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PARTIAL DIFFERENTIAL EQUATIONS

Coverings and Integrable Pseudosymmetries of Differential Equations

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Abstract—We study the problem on the construction of coverings by a given system of differential equations and the description of systems covered by it. This problem is of interest in view of its relationship with the computation of nonlocal symmetries, recursion operators, Bäcklund transformations, and decompositions of systems. We show that the distribution specified by the fibers of the covering is determined by a pseudosymmetry of the system and is integrable in the infinite-dimensional sense. Conversely, every integrable pseudosymmetry of a system defines a covering by this system. The vertical component of the pseudosymmetry is a matrix analog of the evolution differentiation, and the corresponding generating matrix satisfies a matrix analog of the linearization of an equation.

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1. INTRODUCTION

An example of a covering is given by the well-known Cole–Hopf substitution $u = 2w_x/w$, which transforms the heat equation $w_t = w_{xx}$ into the Burgers equation

$$u_t = uu_x + u_{xx}. \tag{1}$$

To construct the inverse transformation, one should understand the variable w as a nonlocal variable given by the system of equations

$$w_x = \frac{1}{2}uw, \qquad w_t = \frac{1}{2}u_x w + \frac{1}{4}u^2 w. \tag{2}$$

To compute the solution $w(t,x)$ from the solution $u = u_0(t,x)$, it suffices to specify an initial value $w(t_0,x_0) = w_0$ and find a solution of system (2) with $u = u_0(t,x)$ and with this initial condition. Thus, to one solution of the Burgers equation (1) there corresponds a set of solutions of the heat equation. One says that the heat equation covers the Burgers equation.

Now consider an arbitrary system of differential equations for a function $u = (u^1,\dots,u^m)$ of the variables $x = (x_1,\dots,x_n)$ of the form

$$G_\alpha(x,u,\dots,u_\sigma,\dots) = 0, \qquad \alpha = 1,\dots,r. \tag{3}$$

Here and in the following, $\sigma = i_1\dots i_k$ is a multi-index, $1 \le i_j \le n$ for all $j = 1,\dots,k$, $|\sigma| = k$, and

$$u_\sigma = \left(\frac{\partial^{|\sigma|}u^1}{\partial x_{i_1}\dots\partial x_{i_k}},\dots,\frac{\partial^{|\sigma|}u^m}{\partial x_{i_1}\dots\partial x_{i_k}}\right).$$

The expression on the left-hand side in (3) means that the function G_α depends on x, u, and finitely many derivatives of the form u_σ. The maximum length $|\sigma|$ of multi-indices σ occurring in Eqs. (3) coincides with the order of the system.

Consider a system of differential equations

$$w_i^j = W_i^j(x, w, u, \ldots, u_\sigma, \ldots), \qquad i = 1, \ldots, n, \qquad j = 1, \ldots, q, \tag{4}$$

for the functions u and $w = (w^1, \ldots, w^q)$ of the variables x such that for any j, i, and s the equation $w_{is}^j = w_{si}^j$ is a corollary of system (3), (4). Then one says that system (3), (4) *covers* system (3). The variables $w^1, \ldots, w^q$ are said to be *nonlocal*, because, to compute their values, one has to integrate system (4) of differential equations with given $u(x)$. The existence of a solution of this system follows from the equality of mixed derivatives.

A symmetry of system (3), (4) depending on the nonlocal variables $w^1, \ldots, w^q$ is called a *nonlocal symmetry* of system (3).

The notion of covering of systems of differential equations was introduced in [1, 2] as a development of the Wahlquist–Estabrook prolongation structures [3]. The language of coverings was used to state some well-known differential substitutions in equations of mathematical physics, the notions of nonlocal symmetry, Bäcklund transformation, and recursion operator [4; 5, Ch. 6], and also dynamic feedback, dynamic linearizability [6], and decomposition [7] of control systems. The problem of finding coverings over a given system is related to the description of integrable systems [8, 9] and was solved by numerous authors (e.g., see [1–5, 8–15]).

The present paper deals with the inverse problem (which is apparently simpler) of the description of coverings by a given system and the search for systems covered by it. We consider invertible transformations of a given system into a system of the form (3), (4) such that the independent and dependent variables of one of the systems are expressed via the independent and dependent variables and the derivatives of the dependent variables with respect to the independent variables of the other system. Such transformations are called $\mathcal{C}$-transformations [5, Ch. 4], or Lie–Bäcklund isomorphisms. When using such transformations, one has to consider the infinite prolongations of the systems to be transformed.

This problem is related to the decomposition problem, because system (3), (4) into which the given system is transformed has a decomposable form. This explains the analogy between our results and the well-known results in [16], where any decomposition of an affine control system is associated with an affine distribution invariant with respect to that system. In our case, every covering by a given system is associated with an invariant integrable distribution on the infinite prolongation of the system.

Coverings by evolution equations with one spatial variable were considered in the paper [17], where the notion of pseudosymmetry of an equation was introduced. We introduce the notion of integrability of a pseudosymmetry, thus generalizing and strengthening the results in [17]. In particular, we prove that coverings by a system of differential equations determine and are determined by integrable pseudosymmetries of that system.

In Sections 2 and 3, we present the notions and assertions of the infinite-dimensional geometry of differential equations needed to state the results of the paper. The main results are stated in Section 4 and proved in Section 5.

2. INFINITE PROLONGATIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

A complete, more general exposition of the theory presented in this section and the next section can be found in the monograph [5, Ch. 3 and Ch. 4].

For $k \geq 0$, let J^k be the finite-dimensional space with coordinates

$$x_i, \quad u^j, \quad u_i^j, \quad \ldots, \quad u_\sigma^j, \quad \ldots, \tag{5}$$

where $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $\sigma = i_1 \ldots i_l$, $l \leq k$. A k times differentiable vector function $s = (s^1(x), \ldots, s^m(x))^{\mathrm{T}}$ and a point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ in a neighborhood of which the function is defined specify a point in J^k with coordinates

$$x_i = a_i, \quad u^j = s^j(a), \quad \ldots, \quad u_\sigma^j = \frac{\partial^{|\sigma|} s^j(a)}{\partial x_{i_1} \ldots \partial x_{i_l}}, \quad \ldots, \quad \sigma = i_1 \ldots i_l,$$

which is called the *k-jet* of $s(x)$ at a. The space J^k is called the *k-jet space* with n independent and m dependent variables.

System (3) of equations of order $\leq k$ defines a surface $\mathcal{E} \subset J^k$. This surface is an invariant object, in contrast to its representation in the form of system (3), because the equations of one and the same surface may be different but equivalent. In what follows, we identify a system of equations of order $\leq k$ with the surface defined by it in the space J^k.

If $s = (s^1(x), \dots, s^m(x))^{\mathrm{T}}$ is a solution of system (3), then s must satisfy all differential corollaries of system (3), in particular, the equations

$$\frac{\partial G_\alpha}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial G_\alpha}{\partial u^j_\sigma} = 0, \qquad \alpha = 1, \dots, r, \qquad i = 1, \dots, n. \tag{6}$$

The union of systems (3) and (6) is called the *first prolongation of system* (3). Define the *lth prolongation of a system* as the first prolongation of its $(l-1)$st prolongation.The corresponding surface in J^{k+l} will be denoted by $\mathcal{E}^{(l)}$. By definition, $\mathcal{E}^{(l)} = (\mathcal{E}^{(l-1)})^{(1)}$.

For any $k \geq 0$ and $s > k$, one had the projection of J^s onto J^k, which "forgets" the coordinates u_σ for $|\sigma| > k$. We denote it by $\pi_{s,k}$. System $\mathcal{E} \subset J^k$ is said to be *formally integrable* if for each positive integer l the surface $\mathcal{E}^{(l)}$ is a submanifold of J^{k+l} and the projection $\pi_{k+l,k+l-1} : \mathcal{E}^{(l)} \to \mathcal{E}^{(l-1)}$ is a vector bundle.

The *space* J^∞ *of infinite jets* is defined as the inverse (projective) limit of the chain of projections

$$J^0 \xleftarrow{\pi_{1,0}} J^1 \leftarrow \cdots \leftarrow J^k \xleftarrow{\pi_{k+1,k}} J^{k+1} \leftarrow \cdots$$

Namely, an element of J^∞ is a sequence of points $\theta_k \in J^k$, $k \geq 0$, such that

$$\theta_0 \xleftarrow{\pi_{1,0}} \theta_1 \hookleftarrow \cdots \hookleftarrow \theta_k \xleftarrow{\pi_{k+1,k}} \theta_{k+1} \hookleftarrow \cdots$$

The *infinite jet of a function* at a point is defined as the sequence of its k-jets at the point. Each point of J^∞ is the infinite jet of some function (see the proof in [5, Ch. 4, Sec. 1.1]). The canonical coordinates on the finite jet spaces generate the *canonical coordinates* (5) on J^∞, where the length of the multi-index σ is an arbitrary positive integer. The projection of J^∞ onto J^l takes each sequence $\{\theta_k\}$ to the point θ_l and is denoted by $\pi_{\infty,l}$.

The set J^∞ is equipped with the structure of an infinite-dimensional smooth manifold. Namely, one defines smooth (infinitely differentiable) functions, vector fields, and differential forms on J^∞. A *smooth function on* J^∞ is a function smoothly depending on finitely many (but arbitrary) coordinates (5). The algebra of smooth functions on J^∞ will be denoted by $\mathcal{F}(J)$. Every derivation of this algebra is a sum (in the general case, infinite) of the form

$$\sum_{i=1}^n g_i \frac{\partial}{\partial x_i} + \sum_{j=1}^m \sum_{|\sigma| \geq 0} g^j_\sigma \frac{\partial}{\partial u^j_\sigma},$$

where g_i and g^j_σ are some smooth functions on J^∞. Each such derivation is a smooth *vector field on* J^∞. For $i = 1, \dots, n$, the vector field

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^m \sum_{|\sigma| \geq 0} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma} \tag{7}$$

is called the *total derivative with respect to* x_i *on* J^∞.

A *differential* 1-*form on* J^∞ is a 1-form depending on finitely many variables (5), i.e., a finite sum

$$\sum_{i=1}^n g_i dx_i + \sum_{j=1}^m \sum_{0 \leq |\sigma| \leq q} g^j_\sigma du^j_\sigma, \qquad g_i, g^j_\sigma \in \mathcal{F}(J),$$

where q is a nonnegative integer. Smooth functions, vector fields, and differential forms on J^∞ are related by the usual algebraic operations. In particular, the Lie derivative of a function g along a vector field X will be denoted by Xg.

We define the *infinite prolongation* $\mathcal{E}^\infty$ (or the *diffiety*) *of a system* $\mathcal{E} \subset J^k$ as the subset of J^∞ formed by the points $\theta = \{\theta_l\} \in J^\infty$ such that for each positive integer l the point θ_{k+l} belongs to $\mathcal{E}^{(l)}$.

The diffiety of system (3) is given by the infinite system of equations

$$D_\sigma G_\alpha = 0, \qquad |\sigma| \geq 0, \qquad \alpha = 1, \dots, r,$$

where $D_\sigma = D_{i_1} \circ \dots \circ D_{i_s}$ for $\sigma = i_1 \dots i_s$.

The structure of an infinite-dimensional smooth manifold on J^∞ is inherited by $\mathcal{E}^\infty$. For example, a *smooth function on* $\mathcal{E}^\infty$ is the restriction to $\mathcal{E}^\infty$ of a smooth function in $\mathcal{F}(J)$. The algebra of smooth functions on $\mathcal{E}^\infty$ is denoted by $\mathcal{F}(\mathcal{E})$. For the case of a system $\mathcal{E}$ of the form (3), for the coordinates on $\mathcal{E}^\infty$ one can take part of the coordinates (5). Let $\mathcal{F}_l(\mathcal{E})$ be the algebra of restrictions to $\mathcal{E}^\infty$ of functions in $\mathcal{F}(J)$ depending only on those coordinates (5) for which $|\sigma| \leq l$. The derivations of the algebra $\mathcal{F}(\mathcal{E})$ are called *vector fields on* $\mathcal{E}^\infty$. It follows from the definition of $\mathcal{E}^\infty$ that the fields (7) are tangent to $\mathcal{E}^\infty$. The restriction of the field D_i to $\mathcal{E}^\infty$ will again be denoted by D_i.

Any infinite solution jet of system (3) lies in $\mathcal{E}^\infty$. The set of all infinite jets of a given solution $s(x)$ is an n-dimensional manifold, which is called the *graph of the solution* $s(x)$ *in* $\mathcal{E}^\infty$. The fields $D_1, \dots, D_n$ are tangent to the graphs of solutions in $\mathcal{E}^\infty$. Further, every n-dimensional submanifold of $\mathcal{E}^\infty$ tangent to these fields is locally the graph of some solution in $\mathcal{E}^\infty$. (The precise statement and proof of this assertion can be found in [5, Ch. 4, Assertion 2.3].) Hence the infinite-dimensional manifold $\mathcal{E}^\infty$ and the fields $D_1, \dots, D_n$ defined on it uniquely determine system (3) and its solutions. The distribution generated by the fields $D_1, \dots, D_n$ on $\mathcal{E}^\infty$ (or on J^∞) is called the *Cartan distribution*. The plane of the Cartan distribution at a point $\theta \in \mathcal{E}^\infty$ will be denoted by $\mathcal{C}_\theta(\mathcal{E})$.

3. TRANSFORMATIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

A *smooth mapping* of a diffiety $\mathcal{E}^\infty$ into a diffiety $\mathcal{S}^\infty$ is a mapping

$$F : \mathcal{E}^\infty \to \mathcal{S}^\infty \tag{8}$$

such that the induced mapping F^* takes smooth functions to smooth functions; i.e., $F^*(\mathcal{F}(\mathcal{S})) \subset \mathcal{F}(\mathcal{E})$, where $F^*(g) = g \circ F$. The mapping (8) is called a *diffeomorphism* if it is smooth and one-to-one and if the inverse mapping is smooth as well.

An arbitrary smooth mapping of diffieties does not preserve differential constraints between variables. The Cartan distribution is the geometric structure defining these constraints. Hence smooth mappings preserving the Cartan distribution are of interest.

A diffeomorphism (8) is called a $\mathcal{C}$-*diffeomorphism* (or a *Lie–Bäcklund isomorphism*) if it preserves the Cartan distribution; i.e.,

$$F_*(\mathcal{C}_\theta(\mathcal{E})) = \mathcal{C}_{F(\theta)}(\mathcal{S}), \qquad \theta \in \mathcal{E}^\infty. \tag{9}$$

Further, systems are said to be $\mathcal{C}$-*diffeomorphic* if their diffieties are related by a $\mathcal{C}$-diffeomorphism. The definition of $\mathcal{C}$-diffeomorphism in a neighborhood of a point $\theta \in \mathcal{E}^\infty$ is obtained if one replaces the manifolds $\mathcal{E}^\infty$ and $\mathcal{S}^\infty$ in the above definitions by neighborhoods of the points $\theta \in \mathcal{E}^\infty$ and $F(\theta) \in \mathcal{S}^\infty$, respectively. Since the n-dimensional integral manifolds of the Cartan distribution coincide with the graphs of solutions of the corresponding system, it follows from condition (9) that each $\mathcal{C}$-diffeomorphism takes the graphs of solutions of one system to the graphs of solutions of the other system. Thus, $\mathcal{C}$-diffeomorphic systems are equivalent systems.

A smooth mapping (8) is called a *covering* if the following conditions hold at each point $\theta \in \mathcal{E}^\infty$.

1. The tangent mapping $F_{*,\theta}$ is a vector space epimorphism.
2. Relation (9) holds.
3. The dimension of the kernel $F_{*,\theta}$ is constant.

The *dimension of the covering* is defined as the dimension of the kernel of $F_{*,\theta}$. For any point $\tilde{\theta} \in \mathcal{S}^\infty$, the set $F^{-1}(\tilde{\theta})$ is called the *fiber of the covering* F. If the mapping (8) is a covering and $\mathcal{E}^\infty$ and $\mathcal{S}^\infty$ are the diffieties of systems $\mathcal{E}$ and $\mathcal{S}$, respectively, then one says that the system $\mathcal{E}$ *covers* the system $\mathcal{S}$, or that F is a *covering of the system* $\mathcal{S}$ by the system $\mathcal{E}$.

Note the following properties of coverings. A composition of coverings is a covering. Every $\mathcal{C}$-diffeomorphism is a covering of dimension zero. The dimension of a covering coincides with the dimension of any fiber of the covering. The graph of each solution of the system $\mathcal{E}$ is taken by the covering (8) to the graph of a solution of the system $\mathcal{S}$. Conversely, for each solution s of the system $\mathcal{S}$ defined in a neighborhood of a point $a \in \mathbb{R}^n$ and for an arbitrary point θ of the fiber $F^{-1}([s]_a^\infty)$, where $[s]_a^\infty$ is the infinite jet of the solution s at the point a, there exists a unique solution $\tilde{s}$ of the system $\mathcal{E}$ such that $[\tilde{s}]_{\tilde{a}}^\infty = \theta$ and the covering (8) takes the graph of the solution $\tilde{s}$ to the graph of the solution s.

Consider a finite-dimensional covering ν of system (3) by some system $\mathcal{E}$. Let $w = (w^1, \ldots, w^q)$ be the coordinates in the fiber of ν in a neighborhood of the point in question. Then the derivatives $w_i^j = D_i(w^j)$ are functions of finitely many coordinates $x, w, u, \ldots, u_\sigma, \ldots$ Hence the system $\mathcal{E}$ has the form (3), (4) in the variables (x, w, u).

4. MAIN RESULTS

Let us state conditions satisfied by the fibers of the covering of system (3) by system (3), (4). Note that for $i = 1, \ldots, n$ the total derivative with respect to x_i in system (3), (4) has the form

$$D_i = \check{D}_i + \sum_{j=1}^{q} W_i^j(x, w, u, \ldots, u_\sigma, \ldots) \frac{\partial}{\partial w^j},$$

where $\check{D}_i$ is the total derivative with respect to x_i in system (3). It is easily seen that the column $X = (\partial/\partial w^1, \ldots, \partial/\partial w^q)^{\mathrm{T}}$ of vector fields on the diffiety of system (3), (4) satisfies the relations

$$[X, D_i] = A_i X, \qquad i = 1, \ldots, n, \tag{10}$$

where $[X, D_i]$ is the column of commutators $[\partial/\partial w^s, D_i]$, $s = 1, \ldots, q$, and $A_i X$ is the product of the function matrix $A_i = (\partial W_i^j/\partial w^s)_{s,j=1,\ldots,q}$ by the column X.

Conditions (10) are not invariant with respect to $\mathcal{C}$-diffeomorphisms. Indeed, a $\mathcal{C}$-diffeomorphism does not necessarily preserve independent variables. At the same time, the total derivatives with respect to the new independent variables define the same Cartan distribution and hence are linear combinations of the fields $D_1, \ldots, D_n$. Hence relations (10) are taken by such a transformation to the relations

$$[X, D_i] = A_i X + B_i D, \quad i = 1, \ldots, n, \tag{11}$$

where B_i is a $q \times n$ matrix of functions on the diffiety $\mathcal{E}^\infty$ of system (3), (4) and D is the column of the vector fields $D_1, \ldots, D_n$.

The columns of vector fields $X = (X_1, \ldots, X_q)^{\mathrm{T}}$ defined on the diffiety $\mathcal{E}^\infty$ of system $\mathcal{E}$ and satisfying relations (11) are called *pseudosymmetries* of the system $\mathcal{E}$ [15, 17]. The distribution generated by the fields X is said to be *invariant with respect to the system* (see [16]).

The relations (11) and the Jacobi identity imply the relations

$$D_j(A_i) - D_i(A_j) + A_j A_i - A_i A_j = 0 \qquad \text{for any } \ i, j, \tag{12}$$

which, in turn, imply that the matrix differential operators $D_i + A_i$ and $D_j + A_j$ commute for any i and j.

If $q = 1$ and $A_i \equiv 0$ for all i, then Eq. (11) means that X_1 defines a higher symmetry of the system. The following two theorems are generalizations of Theorems 2.5 and 3.8 in [5, Ch. 4], characterize higher symmetries, and were proved in [17] for the special case of $1 + 1$ evolution equations.

Theorem 1. *A column $X = (X_1, \dots, X_q)^T$ of vector fields is a pseudosymmetry of the infinite jet space J^∞ if and only if*

$$X = \mathfrak{E}_{\varphi,A} + MD; \tag{13}$$

here φ is a $q \times m$ matrix of arbitrary functions on J^∞, $A = (A_1, \dots, A_n)$ is a tuple of $q \times q$ matrices satisfying relations (12), *MD is the product of a $q \times n$ matrix M of arbitrary functions in $\mathcal{F}(J)$ by a column $D = (D_1, \dots, D_n)^T$ of vector fields, and the term $\mathfrak{E}_{\varphi,A}$ in the canonical coordinates on J^∞ has the form*

$$\mathfrak{E}_{\varphi,A} = \sum_\sigma (D + A)^\sigma(\varphi) \frac{\partial}{\partial u_\sigma}, \tag{14}$$

where $(D + A)^\sigma$, $\sigma = i_1 \dots i_s$, is the composition $(D_{i_1} + A_{i_1}) \circ \dots \circ (D_{i_s} + A_{i_s})$ of $q \times q$ matrix differential operators acting on the function matrix φ and the summand on the right-hand side in (14) *is the product of the resulting matrix by the column $\dfrac{\partial}{\partial u_\sigma} = \left(\dfrac{\partial}{\partial u^1_\sigma}, \dots, \dfrac{\partial}{\partial u^m_\sigma}\right)^{\mathrm{T}}$.*

Note that if an n-tuple A of matrices satisfies relations (12), then the operators $D_i + A_i$, $i = 1, \dots, n$, pairwise commute, and hence the right-hand side of Eq. (14) is independent of the order of elements $i_1, \dots, i_s$ in the multi-indices σ. However, this is not the case if relations (12) are not satisfied. In the general case, we assume that for each $\sigma = i_1 \dots i_s$ the elements $i_1, \dots, i_s$ are arranged in nondescending order; i.e., $i_1 \le i_2 \le \dots \le i_s$.

Matrices $A_1, \dots, A_n$ satisfying relations (12) will be called *coefficient matrices*, the matrix φ will be called the *generating matrix* of the pseudosymmetry (13), and the corresponding columns $\mathfrak{E}_{\varphi,A}$ will be called *evolution pseudosymmetries*.

Theorem 2. *A column $X = (X_1, \dots, X_q)^T$ of vector fields is a pseudosymmetry of a formally integrable system* (3) *if and only if it is the restriction to $\mathcal{E}^\infty$ of a column of the form* (13), *where the restrictions to $\mathcal{E}^\infty$ of the matrices $A_1, \dots, A_n$ satisfy relations* (12) *and the generating matrix $\varphi = (\varphi_{ij})$ satisfies the system of equations*

$$\sum_{j,\sigma} \frac{\partial G_\alpha}{\partial u^j_\sigma} (D + A)^\sigma(\varphi^j)|_{\mathcal{E}^\infty} = 0, \qquad \alpha = 1, \dots, r, \qquad \varphi^j = (\varphi_{1j}, \dots, \varphi_{qj})^{\mathrm{T}}, \tag{15}$$

at the points of $\mathcal{E}^\infty$.

Along with conditions (11), the distribution generated by the fields $\partial/\partial w^1, \dots, \partial/\partial w^q$ satisfies the integrability condition. Note that the manifold $\mathcal{E}^\infty$ on which the distribution is considered is infinite-dimensional. Hence the regularity and involutivity of this distribution do not necessarily imply its integrability. Following the paper [18], where conditions for the integrability of vector fields defining higher symmetries were obtained, we give the following definition.

Definition. A set of vector fields $X_1, \dots, X_q$ on a diffiety $\mathcal{E}^\infty$ is called an *integrable pseudosymmetry* if the following conditions are satisfied.

A. The fields $X_1, \dots, X_q$ generate an involutive distribution on $\mathcal{E}^\infty$.

B. The column $X = (X_1, \dots, X_q)^{\mathrm{T}}$ of vector fields is a pseudosymmetry of the system $\mathcal{E}$.

C. There exists a ring $\mathcal{K}$ of functions on $\mathcal{E}^\infty$ such that $\mathcal{F}_0(\mathcal{E}) \subset \mathcal{K} \subset \mathcal{F}_l(\mathcal{E})$ for some integer $l \ge 0$, the entries of the matrices $A_1, B_1, \dots, A_n, B_n$ in Eq. (11) belong to $\mathcal{K}$, and $X_i(\mathcal{K}) \subset \mathcal{K}$ for each $i = 1, \dots, q$.

Note that conditions A and B in this definition imply the involutivity of the distribution generated by the fields $X_1, \dots, X_q, D_1, \dots, D_n$.

Let $\mathcal{E}$ be a formally integrable system, and let $\mathcal{K}$ be a function ring on the diffiety $\mathcal{E}^\infty$ such that $\mathcal{F}_0(\mathcal{E}) \subset \mathcal{K} \subset \mathcal{F}_l(\mathcal{E})$ for some $l \ge 0$. For $s > 0$, by $D^s\mathcal{K}$ we denote the ring generated by the functions $D_\sigma(f)$, where $f \in \mathcal{K}$ and $|\sigma| \le s$. A point $\theta \in \mathcal{E}^\infty$ is called a *generic point of the ring* $\mathcal{K}$ [18] if the subspaces $\{df|_{\theta'} \in T^*_{\theta'} : f \in \mathcal{K}\}$ and $\{df|_{\theta'} \in T^*_{\theta'} : f \in D^l\mathcal{K}\}$ have constant dimension in some neighborhood of this point.

A *regular point* of an integrable pseudosymmetry is a generic point of the corresponding ring $\mathcal{K}$ at which the fields $X_1, \dots, X_q, D_1, \dots, D_n$ are linearly independent.

We say that an *integrable pseudosymmetry defines a covering* if the fibers of the covering coincide with the maximal integral manifolds of the distribution generated by the fields of the pseudosymmetry.

Theorem 3. *Let $\mathcal{E}$ be a formally integrable system.*

1. *Each integrable pseudosymmetry of the system $\mathcal{E}$ in a neighborhood of a regular point defines a covering by $\mathcal{E}$.*

2. *Each covering by the system $\mathcal{E}$ is defined by some integrable pseudosymmetry of $\mathcal{E}$.*

If an integrable pseudosymmetry defines a covering by the system $\mathcal{E}$ of a system $\mathcal{S}$, then the system $\mathcal{S}$ will be called a *quotient* of $\mathcal{E}$ by this integrable pseudosymmetry (covering).

To construct the covering (8) and the quotient $\mathcal{S}$ corresponding to an integrable pseudosymmetry $X = (X_1, \dots, X_q)$ of $\mathcal{E}$ in a neighborhood of a regular point $\theta \in \mathcal{E}^\infty$, we need to find common first integrals of the vector fields $X_1, \dots, X_q$. The ring of common first integrals of these fields must coincide with the image of the ring $\mathcal{F}(\mathcal{S})$ under the induced mapping F^* of the covering (8). The desired independent variables $z_1, \dots, z_n$ of the quotient must be chosen from common first integrals such that the matrix $(D_i(z_j))$ is nonsingular. In the following, we show that if g is a common first integral of the pseudosymmetry fields, then for each $i = 1, \dots, n$ the total derivative $D_{z_i}(g)$ is a common first integral as well. The dependent variables $u^1, \dots, u^m$ of the quotient must be chosen from the common first integrals in such a way that the following conditions be satisfied:

1. $u^1, \dots, u^m \in \mathcal{K}$.

2. $du^1, \dots, du^m$ are linearly independent at the point θ.

3. The set $\{u^1, \dots, u^m\}$ is a maximal set of common first integrals of the distribution fields such that conditions 1 and 2 are satisfied; i.e., if $g \in \mathcal{K}$ is a common first integral of the vector fields $X_1, \dots, X_q$, then g is a function of $u^1, \dots, u^m$.

The variables $u^1, \dots, u^m$ are related by equations of the form (3). We take functions $w^1, \dots, w^q$ on $\mathcal{E}$ such that the matrix $(X_i(w^j))$ is nonsingular at the point θ for the coordinates in the fiber of the covering, compute their derivatives according to system $\mathcal{E}$, and obtain the equations of system (4).

Example. One can readily verify that the covering of the Burgers equation by the heat equation corresponding to the Cole–Hopf substitution (see the introduction) is determined by the one-dimensional ($q = 1$) pseudosymmetry

$$X_1 = \frac{\partial}{\partial w} + \frac{w_x}{w}\frac{\partial}{\partial w_x} + \frac{w_{xx}}{w}\frac{\partial}{\partial w_{xx}} + \cdots$$

with generating matrix $\varphi = 1$ and coefficient matrices $A_x = w_x/w$ and $A_t = w_{xx}/w$.

5. PROOFS OF THE THEOREMS

First, let us prove that the property of a set of vector fields to be an integrable pseudosymmetry is invariant under $\mathcal{C}$-diffeomorphisms. Consider an integrable pseudosymmetry $(X_1, \dots, X_q)$ of a system $\mathcal{E}$. Any diffeomorphism preserves involutivity and linear independence, and a $\mathcal{C}$-diffeomorphism preserves the Cartan distribution. Hence a $\mathcal{C}$-diffeomorphism $F : \mathcal{E}^\infty \to \mathcal{S}^\infty$ takes the regular involutive distribution generated by the fields $X_1, \dots, X_q, D_1, \dots, D_n$, to the regular involutive distribution generated by the fields $F_*(X_1), \dots, F_*(X_q), \tilde{D}_1, \dots, \tilde{D}_n$, where $\tilde{D}_1, \dots, \tilde{D}_n$ are the total derivatives with respect to the independent variables of the system $\mathcal{S}$. This proves the invariance of condition B. The invariance of condition A can be proved in a similar way.

To prove the invariance of condition C, consider a function ring $\mathcal{K}$ on $\mathcal{E}^\infty$ such that $\mathcal{F}_0(\mathcal{E}) \subset \mathcal{K} \subset \mathcal{F}_l(\mathcal{E})$ and $X_i(\mathcal{K}) \subset \mathcal{K}$ for each $i = 1, \dots, q$. It follows from the smoothness of F that $F^*(\mathcal{F}_0(\mathcal{S})) \subset \mathcal{F}_{r_0}(\mathcal{E})$ for some $r_0 \geq 0$. Since F^{-1} is a $\mathcal{C}$-diffeomorphism, and since a $\mathcal{C}$-diffeomorphism preserves the Cartan distribution, it follows that F^{-1} takes every total derivative $\tilde{D}_i$ on $\mathcal{S}^\infty$ to a linear combination of total derivatives $D_1, \dots, D_n$ on $\mathcal{E}^\infty$. On the other hand, every total derivative D_i on $\mathcal{E}^\infty$ maps $\mathcal{F}_{r_0}(\mathcal{E})$ into $\mathcal{F}_{r_0+1}(\mathcal{E})$. Hence for each function $g \in \mathcal{F}_0(\mathcal{S})$ we have $F^*(\tilde{D}_i(g)) = (F^{-1})_*(\tilde{D}_i)(F^*(g)) \in \mathcal{F}_{r_0+1}(\mathcal{S})$. Each function in $\mathcal{F}_1(\mathcal{S})$ is a function of finitely many functions of the form $g \in \mathcal{F}_0(\mathcal{S})$ and $\tilde{D}_i(g)$, $i = 1, \dots, n$. Consequently, $F^*(\mathcal{F}_1(\mathcal{S})) \subset \mathcal{F}_{r_0+1}(\mathcal{E})$. By induction over s, we prove the relation $F^*(\mathcal{F}_s(\mathcal{S})) \subset \mathcal{F}_{r_0+s}(\mathcal{E})$ for each $s \geq 0$. Likewise, for the $\mathcal{C}$-diffeomorphism F^{-1} there exists an $r_1 \geq 0$ such that for each $s \geq 0$ one has the inclusion $(F^{-1})^*(\mathcal{F}_s(\mathcal{E})) \subset \mathcal{F}_{r_1+s}(\mathcal{S})$, and hence $\mathcal{F}_s(\mathcal{E}) \subset F^*(\mathcal{F}_{s+r_1}(\mathcal{S}))$.

Finally, taking into account the definition of the ring $D^s\mathcal{K}$, we obtain the inclusions

$$F^*(\mathcal{F}_0(\mathcal{S})) \subset \mathcal{F}_{r_0}(\mathcal{E}) \subset D^{r_0}\mathcal{K} \subset \mathcal{F}_{r_0+l}(\mathcal{E}) \subset F^*(\mathcal{F}_{r_1+r_0+l}(\mathcal{S}))$$

and hence $\mathcal{F}_0(\mathcal{S}) \subset (F^{-1})^*(D^{r_0}\mathcal{K}) \subset \mathcal{F}_{r_1+r_0+l}(\mathcal{S})$.

On the other hand, it follows from Eqs. (11) that

$$X(D_i(g)) = D_i(X(g)) + A_i X(g) + B_i D(g) \in D^1\mathcal{K}$$

for each function $g \in \mathcal{K}$, and hence $X_i(D^1\mathcal{K}) \subset D^1\mathcal{K}$ for each $i = 1, \dots, q$. Likewise, $X_i(D^s\mathcal{K}) \subset D^s\mathcal{K}$ and

$$F_*(X_i)((F^{-1})^*(D^s\mathcal{K})) \subset (F^{-1})^*(D^s\mathcal{K}), \qquad s \geq 0.$$

Thus, condition C holds for the fields $F_*(X_1), \dots, F_*(X_q)$ and the ring $(F^{-1})^*(D^{r_0}\mathcal{K})$.

To prove assertion 2 of Theorem 3, recall that if a system $\mathcal{E}$ covers system (3), then, for some choice of independent and dependent variables, the system $\mathcal{E}$ has the form (3), (4). A change of variables in a system is a $\mathcal{C}$-diffeomorphism, and the definitions of covering and integrable pseudosymmetry are invariant under $\mathcal{C}$-diffeomorphisms. Hence it suffices to prove that the column of the fields $\partial/\partial w^1, \dots, \partial/\partial w^q$ is an integrable pseudosymmetry of system (3), (4) defining a covering of system (3) by system (3), (4). For the ring $\mathcal{K}$ in condition C for these fields one can take the ring $\mathcal{F}_0(\mathcal{E})$, i.e., the ring of smooth functions of (t, u, w). Part of condition B has been proved earlier, and the remaining part, condition A, and the fact that the fields $\partial/\partial w^1, \dots, \partial/\partial w^q$ define a covering by system (3), (4) of system (3) are obvious.

The proof of assertion 1 of Theorem 3 consists of the following eight steps.

Step 1. We use the following theorem proved in [18] (see Theorem 1).

Theorem 4. *Let $\mathcal{E}$ be a formally integrable system, and let $\mathcal{K}$ be a function ring on the diffiety $\mathcal{E}^\infty$ of that system such that $\mathcal{F}_0(\mathcal{E}) \subset \mathcal{K} \subset \mathcal{F}_l(\mathcal{E})$ for some $l \in \mathbb{N}$. Then for each generic point of the ring $\mathcal{K}$, there exists a $\mathcal{C}$-diffeomorphism F of some neighborhood of that point into the diffiety $\mathcal{S}^\infty$ of some system $\mathcal{S}$ such that $\mathcal{K} \subset F^*(\mathcal{F}_0(\mathcal{S}))$ and each element of $F^*(\mathcal{F}_0(\mathcal{S}))$ is a function of finitely many elements of $\mathcal{K}$.*

Let $X = (X_1, \dots, X_q)$ be an integrable pseudosymmetry of the system $\mathcal{E}$, let $\mathcal{K}$ be the corresponding ring, and let $\tilde{\theta} \in \mathcal{E}^\infty$ be a regular point of the system. Let us construct a $\mathcal{C}$-diffeomorphism F of some neighborhood $\mathcal{U}^\infty \subset \mathcal{E}^\infty$ of the point $\tilde{\theta}$ into the diffiety $\mathcal{S}^\infty$ of some system $\mathcal{S} \subset J^p$ such that F is related to the ring $\mathcal{K}$ as indicated in Theorem 4. Then, in view of the invariance of integrable pseudosymmetries, the fields $Z_1 = F_*(X_1), \dots, Z_q = F_*(X_q)$ form an integrable pseudosymmetry of the system $\mathcal{S}$. For its ring one can take the ring $\mathcal{F}_0(\mathcal{S})$. Indeed, for each function g in $\mathcal{F}_0(\mathcal{S})$, we have $F^*(g) \in \mathcal{K}$, and hence $X_i(F^*(g)) \in \mathcal{K}$ and

$$Z_i(g) = (F^{-1})^*(X_i(F^*(g))) \in (F^{-1})^*(\mathcal{K}) \subset \mathcal{F}_0(\mathcal{S}), \qquad i = 1, \dots, q;$$

i.e., $Z_i(\mathcal{F}_0(\mathcal{S})) \subset \mathcal{F}_0(\mathcal{S})$ for each $i = 1, \dots, q$. Hence the mapping $\pi_{\infty,0}|_{\mathcal{S}^\infty}$ projects the fields $Z_1, \dots, Z_q$ into some fields $Z_1^0, \dots, Z_q^0$ on $\mathcal{S}^0 = \pi_{\infty,0}(\mathcal{S}^\infty)$.

Step 2. Each function on $\mathcal{S}^1 = \pi_{\infty,1}(\mathcal{S}^\infty)$ is a function of finitely many functions of the form $g \in \mathcal{F}_0(\mathcal{S})$ and $D_i(g)$, $i = 1, \dots, n$, and the fields $Z_1, \dots, Z_q$ satisfy relations similar to (11). Hence for any $j = 1, \dots, q$ and $i = 1, \dots, n$, we have

$$Z_j(D_i(g)) = D_i(Z_j(g)) + \sum_{l=1}^{q} a_{ji}^l Z_l(g) + \sum_{s=1}^{n} b_{ji}^s D_s(g) \in \mathcal{F}_1(\mathcal{S}),$$

where a^l_{ji} and b^s_{ji} are the entries of the matrices A_i and B_i, respectively. Thus, $Z_i(\mathcal{F}_1(\mathcal{S})) \subset \mathcal{F}_1(\mathcal{S})$, and hence the mapping $\pi_{\infty,1}|_{\mathcal{S}^\infty}$ projects the fields $Z_1,\ldots,Z_q$ into some fields $Z^1_1,\ldots,Z^1_q$ on $\mathcal{S}^1$. Since $\pi_{\infty,0}=\pi_{1,0}\circ\pi_{\infty,1}$, it follows that $\pi_{1,0}|_{\mathcal{S}^1}$ projects the fields $Z^1_1,\ldots,Z^1_q$ into the fields $Z^0_1,\ldots,Z^0_q$.

This argument can be generalized to the case of any $s>1$. In particular, $\pi_{\infty,s}|_{\mathcal{S}^\infty}$ projects the fields $Z_1,\ldots,Z_q$ into some fields $Z^s_1,\ldots,Z^s_q$ on $\mathcal{S}^s=\pi_{\infty,s}(\mathcal{S}^\infty)$, and for $k>s$ the mapping $\pi_{k,s}|_{\mathcal{S}^k}$ projects the fields $Z^k_1,\ldots,Z^k_q$ into the fields $Z^s_1,\ldots,Z^s_q$.

Step 3. It follows from the definition of regular point of an integrable pseudosymmetry that the vectors $Z_{1,\theta},\ldots,Z_{q,\theta},D_{1,\theta},\ldots,D_{n,\theta}$ are linearly independent at the point $\theta=F(\tilde{\theta})$. Hence for some $k\ge 0$ the vectors

$$Z^k_{1,\theta_k},\ \ldots,\ Z^k_{q,\theta_k},\ \pi_{\infty,k,*}(D_{1,\theta}),\ \ldots,\ \pi_{\infty,k,*}(D_{n,\theta}) \tag{16}$$

are linearly independent at the point $\theta_k=\pi_{\infty,k}(\theta)$. It follows from the definition of linear independence that the vectors $Z^k_{1,\theta_k},\ldots,Z^k_{q,\theta_k}$ are linearly independent. Since the fields $Z^k_1,\ldots,Z^k_q$ are smooth, it follows that they are linearly independent at every point in some neighborhood of the point $\theta_k\in\mathcal{S}^k$.

Step 4. Since the fields $Z_1,\ldots,Z_q$ are projected by the mapping $\pi_{\infty,k}|_{\mathcal{S}^\infty}$ into the fields $Z^k_1,\ldots,Z^k_q$, it follows that their commutators are projected into the commutators of the fields $Z^k_1,\ldots,Z^k_q$. The latter are fields on $\mathcal{S}^k$, and hence it follows from condition A that the vector fields $Z^k_1,\ldots,Z^k_q$ generate a regular involutive distribution in some neighborhood of the point θ_k. By the Frobenius theorem, this distribution is integrable and has a full set of first integrals.

Step 5. It follows from the linear independence of the vectors (16) that, in a neighborhood of the point θ_k, the fields $Z^k_1,\ldots,Z^k_q$ have a set of common first integrals $z_1,\ldots,z_n$ such that the matrix

$$(D_i(z_j)(\theta))=(\pi_{\infty,k,*}(D_{i,\theta})(z_j))$$

is nonsingular. [Here and in the following, we identify the functions $g\in\mathcal{F}_k(\mathcal{S})$ and $(\pi_{\infty,k})^*(g)\in\mathcal{F}(\mathcal{S})$.] It follows from the smoothness of the fields $D_1,\ldots,D_n$ and the functions $z_1,\ldots,z_n$ that the matrix $(D_i(z_j))$ is nonsingular at each point of some neighborhood of the point $\theta\in\mathcal{S}^\infty$. Consequently, the functions $z_1,\ldots,z_n$ can be taken for the new independent variables.

Step 6. Since $z_1,\ldots,z_n$ are common first integrals of the fields $Z^k_1,\ldots,Z^k_q$ and hence of the fields $Z_1,\ldots,Z_q$, it follows from relations (11) for the fields Z_i that

$$B_i=[Z,D_{z_i}](z)=Z(D_{z_i}(z))-D_{z_i}(Z(z))=0,\qquad i=1,\ldots,n,\qquad z=(z_1,\ldots,z_n)^{\mathrm{T}}.$$

Hence if g is a common first integral of the fields $Z_1,\ldots,Z_q$, then

$$Z(D_{z_i}(g))=D_{z_i}(Z(g))+A_iZ(g)=0,$$

and hence $D_{z_i}(g)$ is a common first integral of these fields for each $i=1,\ldots,n$ as well.

Step 7. We supplement the set $z_1,\ldots,z_n$ with some functions $\phi^1,\ldots,\phi^s\in C^\infty(\mathcal{S}^k)$ to a maximal functionally independent set of common first integrals of the fields $Z^k_1,\ldots,Z^k_q$ in a neighborhood of the point $\theta_k\in\mathcal{S}^k$. Since the vectors $Z^k_{1,\theta_k},\ldots,Z^k_{q,\theta_k}$ are linearly independent, it follows that there exist smooth functions $w^1,\ldots,w^q$ on $\mathcal{S}^k$ such that the matrix $(Z^k_{i,\theta_k}(w_j))$ is nonsingular. It follows from the smoothness of the fields $Z^k_1,\ldots,Z^k_q$ and the functions $w^1,\ldots,w^q$ that the matrix $(Z^k_i(w_j))$ is nonsingular at each point in some neighborhood of the point θ_k. It follows that the functions $z_1,\ldots,z_n,\phi^1,\ldots,\phi^s,w^1,\ldots,w^q$ form a coordinate system in a neighborhood of the point θ_k on $\mathcal{S}^k$.

Since the mapping $\pi_{\infty,k}$ projects the fields $Z_1,\ldots,Z_q$ into the fields $Z^k_1,\ldots,Z^k_q$, it follows that the functions $z_1,\ldots,z_n,\phi^1,\ldots,\phi^s$ [more precisely, their images under the mapping $(\pi_{\infty,k})^*$] are common first integrals of the fields $Z_1,\ldots,Z_q$. Likewise, the functions $w^1,\ldots,w^q$ have the property $\det(Z_i(w_j))\ne 0$. The derivatives $D_{z_i}(\phi^j)$ are common first integrals of the fields $Z_1,\ldots,Z_q$ as well. On the other hand, $D_{z_i}(\phi^j)$ are functions on $\mathcal{S}^{k+1}$, and $\pi_{\infty,k+1}$ projects the fields $Z_1,\ldots,Z_q$ into the fields $Z^{k+1}_1,\ldots,Z^{k+1}_q$. Hence the functions ϕ^j, $D_{z_i}(\phi^j)$ are common first integrals of the fields $Z_1,\ldots,Z_q$. From these functions, choose a maximal functionally independent set in a neighborhood of the point $\theta_{k+1} = \pi_{\infty,k+1}(\theta)$. We supplement it with the functions $z_1, \dots, z_n$ and some functions $\phi^{s+1}, \dots, \phi^{s+s_1} \in C^\infty(\mathcal{S}^{k+1})$ to a maximal functionally independent set of common first integrals of the fields $Z_1^k, \dots, Z_q^k$. Then the functions $z_1, \dots, z_n, \phi^1, \dots, \phi^{s+s_1}, w^1, \dots, w^q$ and part of the derivatives $D_{z_i}(\phi^j)$ form a coordinate system in a neighborhood of the point θ_{k+1} on $\mathcal{S}^{k+1}$, the functions $\phi^{s+1}, \dots, \phi^{s+s_1}$ and part of the derivatives $D_{z_i}(\phi^j)$ being the coordinate functions in the fibers of the projection $\pi_{k+1,k}$. Since one can take the coordinates on $\mathcal{S}^{k+1}$ and part of the derivatives of the coordinates in the fibers $\pi_{k+1,k}$ with respect to the independent variables for the coordinates on $\mathcal{S}^\infty$, it follows that the functions of the form $z_i, \phi^j, w^s, D_\sigma(\phi^j)$ can be taken for the coordinates on $\mathcal{S}^\infty$.

Step 8. The derivatives $D_{z_i}(w^j)$ are functions on $\mathcal{S}^{k+1}$ and hence functions of the abovementioned coordinates on $\mathcal{S}^{k+1}$. We obtain equations of the form

$$D_{z_i}(w^j) = W_i^j(z, w, \phi, D_z(\phi)). \tag{17}$$

The remaining equations in the system $\mathcal{S}$ are equations for the common first integrals of the fields $Z_1, \dots, Z_q$. Indeed, every equation in the system $\mathcal{S}$ can be written in these coordinates in the form

$$G(z, \phi, w, \dots, D_\sigma(\phi), \dots) = 0.$$

The derivative of this equation along the field Z_i has the form

$$\sum_{j=1}^{q} G'_{w^j} Z_i(w^j) = 0.$$

Since the matrix $(Z_i(w^j))_{i,j=1,\dots,q}$ is nonsingular, it follows that $G'_{w^j} \equiv 0$, $j = 1, \dots, q$, and hence the function G is independent of w and satisfies

$$G(z, \phi, \dots, D_\sigma(\phi), \dots) = 0. \tag{18}$$

We apply a similar argument to the remaining equations in the system $\mathcal{S}$. Let us denote the functions ϕ^j by u^j and the functions z_i by x_i. Then Eqs. (18) acquire the form (3), and Eqs. (17) become Eqs. (4). The proof of Theorem 3 is complete.

Proof of Theorem 1. In view of relations (12), straightforward computations readily show that the relation

$$[\mathfrak{E}_{\varphi,A}, D_i](g) = A_i \mathfrak{E}_{\varphi,A}(g)$$

holds for the coordinate functions $g = x_s$ and $g = u_\sigma^j$ for any s, j, and σ. Hence $[\mathfrak{E}_{\varphi,A}, D_i] = A_i \mathfrak{E}_{\varphi,A}$ and the column $X = \mathfrak{E}_{\varphi,A} + MD$ satisfies relations of the form (11).

Conversely, let a column X of vector fields satisfy relations (11). In the canonical coordinates on J^∞, the Cartan distribution is defined by the set of 1-forms

$$d_{\mathcal{C}} u_\sigma^j = du_\sigma^j - \sum_{i=1}^{n} u_{\sigma i}^j \, dx_i, \qquad j = 1, \dots, m, \qquad |\sigma| \geq 0.$$

Set $\varphi_\sigma^j = X \rfloor d_{\mathcal{C}} u_\sigma^j$ and $M_s = X \rfloor dx_s$. Since $D_s \rfloor d_{\mathcal{C}} u_\sigma^j = 0$ and $D_s \rfloor dx_i$ is the zero function for $s \neq i$ and the unit function for $s = i$, it follows that the column $X - \sum_{s=1}^n M_s D_s$ has zero components for $\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}$ and hence

$$X = \sum_{j,\sigma} \varphi_\sigma^j \frac{\partial}{\partial u_\sigma^j} + \sum_{s=1}^{n} M_s D_s$$

for some $\varphi_\sigma^j \in \mathcal{F}(J)$.

We use the infinitesimal Stokes formula and the relation $d(d_{\mathcal{C}} u_\sigma^j) = \sum_{i=1}^n dx_i \wedge d_{\mathcal{C}} u_{\sigma i}^j$ to obtain

$$\begin{aligned} [X, D_i] \rfloor d_{\mathcal{C}} u_\sigma^j &= X \rfloor \left(D_i \rfloor \sum_{s=1}^{n} dx_i \wedge d_{\mathcal{C}} u_{\sigma i}^j \right) + X(D_i \rfloor d_{\mathcal{C}} u_\sigma^j) - D_i(X \rfloor d_{\mathcal{C}} u_\sigma^j) \\ &= X \rfloor d_{\mathcal{C}} u_{\sigma i}^j - D_i(X \rfloor d_{\mathcal{C}} u_\sigma^j) = \varphi_{\sigma i}^j - D_i \varphi_\sigma^j. \end{aligned}$$

The substitution of the left-hand side of relation (11) into the form $d_{\mathcal{C}}u^j_\sigma$ gives the relation

$$A_i X \rfloor d_{\mathcal{C}} u^j_\sigma + MD \rfloor d_{\mathcal{C}} u^j_\sigma = A_i \varphi^j_\sigma.$$

Consequently, $\varphi^j_{\sigma i} = (D_i + A_i)\varphi^j_\sigma$, which proves Eq. (13), where $M = (M_1, \dots, M_n)$, $\varphi = (\varphi^1_\varnothing, \dots, \varphi^m_\varnothing)$ and $\varnothing$ is the empty multi-index.

Proof of Theorem 2. Let X be a column of vector fields on J^∞ of the form (13), (14), let the restrictions of the matrices $A_1, \dots, A_n$ to $\mathcal{E}^\infty$ satisfy relations (12), and let the generating matrix $\varphi = (\varphi_{ij})$ satisfy (15). Since the term MD is tangent to the infinite prolongation of any system, it follows that the tangency of the column X to the diffiety $\mathcal{E}^\infty$ is equivalent to the tangency of the column $\mathfrak{E}_{\varphi,A}$ to the same diffiety. The latter means that the functions $\mathfrak{E}_{\varphi,A}(G_\alpha)$, $\alpha = 1, \dots, r$, are zero on $\mathcal{E}^\infty$, which is equivalent to system (15). Hence the restriction of X to $\mathcal{E}^\infty$ is well defined.

By arguing as in the proof of Theorem 1 while considering the restrictions to $\mathcal{E}^\infty$ of $\mathfrak{E}_{\varphi,A}$, D_i, x_s, and u^j_σ rather than these fields and functions themselves, we obtain the identity

$$[\mathfrak{E}_{\varphi,A}|_{\mathcal{E}^\infty}, D_i|_{\mathcal{E}^\infty}] = A_i|_{\mathcal{E}^\infty} \mathfrak{E}_{\varphi,A}|_{\mathcal{E}^\infty},$$

which means that the restriction of X to $\mathcal{E}^\infty$ is a pseudosymmetry of system (3).

Conversely, let a column X of vector fields on $\mathcal{E}^\infty$ satisfy relations (11). By arguing as in the proof of Theorem 1, we find that the column X is determined by the function columns $\varphi^j = X \rfloor (d_{\mathcal{C}} u^j|_{\mathcal{E}^\infty})$ and $M_s = X \rfloor (dx_s|_{\mathcal{E}^\infty})$, where $X \rfloor (d_{\mathcal{C}} u^j_\sigma|_{\mathcal{E}^\infty}) = (D + A)^\sigma(\varphi^j)|_{\mathcal{E}^\infty}$. In contrast to J^∞, the forms $d_{\mathcal{C}} u^j_\sigma$ on $\mathcal{E}^\infty$ are related by the differential constraints

$$d_{\mathcal{C}} G_\alpha|_{\mathcal{E}^\infty} = \sum_{j,\sigma} \frac{\partial G_\alpha}{\partial u^j_\sigma} d_{\mathcal{C}} u^j_\sigma|_{\mathcal{E}^\infty} = 0, \qquad \alpha = 1, \dots, r.$$

Hence

$$X \rfloor d_{\mathcal{C}} G_\alpha|_{\mathcal{E}^\infty} = \sum_{j,\sigma} \frac{\partial G_\alpha}{\partial u^j_\sigma} (D + A)^\sigma(\varphi^j)|_{\mathcal{E}^\infty} = 0, \qquad \alpha = 1, \dots, r.$$

To complete the proof of Theorem 2, it suffices to note that for the coordinates on $\mathcal{E}^\infty$ one can take the restriction to $\mathcal{E}^\infty$ of part of the coordinates (5); hence X is the restriction to $\mathcal{E}^\infty$ of the column (13).

CONCLUSION

We have obtained a complete description of finite-dimensional coverings by a given system of differential equations. Furthermore, the results of the paper provide a theoretical substantiation of the following algorithm for computing the coverings.

1. Find matrices $A_1, \dots, A_n$ satisfying relations (12).
2. Solve system (15) for φ.
3. From the obtained pseudosymmetries of the form (13), single out those satisfying conditions A and C.
4. Construct the covering corresponding to the obtained integrable pseudosymmetry.

The search for the matrices $A_1, \dots, A_n$ is related to the search for zero curvature representations (see [9]). The solution of a system of the form (15) is similar to the computation of higher symmetries (e.g., see [5, Ch. 4, Sec. 4]). See [18] concerning the verification of condition C. Finally, the construction of a covering and the quotient by the obtained integrable pseudosymmetry has been described above (see Section 4).

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