beta+1), \quad \beta \in [\check{\beta}, 0], \tag{10}$$

and the chain of inequalities

$$\Phi(\beta) \ge (\beta+1)e^{-\pi/2} \ge \beta e^{-2\pi}, \quad \beta \in [\check{\beta}, 0], \tag{11}$$

and the function Ψ *is continuous and strictly increasing and satisfies the relation*

$$\Psi(\beta) = \max_{0\le\eta\le\pi}(\tilde{\Phi}(\beta)e^{-\eta} + \sin\eta), \qquad \beta \in [\check{\beta}, 0], \tag{12}$$

and the inequalities

$$2 > \Psi(\beta) \ge 1, \qquad \beta \in [-1, 0], \tag{13}$$

$$1 > \Psi(\beta) > 0, \qquad \beta \in [\check{\beta}, -1). \tag{14}$$

Proof. The function Φ is strictly increasing, because so is each of the functions $\varphi(\cdot, \theta)$, $\theta \in [-\pi, 0]$. The first inequality in (11) follows from the definition of the function Φ and the relation $\varphi(\beta, -\pi/2) = (\beta+1)e^{-\pi/2}$, and the second inequality follows from the inequality $\beta \ge \check{\beta}$.

Let us verify Eq. (9). Since

$$\varphi'_\theta(\beta, \theta) = \sqrt{2}e^\theta\left(\frac{\beta}{\sqrt{2}} - \sin\left(\theta + \frac{\pi}{4}\right)\right), \qquad \beta \in [\check{\beta}, 0], \qquad \theta \in [-\pi, 0], \tag{15}$$

we conclude that if $\beta \in [-1, 0]$, then the function $\varphi(\beta, \cdot)$ has a unique interior critical point θ_β, in which it has a maximum; if $\beta \in [\check{\beta}, -1)$, then it has a minimum at the point $-3\pi/2 - \theta_\beta < \theta_\beta$ and the maximum at the point $\theta_\beta < -\pi/2$. In the latter case, we note that $\varphi(\beta, \theta_\beta) \ge \varphi(\beta, -\pi/2) = (\beta+1)e^{-\pi/2}$ and $\varphi(\beta, -\pi) = \beta e^{-\pi} \le \beta e^{-2\pi}$ and obtain the desired inequality $\varphi(\beta, -\pi) \le \varphi(\beta, \theta_\beta)$ from the second inequality in the chain of inequalities (11).

Using Eq. (9), we find that $\Phi(-1) = 0$, whence Eq. (10) follows, because the function Φ is increasing.

The function Ψ is continuous and strictly increasing, because so is the function $\tilde{\Phi}$, which in turn follows from the properties of the function Φ and Eq. (10). The second inequality in (13) and the first inequality in (14) follow from the fact that the function Ψ is increasing and from the relation $\Psi(-1) = 1$; the first inequality in (13) follows from the fact that the function Ψ is increasing and from the inequality $\Psi(0) \le e^{-\pi/4}/\sqrt{2} + 1$; and the second inequality in (14) follows from the chain of inequalities

$$\Psi(\beta) \ge \Phi(\beta)e^{3\pi/2} + 1 \ge (\beta+1)e^{\pi} + 1 > 0,$$

in which the first inequality follows from Eq. (8), the second inequality follows from the first inequality in (11), and the third inequality follows from the inequality $\beta \ge \check{\beta}$. Equation (12) for $\beta \in [-1, 0]$ follows from the chain of inequalities

$$\Psi(\beta) \ge \Phi(\beta)e^{-\pi/2} + 1 > \max_{\eta\in[\pi,2\pi]}(\Phi(\beta)e^{-\eta} + \sin\eta),$$

in which the first inequality follows from (8), the second inequality follows from (10) and, for the remaining β, from the second inequality in (14) and from the inequality

$$\max_{\eta\in[\pi,2\pi]}(\Phi(\beta)e^{2\pi-\eta} + \sin\eta) < 0,$$

which in turn follows from Eq. (10). The proof of the lemma is complete.

The following lemma uses a construction similar to those proposed in [17, Lemmas 3 and 4] and [14, Th. 3].

Lemma 2. *Let $s : [1, +\infty) \to [0, 1]$ be an arbitrary continuous function equal to unity on the closed intervals $[\tau_{k+1}e^{-\pi}, \tau_{k+1}e^{-\pi/8}]$, where $\tau_k = \exp(2\pi k)$, $k \geq 0$. To each sequence (α_k) of real numbers we assign the system*

$$\dot{x} = B_\alpha(t)x, \qquad B_\alpha(t) = \begin{pmatrix} q'(t) & s(t)b_\alpha(t) \\ 0 & 0 \end{pmatrix}, \qquad x \in \mathbb{R}^2, \qquad t \geq 1,$$

where $q(t) = t \sin \ln t$, $t \geq 1$, and $b_\alpha(t) = e^{\alpha_k t}$, $t \in [\tau_{k-1}, \tau_k)$, $k \in \mathbb{N}$. Set $\overline{\alpha} = \varlimsup\limits_{k\to\infty} \alpha_k$. Then the following assertions hold.

1. *If $\overline{\alpha} \in (-1, 0)$, then $\lambda_2(B_\alpha) = \Psi(\overline{\alpha})$.*
2. *If $\overline{\alpha} \in (\check{\beta}, -1)$, then $\lambda_1(B_\alpha) = \Psi(\overline{\alpha})$.*

Proof. 1. To prove the first assertion, following [10, Sec. 1.4], we compute the characteristic exponent of the function

$$J_\alpha(t) = e^{q(t)} \int\limits_1^t s(\tau)b_\alpha(\tau)e^{-q(\tau)}d\,\tau, \qquad t \geq 1.$$

Fix an arbitrary number $\varepsilon \in (0, -\overline{\alpha})$ and set $\beta_\varepsilon = \overline{\alpha} + \varepsilon \in (-1, 0)$. Then there exists a $C_\varepsilon > 0$ such that $|b_\alpha(\tau)| \leq C_\varepsilon e^{\beta_\varepsilon \tau}$ for all $\tau \geq 1$. We make the change of variable $\tau = \tau_k e^\theta$ in the integral and obtain the estimate

$$\int\limits_{\tau_k e^{-\pi/2}}^{\tau_k} \exp\{\beta_\varepsilon \tau - q(\tau)\}\,d\tau \leq \tau_k \int\limits_{-\pi/2}^{0} \exp\{\tau_k \varphi(\beta_\varepsilon, \theta)\}\,d\theta \leq \frac{\pi}{2}\tau_k \exp\{\tau_k \Phi(\beta_\varepsilon)\}, \qquad k \geq 1,$$

where the functions φ and Φ are defined by formulas (6) and (7), respectively.

Assume that the inclusion $t \in [\tau_k, \tau_k e^\pi]$ holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned} |J_\alpha(t)| &\leq C_\varepsilon e^{q(t)} \Bigg(\int\limits_1^{\tau_k e^{-\pi/2}} e^{\beta_\varepsilon \tau + \tau}\,d\tau + \int\limits_{\tau_k e^{-\pi/2}}^{\tau_k} e^{\beta_\varepsilon \tau - q(\tau)}\,d\tau + \int\limits_{\tau_k}^{t} d\tau \Bigg) \\ &\leq C_\varepsilon e^{q(t)} \Bigg(\frac{1}{\beta_\varepsilon + 1} \exp\{(\beta_\varepsilon + 1)\tau_k e^{-\pi/2}\} + 2\tau_k \exp\{\tau_k \Phi(\beta_\varepsilon)\} + t \Bigg). \end{aligned}$$

By setting $t = \tau_k e^\eta$ and by applying definition (8), the first inequality in (11), and inequality (13), we obtain the estimate

$$\begin{aligned} |J_\alpha(t)| &\leq C_\varepsilon \Bigg(\frac{1}{\varepsilon} \exp\{((\beta_\varepsilon + 1)e^{-\pi/2-\eta} + \sin\eta)t\} + 2t \exp\{(\Phi(\beta_\varepsilon)e^{-\eta} + \sin\eta)t\} + te^t \Bigg) \\ &\leq \frac{4C_\varepsilon}{\varepsilon} t \exp(\Psi(\beta_\varepsilon)t). \end{aligned}$$

If the inclusion $t \in [\tau_k e^\pi, \tau_{k+1}]$ holds for some $k \in \mathbb{N}$, then

$$|J_\alpha(t)| \leq C_\varepsilon \int\limits_1^t e^\tau\,d\tau \leq C_\varepsilon e^t.$$

Taking into account inequalities (13), we obtain the definitive estimate

$$|J_\alpha(t)| \leq \frac{4C_\varepsilon}{\varepsilon} t \exp(\Psi(\beta_\varepsilon)t), \qquad t \geq \tau_1,$$

from which the inequality $\lambda[J_\alpha] \le \Psi(\beta_\varepsilon)$ follows, where we pass to the limit as $\varepsilon \to 0+$ and obtain the inequality $\lambda[J_\alpha] \le \Psi(\overline{\alpha})$.

Let us establish the opposite inequality. By assumption, the sequence (α_k) has a subsequence (α_{k_j}) converging to $\overline{\alpha}$. Fix an arbitrary $\varepsilon \in (0, \min\{\overline{\alpha}+1, \pi/8\})$ and set $\gamma_\varepsilon = \overline{\alpha} - \varepsilon \in (-1, 0)$. Then there exists a $j_0 \in \mathbb{N}$ such that $\alpha_{k_j} \ge \gamma_\varepsilon$ for all $j \ge j_0$. Let $\eta_0 \in [0, 2\pi]$ be a point at which the maximum in the definition of the number $\Psi(\overline{\alpha})$ is attained. Consider the sequence $t_j = \tau_{k_j} e^{\eta_0}$, $j \in \mathbb{N}$. Note the chain of inequalities $-\pi/2 < \theta_{\gamma_\varepsilon} < \theta_{\gamma_\varepsilon} + \varepsilon < -\pi/8$, from which the relation $s(\tau) = 1$ for all $\tau \in [\tau_{k_j} \exp(\theta_{\gamma_\varepsilon}), \tau_{k_j} \exp(\theta_{\gamma_\varepsilon} + \varepsilon)]$ and the inequality $\tau_{k_j} \exp(\theta_{\gamma_\varepsilon} + \varepsilon) < t_j$ follow. Making the change of variable $\tau = \tau_{k_j} e^\theta$ in the integral and taking into account the fact that, by (15), the function $\varphi(\gamma_\varepsilon, \cdot)$ decreases on the interval $[\theta_{\gamma_\varepsilon}, \theta_{\gamma_\varepsilon} + \varepsilon]$, we obtain the estimate

$$
\begin{aligned}
J_\alpha(t_j) &\ge \exp(t_j \sin\eta_0) \int\limits_{\tau_{k_j} \exp(\theta_{\gamma_\varepsilon})}^{\tau_{k_j} \exp(\theta_{\gamma_\varepsilon}+\varepsilon)} \exp(\gamma_\varepsilon \tau - q(\tau))\, d\tau = \tau_{k_j} \exp(t_j \sin\eta_0) \int\limits_{\theta_{\gamma_\varepsilon}}^{\theta_{\gamma_\varepsilon}+\varepsilon} \exp\{\varphi(\gamma_\varepsilon, \theta)\tau_{k_j}\} e^\theta\, d\theta \\
&\ge \varepsilon \tau_{k_j} e^{-\pi/2} \exp(t_j \sin\eta_0) \min_{\theta \in [\theta_{\gamma_\varepsilon}, \theta_{\gamma_\varepsilon}+\varepsilon]} \exp\{\varphi(\gamma_\varepsilon, \theta)\tau_{k_j}\} \\
&\ge \varepsilon \exp\{(\sin\eta_0 + \varphi(\gamma_\varepsilon, \theta_{\gamma_\varepsilon} + \varepsilon) e^{-\eta_0}) t_j\}
\end{aligned}
$$

for all $j \ge j_0$. Consequently,

$$
\lambda[J_\alpha] \ge \varlimsup_{j\to\infty} \frac{1}{t_j} \ln J_\alpha(t_j) \ge \sin\eta_0 + \varphi(\gamma_\varepsilon, \theta_{\gamma_\varepsilon} + \varepsilon) e^{-\eta_0},
$$

whence, by passing to the limit as $\varepsilon \to 0+$, we obtain the inequality $\lambda[J_\alpha] \ge \Psi(\overline{\alpha})$. Thus, we have established that $\lambda[J_\alpha] = \Psi(\overline{\alpha})$, where $\Psi(\overline{\alpha}) \ge 1$ by virtue of inequality (13).

The vector functions

$$
u_1(t) = \begin{pmatrix} e^{q(t)} \\ 0 \end{pmatrix}, \qquad u_2(t) = \begin{pmatrix} J_\alpha(t) \\ 1 \end{pmatrix}, \qquad t \ge 1,
$$

form a fundamental system of solutions of the system B_α. Now the desired result follows from the relations

$$
\lambda[u_1] = 1 \le \Psi(\overline{\alpha}) = \lambda[J_\alpha] = \lambda[u_2].
$$

The proof of the first assertion is complete.

2. To prove the second assertion, we compute the characteristic exponent of the function

$$
I_\alpha(t) = e^{q(t)} \int\limits_t^\infty s(\tau) b_\alpha(\tau) e^{-q(\tau)}\, d\tau, \qquad t \ge 1.
$$

Let us estimate $\lambda[I_\alpha]$ from above, simultaneously establishing the convergence of the improper integral. Fix an arbitrary number $\varepsilon \in (0, -1-\overline{\alpha})$ and set $\beta_\varepsilon = \overline{\alpha} + \varepsilon \in (\check{\beta}, -1)$. Then there exists a $C_\varepsilon > 0$ such that $|b_\alpha(\tau)| \le C_\varepsilon e^{\beta_\varepsilon \tau}$ for all $\tau \ge 1$. We make the change of variable $\tau = \tau_{k+1} e^\theta$ in the integral and obtain the estimate

$$
\int\limits_{\tau_k e^\pi}^{\tau_{k+1}} \exp\{\beta_\varepsilon \tau - q(\tau)\}\, d\tau \le \tau_{k+1} \int\limits_{-\pi}^{0} \exp\{\tau_{k+1} \varphi(\beta_\varepsilon, \theta)\}\, d\theta \le \pi \tau_{k+1} \exp\{\tau_{k+1} \Phi(\beta_\varepsilon)\}, \qquad k \ge 0,
$$

where the functions φ and Φ are defined by formulas (6) and (7), respectively.

Suppose that the inclusion $t \in [\tau_k, \tau_{k+1}]$ holds for some $k \geq 0$. Then

$$I_\alpha(t) \leq C_\varepsilon e^{q(t)} \left(\int\limits_{\tau_k}^{\tau_k e^\pi} \exp(\beta_\varepsilon \tau)\, d\tau + \int\limits_{\tau_k e^\pi}^{\tau_{k+1}} \exp\{\beta_\varepsilon \tau - q(\tau)\}\, d\tau + \int\limits_{\tau_{k+1}}^{+\infty} \exp\{\beta_\varepsilon \tau + \tau\}\, d\tau \right)$$
$$\leq C_\varepsilon e^{q(t)} \left(-\frac{1}{\beta_\varepsilon} \exp(\beta_\varepsilon \tau_k) + \pi \tau_{k+1} \exp\{\tau_{k+1} \Phi(\beta_\varepsilon)\} - \frac{1}{\beta_\varepsilon + 1} \exp\{(\beta_\varepsilon + 1)\tau_{k+1}\} \right).$$

By setting $t = \tau_k e^\eta$ and by applying inequalities (11) and Definition (8), we obtain the estimate

$$|I_\alpha(t)| \leq C_\varepsilon \left(-\frac{1}{\beta_\varepsilon} \exp\{(\beta_\varepsilon e^{-\eta} + \sin\eta)t\} + \pi e^{2\pi} t \exp\{(\Phi(\beta_\varepsilon) e^{2\pi-\eta} + \sin\eta)t\} \right.$$
$$\left. - \frac{1}{\beta_\varepsilon + 1} \exp\{((\beta_\varepsilon + 1)e^{2\pi-\eta} + \sin\eta)t\} \right) \leq \frac{3C_\varepsilon}{|\beta_\varepsilon + 1|} \pi e^{2\pi} t \exp(\Psi(\beta_\varepsilon)t), \qquad t \geq 1,$$

whence the inequality $\lambda[I_\alpha] \leq \Psi(\beta_\varepsilon)$ follows. We pass to the limit as $\varepsilon \to 0+$ and obtain the inequality $\lambda[I_\alpha] \leq \Psi(\overline{\alpha})$.

Let us establish the opposite inequality. By assumption, the sequence (α_k) has a subsequence (α_{k_j}) converging to $\overline{\alpha}$. Fix an arbitrary number $\varepsilon \in (0, \overline{\alpha} - \check{\beta})$ and set $\gamma_\varepsilon = \overline{\alpha} - \varepsilon \in (\check{\beta}, -1)$. Then there exists a $j_0 \in \mathbb{N}$ such that $\alpha_{k_j} \geq \gamma_\varepsilon$ for all $j \geq j_0$. Let $\eta_0 \in [0, \pi]$ be a point at which the maximum in (12) is attained. Consider the sequence $t_j = \tau_{k_j - 1} e^{\eta_0}$, $j \in \mathbb{N}$. Note the chain of inequalities $-3\pi/4 < \theta_{\gamma_\varepsilon} < \theta_{\gamma_\varepsilon} + \varepsilon < -\pi/8$, which implies the relation $s(\tau) = 1$ for all $\tau \in [\tau_{k_j} \exp(\theta_{\gamma_\varepsilon}), \tau_{k_j} \exp(\theta_{\gamma_\varepsilon} + \varepsilon)]$ and the inequality $\tau_{k_j} \exp(\theta_{\gamma_\varepsilon}) > t_j$. Making the change of variable $\tau = \tau_{k_j} e^\theta$ in the integral and taking into account the fact that, by (15), the function $\varphi(\gamma_\varepsilon, \cdot)$ decreases on the interval $[\theta_{\gamma_\varepsilon}, \theta_{\gamma_\varepsilon} + \varepsilon]$, we obtain the estimate

$$I_\alpha(t_j) \geq \exp(t_j \sin\eta_0) \int\limits_{\tau_{k_j} \exp(\theta_{\gamma_\varepsilon})}^{\tau_{k_j} \exp(\theta_{\gamma_\varepsilon} + \varepsilon)} \exp(\gamma_\varepsilon \tau - q(\tau))\, d\tau = \tau_{k_j} \exp(t_j \sin\eta_0) \int\limits_{\theta_{\gamma_\varepsilon}}^{\theta_{\gamma_\varepsilon} + \varepsilon} \exp\{\varphi(\gamma_\varepsilon, \theta)\tau_{k_j}\} e^\theta\, d\theta$$
$$\geq \varepsilon \tau_{k_j} e^{-3\pi/4} \exp(t_j \sin\eta_0) \min_{\theta \in [\theta_{\gamma_\varepsilon}, \theta_{\gamma_\varepsilon} + \varepsilon]} \exp\{\varphi(\gamma_\varepsilon, \theta)\tau_{k_j}\}$$
$$\geq \varepsilon \exp\{(\sin\eta_0 + \varphi(\gamma_\varepsilon, \theta_{\gamma_\varepsilon} + \varepsilon) e^{2\pi - \eta_0}) t_j\}$$

for all $j \geq j_0$. Consequently,

$$\lambda[I_\alpha] \geq \varlimsup_{j \to \infty} \frac{1}{t_j} \ln I_\alpha(t_j) \geq \sin\eta_0 + \varphi(\gamma_\varepsilon, \theta_{\gamma_\varepsilon} + \varepsilon) e^{2\pi - \eta_0},$$

whence, passing to the limit as $\varepsilon \to 0+$, we obtain the estimate $\lambda[I_\alpha] \geq \Psi(\overline{\alpha})$. Thus, we have established that $\lambda[I_\alpha] = \Psi(\overline{\alpha})$, where $\Psi(\overline{\alpha}) \in (0, 1)$ by virtue of inequality (14).

The vector functions

$$u_1(t) = \begin{pmatrix} -I_\alpha(t) \\ 1 \end{pmatrix}, \qquad u_2(t) = \begin{pmatrix} e^{q(t)} \\ 0 \end{pmatrix}, \qquad t \geq 1,$$

form a fundamental system of solutions of the system B_α. By what has been proved above, we obtain the relations

$$\lambda[u_1] = \Psi(\overline{\alpha}) < 1 = \lambda[u_2],$$

whence the desired result follows. The proof of the lemma is complete.

Proof of the theorem. 1. Necessity. The following formulas were obtained in [6] for the Lyapunov exponents:

$$\lambda_i(A) = \inf_{m \in \mathbb{N}} \sup_{q \in \mathbb{N}} \varphi_i^{mq}(A), \qquad \varphi_i^{mq}(A) = \inf_{L \in G_i(\mathbb{R}^n)} \max_{t \in [m, m+q]} \frac{1}{t} \ln |X_A(t, 0)|_L|, \qquad A \in \mathcal{M}^n,$$

where $X_A(\cdot\,,\cdot)$ is the Cauchy operator of system (1) and $Y|_L$ is the restriction of the mapping Y to the set L. Further, it was proved there that the functions $\varphi_i^{mq} : \mathcal{M}_C^n \to \mathbb{R}$, $m, q \in \mathbb{N}$, are continuous. Thus,

$$f(\mu) = \lambda_i(A(\cdot,\mu)) = \inf_{m\in\mathbb{N}} \sup_{q\in\mathbb{N}} \varphi_i^{mq}(A(\cdot,\mu)), \qquad \mu \in M.$$

By condition (3) the mapping of M into $\mathcal{M}_C^n$ defined by the rule $\mu \mapsto A(\cdot,\mu)$ is continuous. By twice applying the assertion in [5, Sec. 37.1.I], we find that the function f belongs to the class $(^*, G_\delta)$; i.e., the preimage of any ray $[r, +\infty)$ ($r \in \mathbb{R}$) under the mapping f is a G_δ set. Then f is an upper-limit function by Remark 1. The existence of a continuous minorant and a continuous majorant follows from the estimate [11, p. 20]

$$-\|A(\cdot,\mu)\| \le \lambda_i(A(\cdot,\mu)) \le \|A(\cdot,\mu)\|, \qquad \mu \in M,$$

and the inequality

$$|\|A(\cdot,\nu)\| - \|A(\cdot,\mu)\|| \le \|A(\cdot,\nu) - A(\cdot,\mu)\|, \qquad \mu,\nu \in M.$$

2. Sufficiency. Using Theorem 1.4.1 in [18], take an arbitrary infinitely differentiable function $r : \mathbb{R} \to [0,1]$ supported in the interval $(-2\pi, 0)$ and identically equal to unity on the closed interval $[-\pi, -\pi/8]$. Define a function $s : [1, +\infty) \to [0,1]$ by setting

$$s(x) = \sum_{k=1}^{\infty} r\left(\ln \frac{x}{\tau_k}\right), \qquad x \ge 1, \tag{16}$$

where $\tau_k = \exp(2\pi k)$, $k \in \mathbb{N}$. Note that the supports of the terms of the series (16) are pairwise disjoint; therefore, the series (16) converges everywhere, and the function s is infinitely differentiable, is identically equal to unity on the set $\bigcup_{k\in\mathbb{N}}[\tau_k e^{-\pi}, \tau_k e^{-\pi/8}]$, and vanishes in some neighborhood of each of the points τ_k, $k \in \mathbb{N}$.

By assumption, there exist continuous functions $g, h : M \to \mathbb{R}$ satisfying the inequalities

$$g(\mu) \le f(\mu) \le h(\mu), \qquad \mu \in M,$$

and we will assume that $h(\mu) - g(\mu) \ge 1$ for all $\mu \in M$. (Otherwise, h is replaced by $h + 1$.)

Let $f_k : M \to \mathbb{R}$, $k \in \mathbb{N}$, be a sequence of continuous functions satisfying Eq. (4). Without loss of generality, we can assume that the following chain of inequalities holds for each $k \in \mathbb{N}$:

$$g(\mu) \le f_k(\mu) \le h(\mu), \qquad \mu \in M.$$

(Otherwise, the function f_k is replaced by the function $\max\{\min\{f_k(\cdot), h(\cdot)\}, g(\cdot)\}$.)

Set $I = (-1, -1/2)$ if $i \ge 2$ and $I = (\check{\beta}, -1)$ otherwise, where the number $\check{\beta}$ is defined before Lemma 1. Take an arbitrary closed interval $[p_1, p_2] \subset I$ such that $p_1 < p_2$.

Fix a $\mu \in M$. Let $l_\mu : s \mapsto \xi_\mu s + \upsilon_\mu$ be the increasing linear function taking the interval $[g(\mu), h(\mu)]$ to the interval $[\Psi(p_1), \Psi(p_2)]$, where the function Ψ is defined in (8). By Lemma 1, the function $\Psi : [p_1, p_2] \to \mathbb{R}$ is strictly increasing and continuous; consequently, the inverse function is well defined and continuous on the interval $[\Psi(p_1), \Psi(p_2)]$. Define a sequence α^μ by the formula

$$\alpha_k^\mu = \Psi^{-1}(l_\mu(f_k(\mu))), \qquad k \in \mathbb{N}.$$

We set

$$\tilde{A}_\mu(t) = \operatorname{diag}[B_{\alpha^\mu}(t+1), \underbrace{2, \ldots, 2}_{n-2}], \qquad t \ge 0,$$

if $i \le 2$ and

$$\tilde{A}_\mu(t) = \operatorname{diag}[\underbrace{0, \ldots, 0}_{i-2}, B_{\alpha^\mu}(t+1), \underbrace{2, \ldots, 2}_{n-i}], \qquad t \ge 0,$$

if $i > 2$, where B_{α^μ} is the system constructed in Lemma 2 from the function s defined in (16). By Lemma 2, we have the relations

$$\lambda_i(\tilde{A}_\mu) = \lambda_{\min\{i,2\}}(B_{\alpha^\mu}) = \Psi\left(\overline{\lim_{k\to\infty}}\,\alpha_k^\mu\right) = l_\mu\left(\overline{\lim_{k\to\infty}}\,f_k(\mu)\right) = l_\mu(f(\mu)).$$

Let $\tau \mapsto \eta_\mu\tau + \zeta_\mu$ be the inverse function of l_μ. Set $A(t,\mu) = \eta_\mu\tilde{A}_\mu(\eta_\mu t) + \zeta_\mu E$, $t \in \mathbb{R}^+$, where E is the identity matrix. Now if a function $x \neq 0$ is a solution of the system $\tilde{A}_\mu$, then the function $y : t \mapsto x(\eta_\mu t)e^{\zeta_\mu t}$ is a solution of the system $A(\cdot,\mu)$ with $y(0) = x(0)$ and $\lambda[y] = \eta_\mu\lambda[x] + \zeta_\mu$. Therefore, $\lambda_i(A(\cdot,\mu)) = \eta_\mu\lambda_i(\tilde{A}_\mu) + \zeta_\mu = f(\mu)$.

The coefficients of the system $\tilde{A}_\mu$ are bounded (by the number 2) and infinitely differentiable, because the function b_{α^μ} in the definition of the system B_{α^μ} is infinitely differentiable on the intervals (τ_k, τ_{k+1}) and the function s is infinitely differentiable everywhere and vanishes in a neighborhood of each of the points τ_k, $k \in \mathbb{N}$. Therefore, the system $A(\cdot,\mu)$ has infinitely differentiable coefficients and is bounded on the half-line $\mathbb{R}^+$.

Now let us show that the mapping A satisfies condition (3). Given an $\varepsilon \in (0,1)$, take a $T > 1$ such that $e^{-T/2} < \varepsilon$. By virtue of the uniform continuity of the function sb_{α^μ} on the interval $[1, 3T+1]$, there exists a $\delta \in (0,\varepsilon)$ such that the inequality $|s(t')b_{\alpha^\mu}(t') - s(t'')b_{\alpha^\mu}(t'')| < \varepsilon$ holds for any $t', t'' \in [1, 3T+1]$ with $|t' - t''| < \delta$. The relations

$$\xi_\nu = \frac{\Phi(p_2) - \Phi(p_1)}{h(\nu) - g(\nu)} = \frac{1}{\eta_\nu}, \qquad \upsilon_\nu = \Phi(p_1) - \xi_\nu g(\nu), \qquad \zeta_\nu = -\upsilon_\nu\eta_\nu, \qquad \nu \in M, \tag{17}$$

imply that the functions $\nu \mapsto \alpha_k^\nu$, $\nu \mapsto \eta_\nu$, and $\nu \mapsto \zeta_\nu$, $k \in \mathbb{N}$, are continuous. Consequently, there exists a neighborhood U of the point μ such that $|\eta_\nu - \eta_\mu| < \delta/(2T)$ and $|\zeta_\nu - \zeta_\mu| < \varepsilon$ for all $\nu \in U$. Take an $m \in \mathbb{N}$ such that $\tau_m > T$ and a neighborhood $V \subset U$ of the point μ such that $|\alpha_k^\nu - \alpha_k^\mu| < \ln(1+\varepsilon)/\tau_m$ for all $\nu \in V$ and $k = 1, \ldots, m$. Then for any $\nu \in V$, $k = 1, \ldots, m$, and $t \in [\tau_{k-1}, \tau_k)$, we have the chain of inequalities

$$|e^{\alpha_k^\nu t} - e^{\alpha_k^\mu t}| = e^{\alpha_k^\mu t}|e^{(\alpha_k^\nu - \alpha_k^\mu)t} - 1| \le e^{|\alpha_k^\nu - \alpha_k^\mu|t} - 1 < \varepsilon,$$

which implies that $|(sb_{\alpha^\nu})(t) - (sb_{\alpha^\mu})(t)| \le |b_{\alpha^\nu}(t) - b_{\alpha^\mu}(t)| < \varepsilon$ for all $\nu \in V$ and $t \in [1, T]$. At the same time, $0 \le s(t)b_{\alpha^\nu}(t) \le e^{-t/2}$ for all $\nu \in M$ and $t \ge 1$, and hence

$$|(sb_{\alpha^\nu})(t) - (sb_{\alpha^\mu})(\tau)| \le e^{-T/2} < \varepsilon$$

for all $\nu \in M$ and $\tau, t \ge T$.

Inequalities (13) and (14) imply the inequality $\Psi(p_2) - \Psi(p_1) < 1$, and hence $\eta_\nu > 1$ for all $\nu \in M$. Consequently, $|\eta_\nu - \eta_\mu| < \eta_\mu/2$ for all $\nu \in V$. It follows from the preceding that

$$|(sb_{\alpha^\mu})(\eta_\nu t + 1) - (sb_{\alpha^\mu})(\eta_\mu t + 1)| < \varepsilon$$

for all $\nu \in V$ and $t \ge 2T/\eta_\mu$. If $t \in [0, 2T/\eta_\mu]$, then $\eta_\nu t + 1 \in [1, 3T+1]$ and $|\eta_\nu t - \eta_\mu t| < \delta$ for all $\nu \in V$, and hence

$$|(sb_{\alpha^\mu})(\eta_\nu t + 1) - (sb_{\alpha^\mu})(\eta_\mu t + 1)| < \varepsilon.$$

Further, for all $\nu \in V$ one has the chain of inequalities

$$\begin{aligned} |q'(\eta_\nu t + 1) - q'(\eta_\mu t + 1)| &\le \max_{\theta \in [\eta_\mu t + 1, \eta_\nu t + 1]} \left|\frac{\cos(\ln\theta) - \sin(\ln\theta)}{\theta}\right| |\eta_\nu - \eta_\mu| t \\ &\le \frac{2}{(\eta_\mu/2)t + 1}|\eta_\nu - \eta_\mu| t < 2\varepsilon, \qquad t \ge 0. \end{aligned}$$

Since property (3) is independent of the choice of the matrix norm, in the following estimates, we use the *column norm*

$$|T|_1 = \max_{1 \le j \le n} \sum_{i=1}^n |t_{ij}|, \qquad T \in \operatorname{End}\mathbb{R}^n.$$

Since $|\tilde{A}_\mu(t)|_1 \le 2$, $t \in \mathbb{R}^+$, we obtain the following relations for every $\nu \in V$:

$$\begin{aligned}
|A(t,\nu) - A(t,\mu)|_1 &= |(\eta_\nu \tilde{A}_\nu(\eta_\nu t) + \zeta_\nu E) - (\eta_\mu \tilde{A}_\mu(\eta_\mu t) + \zeta_\mu E)|_1 \\
&\le |\zeta_\nu - \zeta_\mu| + \eta_\nu |\tilde{A}_\nu(\eta_\nu t) - \tilde{A}_\mu(\eta_\nu t)|_1 \\
&\quad + |\eta_\nu - \eta_\mu||\tilde{A}_\mu(\eta_\nu t)|_1 + \eta_\mu |\tilde{A}_\mu(\eta_\nu t) - \tilde{A}_\mu(\eta_\mu t)|_1 \\
&\le \varepsilon + 2\eta_\mu |(sb_{\alpha^\nu})(\eta_\nu t + 1) - (sb_{\alpha^\mu})(\eta_\nu t + 1)| + \varepsilon \\
&\quad + \eta_\mu \max\{|(sb_{\alpha^\mu})(\eta_\nu t + 1) - (sb_{\alpha^\mu})(\eta_\mu t + 1)|, |q'(\eta_\nu t + 1) - q'(\eta_\mu t + 1)|\} \\
&\le 2\varepsilon + 4\eta_\mu \varepsilon < 6\eta_\mu \varepsilon, \qquad t \in \mathbb{R}^+.
\end{aligned}$$

3. If the function f is bounded, then for the minorant g and the majorant h in the constructions of part 2 of the proof, we take the constants $g_0 \equiv \inf_{\mu \in M} f(\mu)$ and $h_0 \equiv \sup_{\mu \in M} f(\mu) + 1$, respectively. Then, by Eqs. (17), the values $\eta_\mu \equiv \eta$ and $\zeta_\mu \equiv \zeta$ are independent of μ, and so the mapping constructed in part 2 of the proof satisfies the estimate

$$|A(t,\mu)|_1 \le \eta |\tilde{A}_\mu(\eta t)|_1 + \zeta \le 2\eta + \zeta, \qquad t \in \mathbb{R}^+, \qquad \mu \in M.$$

The proof of the theorem is complete.

The proof of Corollary 3 can be carried out by analogy with parts 2 and 3 of the proof of the theorem. Let us indicate the necessary changes in the constructions. For the functions g and h, just as in part 3 of the proof of the theorem, we take constants. Then $l_\mu \equiv l$, $\eta_\mu \equiv \eta$, and $\zeta_\mu \equiv \zeta$, $\mu \in M = [0,1]$. Set

$$\tilde{\alpha}_k^\mu = \Psi^{-1}(l(f_k(\mu))), \qquad k \in \mathbb{N}, \qquad \mu \in M.$$

As was proved in part 2 of the proof of the theorem, the functions $\mu \mapsto \tilde{\alpha}_k^\mu$, $k \in \mathbb{N}$, are continuous, and therefore, by the Weierstrass theorem [19, Ch. XVI , Sec. 4, Th. 2], for every $k \in \mathbb{N}$ there exists a polynomial p_k such that

$$\sup_{\mu \in M} |p_k(\mu) - \tilde{\alpha}_k^\mu| < 1/k.$$

Set $\alpha_k^\mu = p_k(\mu)$, $k \in \mathbb{N}$, $\mu \in M$. Then for every $t \in \mathbb{R}^+$ the function $\mu \mapsto b_{\alpha^\mu}(t+1)$ is analytic, and hence the function $\mu \mapsto A(t,\mu) \equiv \eta \tilde{A}_\mu(\eta t) + \zeta E$ is analytic as well. Since $\overline{\lim}_{k\to\infty} \alpha_k^\mu = \overline{\lim}_{k\to\infty} \tilde{\alpha}_k^\mu$ for all $\mu \in M$, it follows that the Lyapunov exponents of this family and of the family constructed in part 2 of the proof of the theorem coincide. The proof of the corollary is complete.

Definition 5 [20, Sec. 31.I]. Let X and Y be metric spaces. A function $f : X \to Y$ is said to be *B-measurable of class* 1 if the preimage $f^{-1}(F)$ of every closed set $F \subset Y$ is a G_δ set.

Remark 3. For every metric space X, the set of B-measurable functions $f : X \to \mathbb{R}$ of class 1 coincides with the first Baire class [5, Sec. 38.I].

Lemma 3. *Let X be an uncountable G_δ set in a complete separable metric space, and let $S \subset \mathbb{R}$ be a nonempty bounded Suslin set. Then there exists a B-measurable function $f : X \to \mathbb{R}$ of class* 1 (*of the first Baire class*) *such that $f(X) = S$.*

Proof. By the Aleksandrov–Hausdorff theorem [20, Sec. 33.VI], the set X is homeomorphic to a complete metric space. By [20, Sec. 36.V, Corollary 2], there exists a G_δ subset $N \subset X$ homeomorphic to the space $\mathcal{N}$ of irrational numbers. Using the theorem of [20, Sec. 35.VI], we extend a homeomorphism $h : N \to \mathcal{N}$ to a B-measurable function $\hat{h} : X \to \mathcal{N}$ of class 1. According to [5, Sec. 35.III], there exists a continuous function $g : \mathcal{N} \to \mathbb{R}$ such that $g(\mathcal{N}) = S$. Then $f(X) = S$ and f is B-measurable of class 1 as the composition of a function of class 1 and a continuous function [20, Sec. 31.III, Th. 2]. The proof of the lemma is complete.

Proof of Corollary 1. 1. Let M be a compact space. For every family $A \in \mathcal{A}^n(M)$, it follows from the theorem and Remark 1 that the function $\Lambda_i^A : M \to \mathbb{R}$ belongs to the second Baire class. Then, by [5, Sec. 39.IX], its image $\Lambda_i^A(M)$ is a Suslin set. Further, by the theorem, the function Λ_i^A

has continuous (and hence bounded) minorant and majorant. Consequently, the set $\Lambda_i^A(M)$ is bounded. Finally, the cardinality of $\Lambda_i^A(M)$ does not exceed the cardinality of M. Thus, we obtain the inclusion $\mathfrak{R}_i^n(M) \subset \mathfrak{S} \cap \mathfrak{B} \cap \mathfrak{P}(M)$.

Let us establish the opposite inclusion. Given $S \in \mathfrak{S} \cap \mathfrak{B} \cap \mathfrak{P}(M)$, let us construct a function $f : M \to \mathbb{R}$ of the first Baire class such that $f(M) = S$. If M is uncountable, then the existence of the desired function follows from Lemma 3 (because every compact metric space is complete and separable). Now consider the case in which M is at most countable. Since $S \in \mathfrak{P}(M)$, it follows that there exists a surjection $f : M \to S$. The preimage of every (in particular, open) subset of $\mathbb{R}$ under f is an at most countable union of singletons. Passing to complements, we see that the preimage of every closed subset of $\mathbb{R}$ under f is a G_δ set. Hence the function f belongs to the first Baire class (see Remark 3).

Since every function of the first Baire class is upper-limit, it follows from the theorem that there exists a family $A \in \mathcal{A}^n(M)$ such that $\Lambda_i^A(M) = f(M) = S$. The proof of part 1 of the corollary is complete.

2. Assume that the space M is noncompact and can be represented in the form $K \cup D$, where K is a compact set and D is a countable set. We apply the assertion already proved in part 1 and find that $\Lambda_i^A(K) \in \mathfrak{S} \cap \mathfrak{B} \cap \mathfrak{P}(M)$ for every family $A \in \mathcal{A}^n(M)$. The relation $\Lambda_i^A(M) = \Lambda_i^A(K) \cup \Lambda_i^A(D)$ and the inclusion $\Lambda_i^A(D) \in \mathfrak{C}$ give the inclusion

$$\mathfrak{R}_i^n(M) \subset \{R : R = S \cup C, \quad S \in \mathfrak{S} \cap \mathfrak{B} \cap \mathfrak{P}(M), \quad C \in \mathfrak{C}\}.$$

Let us establish the opposite inclusion. Let $S \in \mathfrak{S} \cap \mathfrak{B} \cap \mathfrak{P}(M)$ and $C \in \mathfrak{C}$. By Lemma 3, there exists a function $\varphi : K \to \mathbb{R}$ of the first Baire class such that $\varphi(K) = S$. Since M is noncompact, it follows that there exists an unbounded continuous function $u : M \to \mathbb{R}^+$. Fix a surjection $c : \mathbb{N} \to C$. Then, by the choice of the function u, there exists a sequence (d_i) of pairwise distinct points of the set $D \setminus K$ such that $u(d_i) > |c_i|$ for all $i \in \mathbb{N}$. Take an arbitrary point $s \in S$ and define a function $f : M \to \mathbb{R}$ by setting

$$f(\mu) = \begin{cases} \varphi(\mu), & \mu \in K, \\ c_i, & \mu = d_i,\ i \in \mathbb{N}, \\ s & \text{for the other } \mu. \end{cases}$$

By construction, $f(M) = S \cup C$, and the functions

$$\mu \mapsto \max\{u(\mu), \sup S\}, \qquad \mu \mapsto \min\{-u(\mu), \inf S\}$$

are a majorant and a minorant, respectively, of f. By Theorem 1 in [20, Sec. 31.IV], the function f is B-measurable of class 1 and hence belongs to the first Baire class (see Remark 3). Since every function of the first Baire class is upper-limit, it follows by the theorem that there exists a family $A \in \mathcal{A}^n(M)$ such that $\Lambda_i^A(M) = f(M) = S \cup C$. The proof of part 2 of the corollary is complete.

3. Assume that the space M is complete and separable and cannot be represented as a union of a compact set and an at most countable set. We denote the set of its condensation points by C [20, Sec. 23.III]. The set $M \setminus C$ is countable, and so the set C is uncountable and noncompact. Hence there exists an unbounded continuous function $u : C \to \mathbb{R}^+$. Take a sequence (c_i) of points of the set C such that the inequality $u(c_i) + 2 < u(c_{i+1})$ is satisfied for each $i \in \mathbb{N}$. Since the set C is closed [20, Sec. 23.IV], it follows by the Tietze theorem [20, Sec. 14.IV] that there exists a continuous extension of the function u to the entire space M; this extension will be denoted by the same letter.

Fix an arbitrary point $s \in S$. For any $i \in \mathbb{N}$, take an open neighborhood U_i of the point c_i in M such that $|u(\mu) - u(c_i)| < 1$ for all $\mu \in U_i$. Set

$$S_i = (S \cap [-u(c_i), u(c_i)]) \cup \{s\}.$$

By Lemma 3, there exists a B-measurable function $f_i : U_i \to \mathbb{R}$ of class 1 such that $f_i(U_i) = S_i$.

Finally, we define a function $f : M \to \mathbb{R}$ by the formula

$$f(\mu) = \begin{cases} f_i(\mu) & \text{for } \mu \in U_i,\ i \in \mathbb{N}, \\ s & \text{for all other } \mu. \end{cases}$$

Note that the neighborhoods U_i, $i \in \mathbb{N}$, are pairwise disjoint, and so the function f is well defined. By Theorem 1 in [20, Sec. 31.IV], the function f is B-measurable of class 1, and hence belongs to the first Baire class (see Remark 3). By construction, the functions $\mu \mapsto \max\{u(\mu)+1, s\}$ and $\mu \mapsto \min\{-u(\mu)-1, s\}$ are a majorant and a minorant, respectively, of the function f. By the choice of the sequence (c_i), we have $\bigcup_{i\in\mathbb{N}} S_i = S$, whence we obtain $f(M) = S$. By applying the theorem, we obtain the desired result. The proof of part 3 and of the entire corollary is complete.

Proof of Corollary 2. Let $A \in \mathcal{A}^n(M)$. Then it follows from the theorem that the function Λ_i^A is upper-limit. By Remark 1, the set $\{\mu \in M : \Lambda_i^A(\mu) \ge q\}$ is a G_δ set for every $q \in \mathbb{R}$. It follows from the obvious relation

$$\{\mu \in M : \Lambda_i^A(\mu) > r\} = \bigcup_{k\in\mathbb{N}} \{\mu \in M : \Lambda_i^A(\mu) \ge r + k^{-1}\}$$

that the set $\{\mu \in M : \Lambda_i^A(\mu) > r\}$ is a $G_{\delta\sigma}$ set.

Conversely, let $S \subset M$ be a G_δ set. Set $f = \chi_S + r - 1$, where $\chi_S : M \to \{0,1\}$ is the characteristic function of the set S. Since the preimage of every ray $[q, +\infty)$, $q \in \mathbb{R}$, under f is a G_δ set, it follows from Remark 1 that the function f is upper-limit. By the theorem, there exists a family $A \in \mathcal{A}^n(M)$ such that $\Lambda_i^A = f$. Then $S = \{\mu \in M : \Lambda_i^A(\mu) \ge r\}$.

Now let $S = \bigcup_{k\in\mathbb{N}} S_k$, where $S_k \subset M$, $k \in \mathbb{N}$, are G_δ sets. Define a function $f : M \to [r, r+1]$ by the formula

$$f(\mu) = r + \sum_{k=1}^{\infty} 2^{-k} \chi_{S_k}(\mu), \qquad \mu \in M. \tag{18}$$

For each $k \in \mathbb{N}$, the preimage of every ray $[q, +\infty)$, $q \in \mathbb{R}$, under the mapping χ_{S_k} is a G_δ set, and hence the function χ_{S_k} can be represented as the limit of a decreasing sequence of functions of the first Baire class by Remark 1. The partial sums of the series (18), as well as its sum f, have the same property, because the series converges uniformly. By Remark 1, the function f is upper-limit. By the theorem, there exists a family $A \in \mathcal{A}^n(M)$ such that $\Lambda_i^A = f$. Then $S = \{\mu \in M : \Lambda_i^A(\mu) > r\}$. The proof of the corollary is complete.

4. CONCLUSION

As said in the introduction, a complete description of the n-tuples $(\Lambda_1, \dots, \Lambda_n)$ of Lyapunov exponents of the families (2) was obtained in [7] for any metric space M for the case in which the families are given by mappings $\mu \mapsto A(\cdot, \mu)$ continuous in the compact-open topology and such that for each $\mu \in M$ the coefficients of system (2) are continuous and bounded on the half-line $\mathbb{R}^+$. A complete description of the same tuples is given in [21, 22] in the absence of the boundedness condition on the half-line for the coefficients of the systems in the family. (One simply needs to discard the requirement for the existence of a semicontinuous minorant.)

In the present paper, for mappings $\mu \mapsto A(\cdot, \mu)$ continuous in the uniform topology and such that for each $\mu \in M$ system (2) has continuous bounded coefficients on the half-line $\mathbb{R}^+$, a complete description of each individual function Λ_i is given. The following two questions remain open. What is each of these functions if the boundedness condition on the half-line for the coefficients of the systems in the family is not satisfied? What are the n-tuples of these functions for families continuous in the uniform topology under the condition that the coefficients of the systems in the family are bounded and without this condition?

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