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INTEGRAL EQUATIONS

Conditions for Well-Posedness of Integral Models of Some Living Systems

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Abstract—We study the existence, uniqueness, and nonnegativity of solutions of a family of delay integral equations used in mathematical models of living systems. Conditions ensuring these properties of solutions on an infinite time interval are obtained. The continuous dependence of solutions on the initial data on finite time intervals is analyzed. Special cases in the form of delay differential and integro-differential equations arising in population dynamics models are presented.

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INTRODUCTION

Mathematical models of living systems use rather involved mathematical techniques. These include integral and differential equations of various types, in particular, delay equations, which take into account the development history of living systems. A necessary condition for a specific differential or integral equation to be usable as a mathematical model of a living system is its well-posedness, that is, the existence and uniqueness of solutions and their continuous dependence on the parameters and the initial and boundary conditions. Models describing the population dynamics of elements (animals, individuals, cells, virus particles, bacteria, molecules, etc.) of a living system involve additional conditions of nonnegativity and boundedness of solutions of these equations on the infinite time interval for nonnegative initial data.

The modern approach to the analysis of properties of solutions of delay integral and differential equations takes into account the structure of the equations to be studied and is based on methods of the theory of operator equations and methods for constructing a priori estimates of solutions [1–6]. Information on the structure of model equations and a priori estimates permit one to prove the nonnegativity and boundedness of solutions and the existence of periodic positive solutions and also study the asymptotic behavior of solutions under a number of conditions. Examples of analysis of these properties of solutions for equations occurring in mathematical models of biophysics and ecology can be found in [7–14].

Mathematical models in the form of integral equations arise, as a rule, when taking into account the following factors for elements of living systems: (i) their age structure; (ii) the existence of several stages of their development with regard to time the elements spend at these stages. Population dynamics models widely use partial differential equations containing functions $\mu(a) \geq 0$ that describe the death rate of individuals in a population depending on age $a \geq 0$. The function $\mu(a)$ determines the survival function $L(a) = \exp(-\int_0^a \mu(s)\,ds)$, which specifies the fraction of individuals that have reached age a. The case of $\mu(a) = \mu_* = \text{const} > 0$ specifies the exponential survival function $L(a) = \exp(-\mu_* a)$ and arises when using ordinary differential equations. Integration of differential equations results in integral equations, which permit one to study the properties of solutions of the models with the use of well-known methods.

For example, the paper [15] deals with the solvability and stability of solutions of a system of nonlinear Volterra integral equations with variable lower limit of integration, and the results are applied to the analysis of age-structured population dynamics models with one and two species. Conditions for the exponential growth or decay of solutions of a nonlinear integral equation arising in the integration of a family of quasilinear differential equations by variation of constants are obtained based on the theory of integral operators in the paper [16], and the results are used to analyze a broad family of age-structured models and models with bounded and unbounded delay. The paper [17] presents an integral epidemic process model constructed with regard to a distributed delay in the latent and immune stages of the disease. For some parameter values, this model reduces to a system of delay differential equations of special structure supplemented with initial conditions. The existence, uniqueness, and nonnegativity of the solution on the half-line $[0,\infty)$ are established.

The paper [17] gives a detailed derivation of population dynamics equations pertaining to stage-dependent models (also known as stage-structured models). Yet another example of such a family of equations is given in the paper [18], where the model equations are derived from the balance relations used in an age-structured populations dynamics model. It is assumed that the birth rate of new individuals and the death rate depend on the individual age and the total population size. To construct the model equations, one uses integral equations, which are then reduced to differential equations of neutral type. Stage-structured delay differential equations are given in the papers [19] (an epidemic model), [20] (a version of the predator–prey model), and [21] (a model of tick population dynamics).

Note that the straightforward representation of stage-structured models in the form of delay differential equations requires imposing certain restrictions on the initial data. By way of example, consider a typical differential equation with given initial condition arising in these models:

$$\frac{dy(t)}{dt} = f(y(t)) - \mu y(t) - e^{-\mu\tau} f(y(t-\tau)), \qquad t \ge 0, \tag{1}$$

$$y(t) = y_0(t), \qquad t \in [-\tau, 0]. \tag{2}$$

The variables in problem (1), (2) have the following meaning: $y(t)$ is the population size at time t, $y_0(t)$ is the initial population size, $\mu > 0$ is a parameter, $f(y)$ is a given function, and τ, $0 < \tau < \infty$, is the duration of some stage of individuals in the population. Assume that the function $f(y)$ is defined in some neighborhood U_0 of the point $y = 0$ and satisfies the Lipschitz condition in U_0 and that the inequality $f(y) \ge 0$ holds for all $y \in U_0$, $y \ge 0$. Further, assume that the function $y_0(t)$ is continuous and nonnegative on the interval $[-\tau, 0]$ and that $y_0(t) \in U_0$ for all t in that interval. Obviously, problem (1), (2) is equivalent to the integral equation [22, p. 300]

$$y(t) = y_0(0) + \int_0^t \left(f(y(s)) - \mu y(s) - e^{-\mu\tau} f(y(s-\tau))\right) ds, \qquad t \ge 0, \tag{3}$$

with the initial condition (2).

It is easily seen that there exists an interval $[0, \tau_0]$ on which there exists a unique continuous solution of problem (1), (2) [or, which is the same, of problem (3), (2)]. [For $t = 0$, by $dy(t)/dt$ we mean the right-hand derivative.] We additionally assume that the function $f(y)$ increases on the interval $\{y \ge 0\} \cap U_0$, the function $y_0(t)$ decreases on the interval $[-\tau, 0]$, and $y_0(0) = 0$. Then there exist μ and τ such that

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = f(y_0(0)) - e^{-\mu\tau} f(y_0(-\tau)) < 0.$$

This inequality, the condition $y(0) = 0$, and the continuity of the function $y(t)$ imply that this function takes negative values for $t \in (0, \tau_1)$, where $0 < \tau_1 \le \tau_0$. The negativity of the function $y(t)$ does not agree with its biological meaning. Consequently, the initial function occurring in condition (2) cannot be prescribed arbitrarily.

On the other hand, Eq. (1) with condition (2) can be integrated by variation of constants. Then problem (1), (2) is replaced by the equation

$$y(t) = \int_{t-\tau}^t e^{-\mu(t-a)} f(y(a))\, da, \qquad t \ge 0, \qquad y(t) = y_0(t), \qquad t \in [-\tau, 0]. \tag{4}$$

It follows from Eq. (4) by the continuity of $y(t)$ that the inequality

$$y_0(0) = \int_{-\tau}^{0} e^{\mu a} f(y_0(a))\, da \tag{5}$$

necessarily holds, which shows that the function $y_0(t)$ used in condition (2) cannot be specified arbitrarily. Further, if $y_0(0) = 0$, then either $f(y) \equiv 0$ or the functions $f(y)$ and $y_0(t)$ have a special form.

Thus, there arises a functional relation between the functions $y_0(t)$ and $f(y)$ in the model (1), (2) and its integral analogs (3), (2), and (4); this relation should be taken into account when studying the properties of the solution $y(t)$. For example, if the continuous dependence of the solution on the initial function $y_0(t)$ or the stability of an equilibrium y_* is studied for problem (1), (2), then one should require that the perturbed initial function $y_1(t)$ is not only "close" to $y_0(t)$ or y_*, respectively, but also satisfies relation (5) in which $y_0(t)$ is replaced with $y_1(t)$. Hence, to use equations of the form (1), (2), one has to impose additional restrictions on the function $y_0(t)$. At the same time, it is apparently more natural and expedient to use an approach in which these restrictions follow from the construction of the model at the stage where the model equations are derived.

The present paper studies one version of the integral model, which is a modification and generalization of models of the form (1), (2) and (4) to the multidimensional case in the presence of delayed variables. The aim of the paper is to derive the model equations and find conditions ensuring the well-posedness of these equations and expressed in terms of the mappings describing the reproductive, death, and migration flow rates of elements of living systems.

1. MODEL EQUATIONS

Consider a living system consisting of elements of m types. Let $x_i(t)$ be the number of elements of ith type at time $t \in \mathbb{R} = (-\infty, \infty)$, and let $x(t) = (x_1(t), \dots, x_m(t))^{\mathrm{T}}$. Let $\omega \geq 0$ be a constant, let $I_\omega = [-\omega, 0]$, and let $x_t : I_\omega \to \mathbb{R}^m$ be the section of the function $x(t)$ at time $t \geq 0$ defined by the rule $x_t(\theta) = x(t + \theta)$, $\theta \in I_\omega$.

Take an i, $1 \leq i \leq m$. Set

$$x_i(t) = \psi_i(t), \qquad t \leq 0, \qquad x_i(t) = x_i^{(0)}(t) + x_i^{(n)}(t), \qquad t \geq 0. \tag{6}$$

The function $\psi_i(t)$ in Eq. (6) describes the number of original elements of the ith type, i.e., elements present in the system prior to $t = 0$. For $t > 0$, the number of original elements of the ith type decreases owing to various causes (ageing, restrictions on the time spent in the system, migration processes, etc.) unrelated to the interaction between elements and is described by the function $x_i^{(0)}(t)$. The mapping $g_i(t, x_t)$ specifies the death rate of elements of the ith type for $t \geq 0$ owing to the interaction between elements of distinct types in the system. The term

$$x_i^{(0)}(t) = \exp\left\{-\int_0^t g_i(s, x_s)\, ds\right\}\psi_i(t) \tag{7}$$

occurring in (6) stands for the number of original elements of the ith type remaining in the system by time $t \geq 0$. The term

$$x_i^{(n)}(t) = \int_0^t P_i(a) \exp\left\{-\int_{t-a}^{t} g_i(s, x_s)\, ds\right\} f_i(t-a, x_{t-a})\, da \tag{8}$$

in (6) stands for the number of elements of the ith type reproduced by the system itself or having entered the system on the time interval $[0, t]$ and survived by time t. The mapping $f_i(t-a, x_{t-a})$

in Eq. (8) describes the production rate of elements of the ith type reproduced by the system or immigrating to the system at time $t-a \geq 0$. The expression

$$P_i(a)\exp\left\{-\int\limits_{t-a}^{t} g_i(s,x_s)\,ds\right\} \tag{9}$$

is the fraction of elements of the ith type entering the system at time $t-a \geq 0$ that are not leaving the system on the time interval $[t-a,t]$. The function $P_i(a)$ occurring in the expression (9) represents the distribution of time spent by elements of the ith type in the system if all interactions between system elements are neglected.

Set

$$\psi(t) = (\psi_1(t),\dots,\psi_m(t))^{\mathrm{T}}, \qquad f(s,x_s) = (f_1(s,x_s),\dots,f_m(s,x_s))^{\mathrm{T}},$$

$$g(s,x_s) = (g_1(s,x_s),\dots,g_m(s,x_s))^{\mathrm{T}}, \qquad G(s,x_s) = \operatorname{diag}[g_1(s,x_s),\dots,g_m(s,x_s)],$$

$$\exp\left\{-\int\limits_0^t G(s,x_s)\,ds\right\} = \operatorname{diag}\left[\exp\left\{-\int\limits_0^t g_1(s,x_s)\,ds\right\},\dots,\exp\left\{-\int\limits_0^t g_m(s,x_s)\,ds\right\}\right],$$

$$\exp\left\{-\int\limits_{t-a}^t G(s,x_s)\,ds\right\} = \operatorname{diag}\left[\exp\left\{-\int\limits_{t-a}^t g_1(s,x_s)\,ds\right\},\dots,\exp\left\{-\int\limits_{t-a}^t g_m(s,x_s)\,ds\right\}\right],$$

$$P(a) = \operatorname{diag}[P_1(a),\dots,P_m(a)].$$

We use relations (6)–(9) and find that the dynamics of the vector $x(t)$ is described by the equations

$$x(t) = \exp\left\{-\int\limits_0^t G(s,x_s)\,ds\right\}\psi(t) + \int\limits_0^t P(a)\exp\left\{-\int\limits_{t-a}^t G(s,x_s)\,ds\right\} f(t-a,x_{t-a})\,da, \quad t \geq 0, \tag{10}$$

$$x(t) = \psi(t), \quad t \in I_\omega. \tag{11}$$

Relations (10) and (11) form a system of nonlinear delay integral equations supplemented with initial conditions. To study problem (10), (11), one can use well-known results given, say, in the survey paper [23] or the monograph [24]. At the same time, to apply general theorems, one should take into account the specific features of the functions and mappings occurring in Eqs. (10) and (11).

2. DEFINITIONS AND ASSUMPTIONS

Let $I, J \subset \mathbb{R}$ be intervals, let $\mathbb{R}_+ = [0,\infty)$ be the nonnegative half-axis, and let $\Omega \subseteq \mathbb{R}^m$ be a domain. Given a vector $\xi = (\xi_1,\dots,\xi_m)^{\mathrm{T}} \in \mathbb{R}^m$, set $\Omega_\xi = [\xi_1,\infty)\times\ldots\times[\xi_m,\infty) \subset \mathbb{R}^m$. We adopt the convention that the inequalities $\xi < 0$, $u > 0$, and $v \leq w$ for vectors $\xi, u, v, w \in \mathbb{R}^m$ are understood componentwise.

Let $C(I,J)$ and $C(I,\Omega)$ be the sets of all continuous functions $y: I \to J$ and $x: I \to \Omega$, respectively. If $x^{(1)}, x^{(2)} \in C(I,\Omega)$, then the inequality $x^{(1)}(t) \leq x^{(2)}(t)$, $t \in I$, is understood as an inequality between the corresponding vectors for each $t \in I$.

Set $I = [a,b]$, where $a < b$ are some numbers. We make the linear space $C(I,\mathbb{R}^m)$ a Banach space by equipping it with the norm

$$\|z\| = \max_{\theta\in I}\|z(\theta)\|_{\mathbb{R}^m}, \qquad z \in C(I,\mathbb{R}^m),$$

where $\|v\|_{\mathbb{R}^m}$ is the norm of a vector $v \in \mathbb{R}^m$; in particular, $\|v\|_{\mathbb{R}^m} = \sum_{i=1}^m |v_i|$. Following [25, p. 11], for a given $\gamma > 0$ we also define a norm on $C(I,\mathbb{R}^m)$ by the formula

$$\|z\|_\gamma = \max_{\theta\in I}(e^{-\gamma\theta}\|z(\theta)\|_{\mathbb{R}^m}), \qquad z \in C(I,\mathbb{R}^m).$$

The norms $\|z\|$ and $\|z\|_\gamma$ are equivalent, because the inequality $e^{-\gamma b}\|z\| \le \|z\|_\gamma \le e^{-\gamma a}\|z\|$ holds for all $z \in C(I,\mathbb{R}^m)$.

We say that a functional $h : C(I_\omega,\mathbb{R}^m_+) \to \mathbb{R}^m_+$ is *isotone* if the inequality $h(z^{(1)}) \le h(z^{(2)})$ holds for any $z^{(1)}, z^{(2)} \in C(I_\omega,\mathbb{R}^m_+)$ such that $z^{(1)}(\theta) \le z^{(2)}(\theta)$, $\theta \in I_\omega$.

We make the following assumptions, which, taken together, will be denoted by $(\mathbf{H_0})$, about the components of functions and mappings occurring in system (10), (11):

(a) The function ψ_i is continuous and nonnegative on the interval $[-\omega,\infty)$ and nonincreasing on the interval $[0,\infty)$, $1 \le i \le m$.

(b) The function P_i is nonincreasing and nonnegative on the interval $[0,\infty)$, and

$$P_i(0) = 1, \qquad 0 < \widehat{P}_i = \int\limits_0^\infty P_i(a)\,da < \infty, \qquad 1 \le i \le m.$$

(c) There exists a vector $\xi \in \mathbb{R}^m$, $\xi < 0$, such that the mappings $f_i, g_i : \mathbb{R}_+ \times C(I_\omega,\Omega_\xi) \to \mathbb{R}$ are continuous; moreover,

$$f_i, g_i : \mathbb{R}_+ \times C(I_\omega,\mathbb{R}^m_+) \to \mathbb{R}_+, \qquad 1 \le i \le m.$$

(d) The mappings f_i, g_i are locally Lipschitz; namely, for each $d \in \mathbb{R}^m$, $d > 0$, and any pair $z_1, z_2 \in C(I_\omega,\Omega_\xi)$ such that $z_1(\theta) \le d$ and $z_2(\theta) \le d$ for $\theta \in I_\omega$, the inequalities

$$|f_i(t,z_1) - f_i(t,z_2)| \le L_f^{(i)}\|z_1 - z_2\|, \qquad |g_i(t,z_1) - g_i(t,z_2)| \le L_g^{(i)}\|z_1 - z_2\|,$$

where $L_f^{(i)} = L_f^{(i)}(\xi,d) > 0$ and $L_g^{(i)} = L_g^{(i)}(\xi,d) > 0$ are Lipschitz constants depending on the vectors ξ and d, $1 \le i \le m$, hold for all $0 \le t < \infty$.

A *solution* of system (10), (11) on an interval $[0,\tau]$, $\tau > 0$, is a function x defined and continuous on the interval $[-\omega,\tau]$ and satisfying Eq. (10) for all $t \in [0,\tau]$ and the initial condition (11).

Let us introduce some notation. If $d \in \mathbb{R}^m$, $d > 0$, then $N_d = [0,d_1] \times \ldots \times [0,d_m] \subset \mathbb{R}^m_+$. Fix a $\tau > 0$. By $C_{\psi,\tau} \subset C([-\omega,\tau],\mathbb{R}^m)$ we denote the set of all functions $x \in C([-\omega,\tau],\mathbb{R}^m)$ such that $x(t) = \psi(t)$, $t \in I_\omega$. Let $C_{\psi,\tau,0}$ be the set of (componentwise) nonnegative functions $x \in C_{\psi,\tau}$. Let $\upsilon = \upsilon(t) = (\upsilon_1(t),\ldots,\upsilon_m(t))^{\mathrm{T}}$ be a function with positive components defined on the interval $[-\omega,\tau]$. Further, we define $C_{\psi,\tau,0,\upsilon}$ as the set of functions $x \in C_{\psi,\tau}$ satisfying the inequalities $0 \le x(t) \le \upsilon(t)$, $t \in [-\omega,\tau]$. Note that the sets $C_{\psi,\tau,0,\upsilon}$, $C_{\psi,\tau,0}$, and $C_{\psi,\tau}$ are closed in the space $C([-\omega,\tau],\mathbb{R}^m)$. Consequently, they are complete metric spaces in the metric generated by the norm $\|\cdot\|$ as well as in the metrics generated by the norms $\|\cdot\|_\gamma$. Further, let E be the identity matrix, and let $\widehat{P} = \operatorname{diag}[\widehat{P_1},\ldots,\widehat{P_m}]$.

Consider the operator F that takes each function $x \in C_{\psi,\tau,0}$ to a function $F(x) \in C_{\psi,\tau,0}$ by the formulas

$$F(x)(t) = \psi(t), \qquad t \in I_\omega, \tag{12}$$

$$F(x)(t) = \exp\left\{-\int\limits_0^t G(s,x_s)\,ds\right\}\psi(t) + \int\limits_0^t P(a)\exp\left\{-\int\limits_{t-a}^t G(s,x_s)\,ds\right\}f(t-a,x_{t-a})\,da$$

$$= \exp\left\{-\int\limits_0^t G(s,x_s)\,ds\right\}\psi(t) + \int\limits_0^t P(t-\alpha)\exp\left\{-\int\limits_\alpha^t G(s,x_s)\,ds\right\}f(\alpha,x_\alpha)\,d\alpha, \quad t \in [0,\tau]. \tag{13}$$

Let us introduce additional assumptions taking into account the specific features of the mappings $f(t,x_t)$ and $g(t,x_t)$ in various models of living systems. In these assumptions, inequalities for $f(t,z)$ for given (t,z) are understood componentwise:

$(\mathbf{H_1})$ The mapping f is bounded on the set $\mathbb{R}_+ \times C(I_\omega,\mathbb{R}^m_+)$; namely, there exists a vector $q \in \mathbb{R}^m_+$ such that the inequality $f(t,z) \le q$ holds for all $(t,z) \in \mathbb{R}_+ \times C(I_\omega,\mathbb{R}^m_+)$.

($\mathbf{H_2}$) The mapping f is linearly majorized on the set $\mathbb{R}_+ \times C(I_\omega, \mathbb{R}^m_+)$; namely, the estimate

$$f(t,z) \le p + \int_{-\omega}^{0} d\nu(\theta)\, z(\theta)$$

holds for all $(t,z) \in \mathbb{R}_+ \times C(I_\omega, \mathbb{R}^m_+)$, where $p \in \mathbb{R}^m_+$, ν is the $m \times m$ matrix whose entries $\nu_{ij}(\theta)$ are defined and nondecreasing on the interval I_ω and the matrix $\Delta\nu = \nu(0) - \nu(-\omega)$ has at least one positive entry.

($\mathbf{H_3}$) The mapping f is h-majorized on the set $\mathbb{R}_+ \times C(I_\omega, \mathbb{R}^m_+)$; namely, the estimate $f(t,z) \le h(z)$, where $h : C(I_\omega, \mathbb{R}^m_+) \to \mathbb{R}^m_+$ is a continuous isotone functional, holds for all $(t,z) \in \mathbb{R}_+ \times C(I_\omega, \mathbb{R}^m_+)$.

($\mathbf{H_4}$) The mapping f is r,g-majorized on the set $\mathbb{R}_+ \times C(I_\omega, \mathbb{R}^m_+)$; namely, there exists a vector $r \in \mathbb{R}^m$ with components $r_i > 0$, $1 \le i \le m$, such that the estimate $f(t,z) \le \operatorname{diag}[r_1, \dots, r_m] g(t,z)$ holds for all $(t,z) \in \mathbb{R}_+ \times C(I_\omega, \mathbb{R}^m_+)$.

3. AUXILIARY RESULTS

Let us proceed to the study of the properties of the operator F given by formulas (12) and (13).

Lemma 1. *Let assumption* (H_0) *and* (H_1) *be satisfied, and let*

$$v^{(0)} = \max\Big\{ \sup_{t \in I_\omega} \psi(t);\ \psi(0) + \widehat{P}q \Big\} \in \mathbb{R}^m. \tag{14}$$

Then for each $\tau > 0$ the set $C_{\psi,\tau,0,v^{(0)}}$ is F-invariant, and there exists a constant $\gamma > 0$ such that the operator F is a contraction in the norm $\|\cdot\|_\gamma$ on the set $C_{\psi,\tau,0,v^{(0)}}$.

Proof. Fix a $\tau > 0$. Let $x \in C_{\psi,\tau,0}$. In view of Eq. (14), we obtain the inequalities

$$0 \le F(x)(t) = \psi(t) \le \sup_{s \in I_\omega} \psi(s) \le v^{(0)} \quad \text{for} \quad t \in I_\omega.$$

If $t \in [0,\tau]$, then

$$0 \le F(x)(t) \le \psi(0) + \int_0^t P(a) q\, da \le \psi(0) + \widehat{P}q \le v^{(0)}. \tag{15}$$

Consequently, the inclusion $F(x) \in C_{\psi,\tau,0,v^{(0)}}$ holds for each function $x \in C_{\psi,\tau,0,v^{(0)}} \subset C_{\psi,\tau,0}$.

Let $x^{(1)}, x^{(2)} \in C_{\psi,\tau,0,v^{(0)}}$. Since $0 \le x^{(k)}(t) \le v^{(0)}$, $t \in [-\omega,\tau]$, $k = 1,2$, we can take the vector $d = v^{(0)} > 0$ for the vector d occurring in the definition of the local Lipschitz property of f and g. The Lipschitz constants $L_f^{(i)}$ and $L_g^{(i)}$ of the components of the mappings f and g depend on ξ and $v^{(0)}$, $1 \le i \le m$. In view of the boundedness condition for f, we have the inequalities $0 \le f_i(t,z) \le q_i$, $1 \le i \le m$, for all $(t,z) \in [0,\tau] \times C(I_\omega, N_{v^{(0)}})$.

Fix an i, $1 \le i \le m$, and some constant $\gamma > 0$ and estimate the expression

$$e^{-\gamma t} |F_i(x^{(1)})(t) - F_i(x^{(2)})(t)|, \qquad t \in [-\omega,\tau].$$

By the definition of F,

$$e^{-\gamma t} |F_i(x^{(1)})(t) - F_i(x^{(2)})(t)| = 0$$

for each $t \in I_\omega$. For $t \in [0, \tau]$, we have

$$e^{-\gamma t}(F_i(x^{(1)})(t) - F_i(x^{(2)})(t)) = e^{-\gamma t}B_i(t) + e^{-\gamma t}H_i(t)$$
$$= e^{-\gamma t}\biggl(\exp\biggl\{-\int_0^t g_i(s, x_s^{(1)})\,ds\biggr\} - \exp\biggl\{-\int_0^t g_i(s, x_s^{(2)})\,ds\biggr\}\biggr)\psi_i(t)$$
$$+ e^{-\gamma t}\int_0^t P_i(t-\alpha)\biggl(\exp\biggl\{-\int_\alpha^t g_i(s, x_s^{(1)})\,ds\biggr\}f_i(\alpha, x_\alpha^{(1)})$$
$$- \exp\biggl\{-\int_\alpha^t g_i(s, x_s^{(2)})\,ds\biggr\}f_i(\alpha, x_\alpha^{(2)})\biggr)\,d\alpha.$$

Let us estimate $|B_i(t)|$. Since $|e^{-w_1} - e^{-w_2}| \le |w_1 - w_2|$ for all $w_1, w_2 \in \mathbb{R}_+$, we use the nonnegativity of the functions $g_i(s, x_s^{(1)})$ and $g_i(s, x_s^{(2)})$, set $\nu(t) = \|x^{(1)}(t) - x^{(2)}(t)\|_{\mathbb{R}^m}$, and arrive at the relations

$$|B_i(t)| = \biggl|\exp\biggl\{-\int_0^t g_i(s, x_s^{(1)})\,ds\biggr\} - \exp\biggl\{-\int_0^t g_i(s, x_s^{(2)})\,ds\biggr\}\biggr|\psi_i(t)$$
$$\le \int_0^t |g_i(s, x_s^{(1)}) - g_i(s, x_s^{(2)})|\,ds\,\psi_i(t) \le \int_0^t L_g^{(i)}\|x_s^{(1)} - x_s^{(2)}\|\,ds\,\psi_i(t)$$
$$\le L_g^{(i)}\psi_i(0)\int_0^t e^{\gamma s}e^{-\gamma s}\max_{\theta\in I_\omega}(\nu(s+\theta))\,ds \le L_g^{(i)}\psi_i(0)\int_0^t e^{\gamma s}\max_{\theta\in I_\omega}(e^{-\gamma(s+\theta)}\nu(s+\theta))\,ds$$
$$\le L_g^{(i)}\psi_i(0)\int_0^t e^{\gamma s}\max_{s\in[0,t]}\max_{\theta\in I_\omega}(e^{-\gamma(s+\theta)}\nu(s+\theta))\,ds$$
$$\le L_g^{(i)}\psi_i(0)\int_0^t e^{\gamma s}\max_{s+\theta\in[-\omega,\tau]}(e^{-\gamma(s+\theta)}\nu(s+\theta))\,ds$$
$$= e^{\gamma t}\frac{L_g^{(i)}\psi_i(0)}{\gamma}\|x^{(1)} - x^{(2)}\|_\gamma(1 - e^{-\gamma t}), \qquad t \in [0, \tau].$$

Further, we obtain the relations

$$|H_i(t)| = \biggl|\int_0^t P_i(t-\alpha)\biggl(\exp\biggl\{-\int_\alpha^t g_i(s, x_s^{(1)})\,ds\biggr\}f_i(\alpha, x_\alpha^{(1)}) - \exp\biggl\{-\int_\alpha^t g_i(s, x_s^{(2)})\,ds\biggr\}f_i(\alpha, x_\alpha^{(2)})\biggr)\,d\alpha\biggr|$$
$$\le \int_0^t P_i(t-\alpha)\biggl|\exp\biggl\{-\int_\alpha^t g_i(s, x_s^{(1)})\,ds\biggr\}f_i(\alpha, x_\alpha^{(1)}) - \exp\biggl\{-\int_\alpha^t g_i(s, x_s^{(2)})\,ds\biggr\}f_i(\alpha, x_\alpha^{(2)})\biggr|\,d\alpha$$
$$\le \int_0^t P_i(t-\alpha)\biggl(L_f^{(i)}\|x_\alpha^{(1)} - x_\alpha^{(2)}\| + q_i\int_\alpha^t L_g^{(i)}\|x_s^{(1)} - x_s^{(2)}\|\,ds\biggr)\,d\alpha$$
$$\le L_f^{(i)}\int_0^t e^{\gamma\alpha}e^{-\gamma\alpha}\|x_\alpha^{(1)} - x_\alpha^{(2)}\|\,d\alpha + q_iL_g^{(i)}\int_0^t P(t-\alpha)\biggl(\int_\alpha^t e^{\gamma s}e^{-\gamma s}\|x_s^{(1)} - x_s^{(2)}\|\,ds\biggr)\,d\alpha$$

for $t \in [0, \tau]$.

It follows that

$$e^{-\gamma t}|H_i(t)| \le \frac{L_f^{(i)}}{\gamma}\|x^{(1)}-x^{(2)}\|_\gamma(1-e^{-\gamma t}) + \ e^{-\gamma t} q_i L_g^{(i)}\|x^{(1)}-x^{(2)}\|_\gamma \int\limits_0^t P_i(t-\alpha)\left(\int\limits_\alpha^t e^{\gamma s}\,ds\right)d\alpha$$

$$\le \frac{L_f^{(i)}}{\gamma}\|x^{(1)}-x^{(2)}\|_\gamma(1-e^{-\gamma t}) + \frac{q_i L_g^{(i)}\widehat{P}_i}{\gamma}\|x^{(1)}-x^{(2)}\|_\gamma, \qquad t\in[0,\tau].$$

We combine the resulting estimates and obtain the inequality

$$e^{-\gamma t}|F_i(x^{(1)})(t)-F_i(x^{(2)})(t)| \le \frac{1}{\gamma}(L_g^{(i)}\psi_i(0)+L_f^{(i)}+q_iL_g^{(i)}\widehat{P}_i)\|x^{(1)}-x^{(2)}\|_\gamma$$

for all $t\in[0,\tau]$.

Let $A>1$ be some constant. We define a constant $\gamma>0$ by the formula

$$\gamma = A\sum_{i=1}^m (L_g^{(i)}\psi_i(0)+L_f^{(i)}+q_iL_g^{(i)}\widehat{P}_i). \tag{16}$$

We set $0<\lambda=1/A<1$ and arrive at the estimate

$$\|F(x^{(1)})-F(x^{(2)})\|_\gamma = \max_{t\in[-\omega,\tau]}(e^{-\gamma t}\|F(x^{(1)})(t)-F(x^{(2)})(t)\|_{\mathbb{R}^m})$$

$$= \max_{t\in[-\omega,\tau]}\left(e^{-\gamma t}\sum_{i=1}^m |F_i(x^{(1)})(t)-F_i(x^{(2)})(t)|\right) \le \lambda\|x^{(1)}-x^{(2)}\|_\gamma$$

in view of Eq. (16). Consequently, F is a contraction operator in the norm $\|\cdot\|_\gamma$ on the set $C_{\psi,\tau,0,\upsilon^{(0)}}$. Since τ has been chosen arbitrarily, this completes the proof of the lemma.

Lemma 2. *Let assumptions* (H_0) *and* (H_2) *be satisfied, and let*

$$\upsilon(t)=ce^{\eta t}, \quad t\in\mathbb{R}, \quad \eta\in\mathbb{R}, \quad \eta>0, \quad c\in\mathbb{R}^m, \quad c>0. \tag{17}$$

Then there exists a number η and a vector c in (17) *such that, for each $\tau>0$, the set $C_{\psi,\tau,0,\upsilon}$ is F-invariant and there exists a constant $\gamma>0$ such that F is a contraction operator in the norm $\|\cdot\|_\gamma$ on the set $C_{\psi,\tau,0,\upsilon}$.*

Proof. Consider the function $\upsilon(t)$ given by (17). Let us show that there exist $\eta>0$ and $c>0$ such that the inequalities

$$0\le\psi(t)+\int\limits_0^t P(t-\alpha)\left(p+\int\limits_{-\omega}^0 d\nu(\theta)\upsilon(\alpha+\theta)\right)d\alpha \le \upsilon(t), \qquad 0\le t<\infty, \tag{18}$$

are satisfied. The left inequality in (18) is obvious. Let us prove the right inequality. Note that all entries of the matrix $\Delta\nu$ are nonnegative by assumption (H_2). Let us write the preliminary estimate

$$\psi(t)+\int\limits_0^t P(t-\alpha)\left(p+\int\limits_{-\omega}^0 d\nu(\theta)\upsilon(\alpha+\theta)\right)d\alpha \le \psi(0)+\int\limits_0^t P(t-\alpha)\left(p+\int\limits_{-\omega}^0 d\nu(\theta)ce^{\eta(\alpha+\theta)}\right)d\alpha$$

$$\le \psi(0)+\widehat{P}p+\int\limits_0^t P(t-\alpha)e^{\eta\alpha}\,d\alpha\Delta\nu c \le \psi(0)+\widehat{P}p+\int\limits_0^t Ie^{\eta\alpha}\,d\alpha\Delta\nu c$$

$$\le \psi(0)+\widehat{P}p+\eta^{-1}e^{\eta t}\Delta\nu c, \qquad 0\le t<\infty.$$

For the last expression in this estimate, consider the inequality

$$\psi(0) + \widehat{P}p + \eta^{-1}\Delta\nu c e^{\eta t} \le c e^{\eta t}, \qquad 0 \le t < \infty. \tag{19}$$

We introduce the matrix $Q(\eta) = E - \eta^{-1}\Delta\nu$ and rewrite inequality (19) in the form

$$e^{\eta t} Q(\eta) c \ge \psi(0) + \widehat{P}p, \qquad 0 \le t < \infty. \tag{20}$$

The matrix $Q(\eta)$ has a special structure; namely, its off-diagonal entries are nonpositive. It is easily seen that there exists an $\eta_0 > 0$ such that

$$Q(\eta_0)(1, \dots, 1)^{\mathrm{T}} > 0. \tag{21}$$

According to the theory of matrices of special form [26, Part 6], it follows from inequality (21) that the matrix $Q(\eta_0)$ is a nondegenerate M-matrix. By the properties of M-matrices, we find that the matrix $Q^{-1}(\eta_0)$ exists and all of its entries are nonnegative. Obviously, each row of this matrix contains at least one positive entry. Let $u^{(0)} \in \mathbb{R}^m$, $u^{(0)} > 0$, be a given vector. Then the vector $c = c^{(0)} = Q^{-1}(\eta_0)(\psi(0) + \widehat{P}p + u^{(0)}) > 0$ satisfies inequality (20) with $\eta = \eta_0$ for all $t \ge 0$. Consequently, the function $\upsilon(t) = c^{(0)} \exp(\eta_0 t)$ satisfies inequalities (19) and (18). Consider inequality (18) together with the inequality

$$\psi(t) \le \upsilon(t) = c e^{\eta t}, \qquad t \in I_\omega. \tag{22}$$

We can ensure that inequality (22) is satisfied by an appropriate choice of the vector $u^{(0)}$ in the expression for the vector $c = c^{(0)}$. Obviously, there exists a vector $u^{(0)}$ such that the vector $c^{(0)}$ satisfies the inequality

$$c^{(0)} \ge \max_{t \in I_\omega} (e^{-\eta_0 t} \psi(t)),$$

which guarantees that inequality (22) holds. As a result, we find that the function $\upsilon = \upsilon(t) = c^{(0)} \exp(\eta_0 t)$ satisfies inequalities (18) and (22).

Fix a $\tau > 0$. Assume that $x \in C_{\psi,\tau,0,\upsilon}$. Since the mapping f is linearly majorized, we have

$$0 \le F(x)(t) = \psi(t) \le \upsilon(t), \qquad t \in I_\omega,$$

$$0 \le F(x)(t) \le \psi(t) + \int_0^t P(t-\alpha) f(\alpha, x_\alpha)\, d\alpha \le \psi(t) + \int_0^t P(t-\alpha)\biggl(p + \int_{-\omega}^0 d\nu(\theta) x(\alpha+\theta)\biggr) d\alpha$$

$$\le \psi(t) + \int_0^t P(t-\alpha)\biggl(p + \int_{-\omega}^0 d\nu(\theta) \upsilon(\alpha+\theta)\biggr) d\alpha \le \upsilon(t), \qquad t \in [0, \tau].$$

Consequently, $F(x) \in C_{\psi,\tau,0,\upsilon}$ for each $x \in C_{\psi,\tau,0,\upsilon}$.

Let $x^{(1)}, x^{(2)} \in C_{\psi,\tau,0,\upsilon}$. Since $0 \le x^{(k)}(t) \le \upsilon(t)$, $t \in [-\omega, \tau]$, $k = 1, 2$, it follows that the vector $d = \upsilon(\tau) > 0$ can be taken for the vector d occurring in the definition of the local Lipschitz property of the mappings f and g. The Lipschitz constants $L_f^{(i)}$ and $L_g^{(i)}$ of the components of f and g depend on ξ and $\upsilon(\tau)$, $1 \le i \le m$. Since the mapping f is linearly majorized, we have the inequality $0 \le f(t, z) \le p + \Delta\nu d$ for all $(t, z) \in [0, \tau] \times C(I_\omega, N_d)$, which in componentwise form becomes

$$0 \le f_i(t, z) \le S_f^{(i)} = p_i + (\Delta\nu d)_i, \qquad 1 \le i \le m.$$

Each of the constants $S_f^{(i)} > 0$ depends on $\upsilon(\tau)$, $1 \le i \le m$. By reproducing the argument in the proof of Lemma 1 and by taking $0 < \lambda = 1/A < 1$ and

$$\gamma = A \sum_{i=1}^m (L_g^{(i)} \psi_i(0) + L_f^{(i)} + S_f^{(i)} L_g^{(i)} \widehat{P}_i),$$

where $A > 1$ is some constant, we arrive at the inequality

$$\|F(x^{(1)}) - F(x^{(2)})\|_\gamma \le \lambda \|x^{(1)} - x^{(2)}\|_\gamma.$$

Thus, F is a contraction operator in the norm $\|\cdot\|_\gamma$ on the set $C_{\psi,\tau,0,v}$. Since τ has been chosen arbitrarily, this completes the proof of the lemma.

Lemma 3. *Let assumptions* (H_0) *and* (H_3) *be satisfied, and assume that there exists a vector* $w^{(0)} \in \mathbb{R}^m$, $w^{(0)} > 0$, *such that*

$$\psi(t) \le w^{(0)}, \quad t \in I_\omega, \quad \psi(t) + \int\limits_0^t P(a)\, da h(w^{(0)}) \le w^{(0)}, \quad t \in [0, \infty). \tag{23}$$

Then for each $\tau > 0$ *the set* $C_{\psi,\tau,0,w^{(0)}}$ *is* F*-invariant, and there exists a constant* $\gamma > 0$ *such that* F *is a contraction operator in the norm* $\|\cdot\|_\gamma$ *on the set* $C_{\psi,\tau,0,w^{(0)}}$.

Proof. Fix a $\tau > 0$. Let $x \in C_{\psi,\tau,0,w^{(0)}}$. Since h is isotone, we arrive at the inequalities $0 \le h(x_t) \le h(w^{(0)})$, $t \in [0, \tau]$. We use inequalities (23) and the fact that f is h-majorized and find that $0 \le F(x)(t) = \psi(t) \le w^{(0)}$ for $t \in I_\omega$ and the estimates

$$0 \le F(x)(t) \le \psi(t) + \int\limits_0^t P(t-\alpha) f(\alpha, x_\alpha)\, d\alpha \le \psi(t) + \int\limits_0^t P(t-\alpha) h(x_\alpha)\, d\alpha$$
$$\le \psi(t) + \int\limits_0^t P(t-\alpha) h(w^{(0)})\, d\alpha \le \psi(t) + \int\limits_0^t P(a)\, da\, h(w^{(0)}) \le w^{(0)}$$

hold for $t \in [0, \tau]$. Consequently, $F(x) \in C_{\psi,\tau,0,w^{(0)}}$ for each $x \in C_{\psi,\tau,0,w^{(0)}}$.

Let $x^{(1)}, x^{(2)} \in C_{\psi,\tau,0,w^{(0)}}$. Since $0 \le x^{(k)}(t) \le w^{(0)}$, $t \in [-\omega, \tau]$, $k = 1, 2$, we can take the vector $d = w^{(0)} > 0$ for the vector d occurring in the definition of the local Lipschitz property of the mappings f and g. The Lipschitz constants $L_f^{(i)}$ and $L_g^{(i)}$ of the components of f and g depend on ξ and $w^{(0)}$, $1 \le i \le m$. Since the mapping f is continuous, we obtain $0 \le f_i(t, z) \le M_f^{(i)}$ for all $(t, z) \in [0, \tau] \times C(I_\omega, N_{w^{(0)}})$, where $M_f^{(i)} > 0$ are some constants depending on τ and $w^{(0)}$, $1 \le i \le m$. By reproducing the argument in the proof of Lemma 1 and by taking $0 < \lambda = 1/A < 1$ and

$$\gamma = A \sum_{i=1}^m (L_g^{(i)} \psi_i(0) + L_f^{(i)} + M_f^{(i)} L_g^{(i)} \widehat{P}_i),$$

where $A > 1$ is some constant, we arrive at the inequality

$$\|F(x^{(1)}) - F(x^{(2)})\|_\gamma \le \lambda \|x^{(1)} - x^{(2)}\|_\gamma.$$

As a result, we find that F is a contraction operator in the norm $\|\cdot\|_\gamma$ on the set $C_{\psi,\tau,0,w^{(0)}}$; since τ has been chosen arbitrarily, this completes the proof of the lemma.

Lemma 4. *Let assumptions* (H_0) *and* (H_4) *be satisfied, and let*

$$u^{(0)} = \max\Big\{\sup_{t \in I_\omega} \psi(t); r\Big\} \in \mathbb{R}^m. \tag{24}$$

Then for each $\tau > 0$ *the set* $C_{\psi,\tau,0,u^{(0)}}$ *is* F*-invariant, and there exists a constant* $\gamma > 0$ *such that* F *is a contraction operator in the norm* $\|\cdot\|_\gamma$ *on the set* $C_{\psi,\tau,0,u^{(0)}}$.

Proof. Fix a $\tau > 0$. Let $x \in C_{\psi,\tau,0,u^{(0)}}$. By (24), $0 \le F(x)(t) = \psi(t) \le \sup_{s\in I_\omega} \psi(s) \le u^{(0)}$ for $t \in I_\omega$. Let $t \in [0, \tau]$. Since the mapping f is r, g-majorized, we obtain

$$\begin{aligned}
0 \le F(x)(t) &= \exp\Big\{-\int_0^t G(s, x_s)\,ds\Big\}\Big(\psi(t) + \int_0^t P(t-\alpha)\exp\Big\{\int_0^\alpha G(s, x_s)\,ds\Big\} f(\alpha, x_\alpha)\,d\alpha\Big)\\
&\le \exp\Big\{-\int_0^t G(s, x_s)\,ds\Big\}\Big(\psi(0) + \int_0^t \exp\Big\{\int_0^\alpha G(s, x_s)\,ds\Big\}\operatorname{diag}[r_1,\dots,r_m]\, g(\alpha, x_\alpha)\,d\alpha\Big)\\
&= \exp\Big\{-\int_0^t G(s, x_s)\,ds\Big\}\Big(\psi(0) + \int_0^t d\exp\Big\{\int_0^\alpha G(s, x_s)\,ds\Big\} r\Big)\\
&= \exp\Big\{-\int_0^t G(s, x_s)\,ds\Big\}\Big(\psi(0) + \Big(\exp\Big\{\int_0^t G(s, x_s)\,ds\Big\} - E\Big) r\Big)\\
&= \exp\Big\{-\int_0^t G(s, x_s)\,ds\Big\}\psi(0) + \Big(E - \exp\Big\{-\int_0^t G(s, x_s)\,ds\Big\}\Big) r \le u^{(0)}.
\end{aligned}$$

Consequently, $F(x) \in C_{\psi,\tau,0,u^{(0)}}$ for each $x \in C_{\psi,\tau,0,u^{(0)}} \subset C_{\psi,\tau,0}$. By analogy with the proofs of Lemmas 1–3, one can readily see that F is a contraction operator in the norm $\|\cdot\|_\gamma$ on the set $C_{\psi,\tau,0,u^{(0)}}$. Since τ has been chose arbitrarily, this completes the proof of the lemma.

Remark. Lemmas 1, 2, and 4 hold for an arbitrary function ψ satisfying the conditions in assumption (H_0). This follows from the construction of the vector $v^{(0)}$, the function $v(t)$, and the vector $u^{(0)}$ given by formulas (14), (17), and (24), respectively. For Lemma 3, the restriction on the function ψ is important, because the vector $w^{(0)}$ may satisfy the second inequality in (23) but not the first inequality for an arbitrary function ψ.

4. MAIN RESULTS

Let us proceed to the analysis of system (10), (11) under assumptions (H_0) and (H_1)–(H_4).

Lemma 5. *Let assumption* (H_0) *be satisfied. If system* (10), (11) *has a solution* $x^{(*)} \in C_{\psi,\tau,0}$, *then* $x^{(*)}$ *is the unique solution of this system in* $C_{\psi,\tau}$.

Proof. Let $x^{(*)} \in C_{\psi,\tau,0}$ be a solution of system (10), (11) on the interval $[0, \tau]$. Obviously, $x^{(*)}$ is a fixed point of F. Assume that $x \in C_{\psi,\tau}$ is another solution of system (10), (11) on this interval. By the definition of solution,

$$\begin{aligned}
&x(t) = \psi(t) \ge 0, \qquad t \in I_\omega,\\
&x(t) = \exp\Big\{-\int_0^t G(s, x_s)\,ds\Big\}\psi(t) + \int_0^t P(t-\alpha)\exp\Big\{-\int_\alpha^t G(s, x_s)\,ds\Big\} f(\alpha, x_\alpha)\,d\alpha, \quad t \in [0, \tau].
\end{aligned}$$

Further, the solution x may have negative components. This means that there may exist $0 \le t_1 < \tau_1 \le \tau$ such that $x(t_1) \ge 0$ and the inequalities $\xi_j \le x_j(t) < 0$, $t \in (t_1, \tau_1]$, hold for some $j = 1, \dots, m$, where the vector $\xi < 0$ is indicated under assumption (H_0). It follows from the continuity of the solutions $x^{(*)}$ and x that there exists a vector $d = (d_1, \dots, d_m)^{\mathrm T} \in \mathbb{R}^m$, $d > 0$, such that

$$x^{(*)}(t), x(t) \in N_{\xi,d} = [\xi_1, d_1] \times \dots \times [\xi_m, d_m] \subset \mathbb{R}^m, \qquad x^{(*)}(t) \ge 0, \qquad t \in [-\omega, \tau]. \tag{25}$$

Fix a $1 \le i \le m$. Let us estimate the expression $|x_i^{(*)}(t) - x_i(t)|$, $t \in [-\omega, \tau]$. We introduce constants $L_f^{(i)}$, $L_g^{(i)}$, $M_f^{(i)}$, and $M_g^{(i)}$, which characterize the numerical values of the mappings f and g in view of (25). Here $L_f^{(i)}$ and $L_g^{(i)}$ are the Lipschitz constants of f and g. The constant $M_f^{(i)} = \max_{0\le t\le\tau} |f_i(t, x_t)|$ estimates $|f_i|$, while the constant

$$M_g^{(i)} = \max_{0\le t\le\tau} \exp\left\{ \max\left(0; \int_0^t (g_i(s, x_s^{(*)}) - g_i(s, x_s))\, ds\right)\right\}$$

characterizes the contribution of the solutions $x^{(*)}$ and x into the estimate of the absolute values of the expressions

$$\exp\left\{-\int_0^t g_i(s, x_s^{(*)})\, ds\right\} - \exp\left\{-\int_0^t g_i(s, x_s)\, ds\right\}, \qquad t \in [0, \tau],$$

and

$$\exp\left\{-\int_\alpha^t g_i(s, x_s^{(*)})\, ds\right\} - \exp\left\{-\int_\alpha^t g_i(s, x_s)\, ds\right\}, \qquad 0 \le \alpha \le t,$$

with regard to the fact that some components of the solution x may be negative. The functions $x_i^{(*)}(t)$ and $x_i(t)$ satisfy the relations

$$x_i^{(*)}(t) - x_i(t) = 0, \qquad t \in I_\omega, \qquad x_i^{(*)}(t) - x_i(t) = Z_i^{(1)}(t) + Z_i^{(2)}(t), \qquad t \in [0, \tau], \tag{26}$$

where

$$\begin{aligned} Z_i^{(1)}(t) &= \left(\exp\left\{-\int_0^t g_i(s, x_s^{(*)})\, ds\right\} - \exp\left\{-\int_0^t g_i(s, x_s)\, ds\right\}\right)\psi_i(t) \\ &= \exp\left\{-\int_0^t g_i(s, x_s^{(*)})\, ds\right\}\left(1 - \exp\left\{\int_0^t (g_i(s, x_s^{(*)}) - g_i(s, x_s))\, ds\right\}\right)\psi_i(t), \\ Z_i^{(2)}(t) &= \int_0^t P_i(t-\alpha)\left(\exp\left\{-\int_\alpha^t g_i(s, x_s^{(*)})\, ds\right\} f_i(\alpha, x_\alpha^{(*)}) - \exp\left\{-\int_\alpha^t g_i(s, x_s)\, ds\right\} f_i(\alpha, x_\alpha)\right) d\alpha \\ &= \int_0^t P_i(t-\alpha) \exp\left\{-\int_\alpha^t g_i(s, x_s^{(*)})\, ds\right\}(f_i(\alpha, x_\alpha^{(*)}) - f_i(\alpha, x_\alpha))\, d\alpha \\ &\quad + \int_0^t P_i(t-\alpha)\left(\exp\left\{-\int_\alpha^t g_i(s, x_s^{(*)})\, ds\right\} - \exp\left\{-\int_\alpha^t g_i(s, x_s)\, ds\right\}\right) f_i(\alpha, x_\alpha)\, d\alpha. \end{aligned}$$

Let us estimate $|Z_i^{(1)}(t)|$ using the finite increment formula for the function $\exp(-w)$, $w \in [a, b]$, where $[a, b]$ is some closed interval on the real line. Fix a $t \in [0, \tau]$. Then

$$\begin{aligned} Z_i^{(1)}(t) &= \exp\left\{-\int_0^t g_i(s, x_s^{(*)})\, ds\right\}\psi_i(t)\left(e^0 - \exp\left\{\int_0^t (g_i(s, x_s^{(*)}) - g_i(s, x_s))\, ds\right\}\right) \\ &= \exp\left\{-\int_0^t g_i(s, x_s^{(*)})\, ds\right\}\psi_i(t)(-e^{y_i(t)}) \int_0^t (g_i(s, x_s^{(*)}) - g_i(s, x_s))\, ds, \end{aligned}$$

where the variable $y_i(t)$ lies in the range between 0 and $\int_0^t (g_i(s,x_s^{(*)}) - g_i(s,x_s))\,ds$. Consequently,

$$|Z_i^{(1)}(t)| = \exp\left\{-\int_0^t g_i(s,x_s^{(*)})\,ds\right\}\psi_i(t)e^{y_i(t)}\left|\int_0^t (g_i(s,x_s^{(*)}) - g_i(s,x_s))\,ds\right|, \qquad t \in [0,\tau].$$

Hence for each $t \in [0,\tau]$ one has the inequalities

$$\exp\left\{-\int_0^t g_i(s,x_s^{(*)})\,ds\right\} \le 1, \qquad \psi_i(t) \le \psi_i(0),$$

$$e^{y_i(t)} \le \exp\left\{\max\left(0; \int_0^t (g_i(s,x_s^{(*)}) - g_i(s,x_s))\,ds\right)\right\},$$

$$|Z_i^{(1)}(t)| \le M_g^{(i)}\psi_i(0)\int_0^t |g(s,x_s^{(*)}) - g(s,x_s)|\,ds \le M_g^{(i)}\psi_i(0)L_g^{(i)}\int_0^t \|x_s^{(*)} - x_s\|\,ds.$$

For the expression $|Z_i^{(2)}(t)|$, we obtain the similar estimate

$$|Z_i^{(2)}(t)| \le \int_0^t P_i(t-\alpha)\left(L_f^{(i)}\|x_\alpha^{(*)} - x_\alpha\| + M_g^{(i)}M_f^{(i)}L_g^{(i)}\int_\alpha^t \|x_s^{(*)} - x_s\|\,ds\right)d\alpha$$

$$\le (L_f^{(i)} + M_g^{(i)}M_f^{(i)}L_g^{(i)}\widehat{P}_i)\int_0^t \|x_s^{(*)} - x_s\|\,ds, \qquad t \in [0,\tau].$$

By virtue of relations (26) and the estimates for the expressions $|Z_i^{(1)}(t)|$ and $|Z_i^{(2)}(t)|$, we arrive at the relations

$$\|x^{(*)}(t) - x(t)\|_{\mathbb{R}^m} = 0, \qquad t \in I_\omega,$$

$$\|x^{(*)}(t) - x(t)\|_{\mathbb{R}^m} \le M\int_0^t \|x_s^{(*)} - x_s\|\,ds, \qquad t \in [0,\tau],$$

where

$$M = \sum_{i=1}^m (M_g^{(i)}\psi_i(0)L_g^{(i)} + L_f^{(i)} + M_g^{(i)}M_f^{(i)}L_g^{(i)}\widehat{P}_i) > 0.$$

Note that the constant M depends on τ, ξ, and d but is independent of t. Let $t \in [0,\tau]$ and $\theta \in I_\omega$. Then

$$\|x^{(*)}(t+\theta) - x(t+\theta)\|_{\mathbb{R}^m} = 0, \qquad t+\theta \le 0,$$

$$\|x^{(*)}(t+\theta) - x(t+\theta)\|_{\mathbb{R}^m} \le M\int_0^{t+\theta} \|x_s^{(*)} - x_s\|\,ds \le M\int_0^t \|x_s^{(*)} - x_s\|\,ds, \qquad t+\theta \ge 0,$$

$$\|x_t^{(*)} - x_t\| = \max_{\theta \in I_\omega}\|x^{(*)}(t+\theta) - x(t+\theta)\|_{\mathbb{R}^m} \le M\int_0^t \|x_s^{(*)} - x_s\|\,ds.$$

Let $u(t) = \|x_t^{(*)} - x_t\|$, $t \in [0, \tau]$. We apply the Bellman–Gronwall lemma to the function $u(t)$ in the last relations and obtain $u \equiv 0$. Consequently, $x^{(*)}(t) = x(t)$ for all $t \in [-\omega, \tau]$, which completes the proof of the lemma.

Theorem 1. *Let assumptions* (H_0) *and* (H_1) *be satisfied. Then* (i) *system* (10), (11) *has a unique solution* $x \in C_{\psi,\tau}$ *on any interval* $[0, \tau]$, *and* $x \in C_{\psi,\tau,0,v^{(0)}}$, *where the vector* $v^{(0)}$ *is given by* (14); (ii) *the solution of system* (10), (11) *on each integral* $[0, \tau]$ *continuously depends on the variations in the function* ψ *on the interval* $[-\omega, \tau]$.

Proof. Let us prove (i). Fix a $\tau > 0$. We take into account Lemma 1, use the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_\gamma$, and apply the Banach fixed point theorem to the operator F. We find that there exists a unique function $x^{(*)} \in C_{\psi,\tau,0,v^{(0)}}$ such that $F(x^{(*)}) = x^{(*)}$. By Lemma 5, system (10), (11) does not have solutions other than $x^{(*)}$ on the interval $[0, \tau]$. Since the number $\tau > 0$ is arbitrary, it follows that system (10), (11) has a unique solution $x = x^{(*)}$ on each finite interval $[0, \tau]$.

Let us prove (ii). Fix an interval $[0, \tau]$. Let $\psi^{(1)}$ and $\psi^{(2)}$ be two functions each of which satisfies the conditions imposed on the function ψ in assumption (H_0). The operators $F^{(1)}$ and $F^{(2)}$ corresponding to these functions coincide with the operator F for $\psi = \psi^{(1)}$ and $\psi = \psi^{(2)}$. The solutions of system (10), (11) with $F = F^{(1)}$ and $F = F^{(2)}$ will be denoted by $x^{(1)}$ and $x^{(2)}$, respectively. We assume that the function $\psi^{(1)}$ is fixed, while the function $\psi^{(2)}$ may vary. Let us show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $\|\psi^{(2)} - \psi^{(1)}\| < \delta$ implies the inequality $\|x^{(2)} - x^{(1)}\| < \varepsilon$. In view of the remark in Section 3, note that the vector $v^{(0)}$ given by (14) can be chosen to ensure that inequalities (15) simultaneously hold for both functions $\psi^{(1)}$ and $\psi^{(2)}$. Then the inequalities $0 \le x^{(1)}(t)$ and $x^{(2)}(t) \le v^{(0)}$ hold for all $t \in [-\omega, \tau]$.

Let $\gamma > 0$ be some constant. Fix $1 \le i \le m$. For all $t \in [-\omega, 0]$, one has the estimate

$$e^{-\gamma t}|x_i^{(2)}(t) - x_i^{(1)}(t)| = e^{-\gamma t}|\psi_i^{(2)}(t) - \psi_i^{(1)}(t)| < e^{\gamma\omega}\delta.$$

For $t \in [0, \tau]$, we have the following estimates obtained by analogy with the estimates in Lemma 1:

$$\begin{aligned} e^{-\gamma t}|x_i^{(2)}(t) - x_i^{(1)}(t)| &= e^{-\gamma t}|F_i^{(2)}(x^{(2)})(t) - F_i^{(1)}(x^{(1)})(t)| \\ &\le e^{-\gamma t}|\psi_i^{(2)}(t) - \psi_i^{(1)}(t)| + \frac{L_g^{(i)}\psi_i^{(1)}(0)}{\gamma}\|x^{(2)} - x^{(1)}\|_\gamma \\ &\quad + \frac{L_f^{(i)}}{\gamma}\|x^{(2)} - x^{(1)}\|_\gamma + \frac{q_i L_g^{(i)}\widehat{P}_i}{\gamma}\|x^{(2)} - x^{(1)}\|_\gamma. \end{aligned}$$

Let $A > 0$ be some constant. Set

$$\gamma = \sum_{i=1}^{m}(L_g^{(i)}\psi_i^{(1)}(0) + L_f^{(i)} + q_i L_g^{(i)}\widehat{P}_i) + A, \qquad 0 < q = \frac{\gamma - A}{\gamma} < 1.$$

By combining the estimates on the intervals $[-\omega, 0]$ and $[0, \tau]$, we arrive at the inequality

$$\|x^{(2)} - x^{(1)}\|_\gamma < e^{\gamma\omega}m\delta + \delta + q\|x^{(2)} - x^{(1)}\|_\gamma,$$

whence it follows that

$$\|x^{(2)} - x^{(1)}\| < \frac{e^{\gamma(\omega+\tau)}m + e^{\gamma\tau}}{1 - q}\delta.$$

We take

$$\delta = \varepsilon(1 - q)/(e^{\gamma(\omega+\tau)}m + e^{\gamma\tau})$$

and arrive at the desired inequality $\|x^{(2)} - x^{(1)}\| < \varepsilon$. The proof of the theorem is complete.

Using Lemmas 2–5, the remark in Section 3, and the scheme of proof of Theorem 1, one can readily verify that the following theorems hold.

Theorem 2. *Let assumptions* (H_0) *and* (H_2) *be satisfied. Then* (i) *system* (10), (11) *has a unique solution* $x \in C_{\psi,\tau}$ *on each interval* $[0,\tau]$, *and* $x \in C_{\psi,\tau,0,\upsilon}$, *where the function* $\upsilon(t)$ *is defined by Eq.* (17) *with parameters* η *and* c *ensuring that inequalities* (18) *and* (22) *hold*; (ii) *the solution of system* (10), (11) *on each interval* $[0,\tau]$ *continuously depends on the variations in the function* ψ *on the interval* $[-\omega,\tau]$.

Theorem 3. *Let assumptions* (H_0) *and* (H_3) *be satisfied. Assume that there exists a vector* $w^{(0)} \in \mathbb{R}^m$, $w^{(0)} > 0$, *satisfying inequalities* (23). *Then* (i) *system* (10), (11) *has a unique solution* $x \in C_{\psi,\tau}$ *on each interval* $[0,\tau]$, *and* $x \in C_{\psi,\tau,0,w^{(0)}}$; (ii) *the solution of system* (10), (11) *on each interval* $[0,\tau]$ *continuously depends on the variations in the function* ψ *on the interval* $[-\omega,\tau]$ *under the condition that the vector* $w^{(0)}$ *and the function* ψ *satisfy inequalities* (23).

Theorem 4. *Let assumptions* (H_0) *and* (H_4) *be satisfied. Then* (i) *system* (10), (11) *has a unique solution* $x \in C_{\psi,\tau}$ *on each interval* $[0,\tau]$, *and* $x \in C_{\psi,\tau,0,u^{(0)}}$, *where the vector* $u^{(0)}$ *is given by* (24); (ii) *the solution of system* (10), (11) *on each interval* $[0,\tau]$ *continuously depends on the variations in the function* ψ *on the interval* $[-\omega,\tau]$.

5. SOME SPECIAL CASES

Let us give three examples of system (10), (11) reducible to an equivalent Cauchy problem for systems of delay functional-differential equations. Let $x(t)$ be a continuous function satisfying Eq. (10) and the initial condition (11) on any interval $[0,\tau]$, $\tau > 0$, and let $dx(t)/dt$ be the right-hand derivative of $x(t)$ (taken componentwise).

Example 1. Let $\mu = \mathrm{diag}[\mu_1,\dots,\mu_m]$, $\mu_i > 0$, $i = 1,\dots,m$, be some constant, and let

$$\psi_i(t) = e^{-\mu_i t}\psi_i(0), \quad t \ge 0, \quad P_i(a) = e^{-\mu_i a}, \quad a \ge 0, \quad 1 \le i \le m.$$

We differentiate Eq. (10), take into account condition (11), and arrive at the Cauchy problem

$$\frac{dx(t)}{dt} = f(t,x_t) - (\mu + G(t,x_t))x(t), \quad t \ge 0, \quad x(t) = \psi(t), \quad t \in I_\omega. \tag{27}$$

Equations of the form (27) are widely used in mathematical models of living systems. Examples of such models can be found in the papers [7–14], the survey [27], and the monographs [28, 29].

Note that assumptions (H_0) and (H_1) or (H_0) and (H_2) hold for a broad class of epidemic models and models of Lotka–Volterra type given in the form (27). Hence we obtain the global solvability of these models and the nonnegativity of solutions for nonnegative initial data (e.g., see [7, 8]). The paper [14] gives an example of a model in which it is important to use a vector $\xi < 0$ occurring in the conditions of assumption (H_0).

Example 2. Fix $1 \le i \le m$. Assume that

$$P_i(a) = 1, \qquad a \in [0,\sigma_i), \qquad P_i(a) = 0, \qquad a \in [\sigma_i,\infty),$$

where $\sigma_i > 0$ is some constant. Set

$$\psi_i(t) = \int_0^{\sigma_i} \varphi_i(t-a)\,da, \quad t \in I_\omega,$$

$$\psi_i(t) = \int_t^{\sigma_i} \varphi_i(t-a)\,da, \quad t \in [0,\sigma_i), \quad \psi_i(t) = 0, \quad t \in [\sigma_i,\infty),$$

where the function φ_i is continuous and nonnegative on the interval $(-\omega - \sigma_i, 0]$. We successively differentiate each equation in system (10) on the intervals $[0, \sigma_i)$ and $[\sigma_i, \infty)$, take into account condition (11), and obtain the Cauchy problem for the function $x(t)$ in the componentwise form

$$\frac{dx_i(t)}{dt} = f_i(t, x_t) - g_i(t, x_t)x_i(t) - \exp\left\{-\int_0^t g_i(s, x_s)\, ds\right\}\varphi_i(t - \sigma_i), \qquad 0 \le t < \sigma_i, \tag{28}$$

$$\frac{dx_i(t)}{dt} = f_i(t, x_t) - g_i(t, x_t)x_i(t) - \exp\left\{-\int_{t-\sigma_i}^t g_i(s, x_s)\, ds\right\}f_i(t - \sigma_i, x_{t-\sigma_i}), \qquad t \ge \sigma_i, \tag{29}$$

$$x_i(t) = \int_{t-\sigma_i}^t \varphi_i(s)\, ds, \qquad t \in I_\omega, \qquad 1 \le i \le m. \tag{30}$$

Example 3. Assume that, for each $1 \le i \le m$, the function $\psi_i(t)$ is continuously differentiable on the interval $[0, \infty)$ and the function $P_i(a)$ has the form

$$P_i(a) = \int_a^\infty \varrho_i(s)\, ds, \qquad a \in [0, \infty),$$

where the function $\varrho_i(s)$ is nonnegative and continuous on the interval $[0, \infty)$ and $\int_0^\infty \varrho_i(s)\, ds = 1$. Set $\varrho(s) = \operatorname{diag}[\varrho_1(s), \ldots, \varrho_m(s)]$. We differentiate Eq. (10) componentwise, take into account condition (11), and arrive at the Cauchy problem

$$\frac{dx(t)}{dt} = f(t, x_t) - G(t, x_t)x(t) - \beta(t, x_t), \qquad t \ge 0, \tag{31}$$

$$\beta(t, x_t) = \exp\left\{-\int_0^t G(s, x_s)\, ds\right\}\left(-\frac{d\psi(t)}{dt}\right) + \int_0^t \varrho(t - \alpha)\exp\left\{-\int_\alpha^t G(s, x_s)\, ds\right\}f(\alpha, x_\alpha)\, d\alpha, \tag{32}$$

$$x(t) = \psi(t), \qquad t \in I_\omega. \tag{33}$$

Examples of equations of the form (28)–(33) arising in various models can be found in [17–21] and [30–35]. The structure of Eqs. (10), (11), and (28)–(33) for a special choice of the mappings f and g and the function ψ is closest to the model described in [17].

Equations (28)–(30) and (31)–(33) are a generalization of Eqs. (1), (2), and (4) from the viewpoint of well-posedness of initial conditions. It follows from our results that the analysis of Eqs. (29) together with Eqs. (28) and the initial conditions (30) eliminates the problem of possible functional dependence between the initial function $\psi(t)$ and the mapping $f(t, x_t)$ as well as the problem of possible negativity of the solution $x(t)$ for nonnegative initial data. This conclusion remains valid for Eqs. (31)–(33).

6. CONCLUSION

We establish a set of conditions guaranteeing the well-posedness of the family of integral equations constructed here. Assumption (H_0), together with each of assumptions (H_1)–(H_4) enables one to use Eqs. (10) and (11) as an adequate mathematical model for studying living systems. Note that the equations of the integral model also admit other assumptions supplementing (H_1)–(H_4) and taking into account the specific properties of the mappings $f(t, x_t)$, $g(t, x_t)$ in concrete cases.

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