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ORDINARY DIFFERENTIAL EQUATIONS

Analog of the First Fredholm Theorem for Higher-Order Nonlinear Differential Equations

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Abstract—We study the existence of solutions continuously depending on a parameter for higher-order nonlinear ordinary differential equations with linear boundary conditions. In particular, we prove a theorem of Fredholm type providing tests for the unique solvability of this problem.

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In the rectangle $\Omega = [0,a] \times [0,b]$, consider the differential equation

$$\frac{\partial^n u}{\partial y^n} = p\left(x, y, u, \frac{\partial u}{\partial y}, \dots, \frac{\partial^{n-1} u}{\partial y^{n-1}}\right) + q(x,y) \tag{1}$$

with the boundary conditions

$$h_i(u(x,\cdot))(x) = c_i(x) \qquad (i = 1, \dots, n), \tag{2}$$

where $p : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $q : \Omega \to \mathbb{R}$, and $c_i : [0,a] \to \mathbb{R}$ $(i = 1, \dots, n)$ are continuous functions and $h_i : C^{n-1}([0,b]) \to C([0,a])$ $(i = 1, \dots, n)$ are bounded linear operators. Further, the function p is continuously differentiable with respect to the phase variables, and

$$p(x, y, 0, \dots, 0) \equiv 0. \tag{3}$$

A function $u : \Omega \to \mathbb{R}$ is called a *solution of the differential equation* (1) if it is continuous, has continuous partial derivatives $\partial^k u(x,y)/\partial y^k$ $(k = 1, \dots, n)$, and satisfies the equation at each point of Ω. A solution of Eq. (1) satisfying the boundary conditions (2) is called a *solution of problem* (1), (2).

Problem (1), (2) arises when studying the well-posedness of initial and boundary value problems for ordinary differential equations (e.g., see [1–3] and the bibliography therein). It also has applications in the theory of initial–boundary value problems for higher-order hyperbolic equations (see [4–6]). Nevertheless, this problem is so far insufficiently studied.

In the present paper, we find tests for the unique solvability of problem (1), (2) and the stability of its solution under small perturbations of the functions q and c_i $(i = 1, \dots, n)$.

Prior to stating the main results, let us present the notation and definitions adopted in the paper:

$C(I)$ is the Banach space of continuous functions $v : I \to \mathbb{R}$ with the norm

$$\|v\|_{C(I)} = \max\{|v(t)| : t \in I\}.$$

$C^m(I)$ is the Banach space of m times continuously differentiable functions $u : I \to \mathbb{R}$ with the norm

$$\|v\|_{C^m(I)} = \sum_{i=0}^{m} \|v^{(i)}\|_{C(I)}.$$

$C^{0,n}(\Omega)$ is the space of continuous functions $u : \Omega \to \mathbb{R}$ with continuous partial derivatives $\partial^k u(x,y)/\partial y^k$ $(k = 1, \dots, n)$.

Definition 1. We say that a continuous vector function $(p_{11}, \dots, p_{1n}; p_{21}, \dots, p_{2n}) : \Omega \to \mathbb{R}^{2n}$ belongs to the set $U_{h_1,\dots,h_n}(\Omega)$ if, for any $x \in [0,a]$ and arbitrary measurable functions $p_i : [0,b] \to \mathbb{R}$ $(i = 1, \dots, n)$ satisfying the inequalities

$$p_{1i}(x,y) \le p_i(y) \le p_{2i}(x,y) \qquad (i = 1, \dots, n) \tag{4}$$

almost everywhere on $[0,b]$, the boundary value problem

$$v^{(n)} = \sum_{i=1}^{n} p_i(y) v^{(i-1)}, \tag{5}$$

$$h_i(v)(x) = 0 \qquad (i = 1, \dots, n) \tag{6}$$

has only the trivial solution.

Definition 2. An operator $g : C([0,a]; \mathbb{R}^n) \times C(\Omega) \to C^{0,n}(\Omega)$ is called the *Green's operator* of the boundary value problem

$$\frac{\partial^n u}{\partial y^n} = p\left(x, y, u, \frac{\partial u}{\partial y}, \dots, \frac{\partial^{n-1} u}{\partial y^{n-1}}\right), \tag{1_0}$$

$$h_i(u(x,\cdot))(x) = 0 \quad (i = 1, \dots, n) \quad \text{for} \quad 0 \le x \le a \tag{2_0}$$

if the function

$$u(x,y) = g(c_1, \dots, c_n, q)(x,y)$$

is a solution of problem (1), (2) for arbitrary $c_i \in C([0,a])$ $(i = 1, \dots, n)$ and $q \in C(\Omega)$.

The following theorem is a counterpart of the first Fredholm theorem for the nonlinear problem (1), (2).

Theorem 1. *If the inequalities*

$$p_{1i}(x,y) \le \frac{\partial p(x,y,z_1,\dots,z_n)}{\partial z_i} \le p_{2i}(x,y) \qquad (i = 1, \dots, n), \tag{7}$$

where

$$(p_{11}, \dots, p_{1n}; p_{21}, \dots, p_{2n}) \in U_{h_1,\dots,h_n}(\Omega), \tag{8}$$

hold on the set $\Omega \times \mathbb{R}^n$, *then problem* (1), (2) *is uniquely solvable for arbitrary* $c_i \in C([0,a])$ $(i = 1, \dots, n)$ *and* $q \in C(\Omega)$.

Theorem 2. *If conditions* (7) *and* (8) *are satisfied, then there exists a positive constant* r *such that the inequality*

$$\begin{aligned} &\|g(c_{11}, \dots, c_{1n}, q_1)(x,\cdot) - g(c_{21}, \dots, c_{2n}, q_2)(x,\cdot)\|_{C^{n-1}([0,b])} \\ &\qquad \le r\left(\sum_{i=1}^{n} |c_{1i}(x) - c_{2i}(x)| + \int_0^b |q_1(x,t) - q_2(x,t)|\, dt\right), \end{aligned} \tag{9}$$

where g *is the Green's operator of problem* (1_0), (2_0), *is satisfied on the interval* $[0,a]$ *for any* $c_{ji} \in C([0,a])$ $(i = 1, \dots, n)$ *and* $q_j \in C(\Omega)$ $(j = 1, 2)$.

According to Theorem 2, if conditions (7) and (8) are satisfied, then problem (1), (2) is well posed in the sense that small changes in the functions c_i $(i = 1, \dots, n)$ and q result in small changes in the solution of problem (1), (2).

To prove the theorems, we need the following Lemmas 1 and 2.

Lemma 1. *Let condition* (8) *be satisfied. Then there exists a positive constant r such that if $x \in [0, a]$ and $p_i \in C([0, b])$ $(i = 1, \dots, n)$ are functions satisfying inequalities* (4) *on the interval $[0, b]$, then an arbitrary function $w \in C^n([0, b])$ admits the estimate*

$$\|w\|_{C^{n-1}([0,b])} \le r\left(c_0(x) + \int\limits_a^b |q_0(y)|\, dt\right),$$

where

$$q_0(y) = w^{(n)}(y) - \sum_{i=1}^n p_i(y) w^{(i-1)}(y), \qquad c_0(x) = \sum_{i=1}^n |h_i(w)(x)|.$$

Proof. Assume that the lemma is false. Then for each positive integer k there exists a number $x_k \in [0, a]$ and functions $w_k \in C^n([0, b])$, $p_{ik} \in C([0, b])$ $(i = 1, \dots, n)$ such that

$$p_{1i}(x_k, y) \le p_{ik}(y) \le p_{2i}(x_k, y) \quad \text{for} \quad 0 \le y \le b \quad (i = 1, \dots, n), \tag{10}$$

$$\|w_k\|_{C^{n-1}([0,b])} > k\left(c_k + \int\limits_a^b |q_k(y)|\, dy\right),$$

where

$$w_k^{(n)}(y) - \sum_{i=1}^n p_{ik}(y) w_k^{(i-1)}(y) = q_k(y), \qquad c_k = \sum_{i=1}^n |h_i(w_k)(x_k)|.$$

Set

$$v_k(y) = \frac{w_k(y)}{\|w_k(y)\|_{C^{n-1}([0,b])}}, \qquad \overline{p}_{ik}(y) = \int\limits_0^y p_{ik}(t)\, dt \qquad (i = 1, \dots, n),$$

$$v_k^{(n)}(y) - \sum_{i=1}^n p_{ik}(y) v_k^{(i-1)}(y) = \varepsilon_k(y),$$

$$\sum_{i=1}^n |h_i(v_k)(x_k)| = \delta_k. \tag{11}$$

Then

$$\int\limits_0^b |\varepsilon_k(t)|\, dt < \frac{1}{k}, \qquad \delta_k < \frac{1}{k}, \tag{12}$$

$$\|v_k\|_{C^{n-1}([0,b])} = 1, \quad |v_k^{(n-1)}(y_1) - v_k^{(n-1)}(y_2)| \le \varrho|y_1 - y_2| + \frac{1}{k} \quad \text{for} \quad 0 \le y_i \le b \quad (i=1,2), \tag{13}$$

$$|\overline{p}_{ik}(y_1) - \overline{p}_{ik}(y_2)| \le \varrho|y_1 - y_2| \quad \text{for} \quad 0 \le y_i \le b \quad (i = 1, 2), \tag{14}$$

where

$$\varrho = \max\left\{\sum_{i=1}^n (|p_{1i}(x, y)| + |p_{2i}(x, y)|) : (x, y) \in \Omega\right\}.$$

In view of the Arzelà–Ascoli lemma and conditions (13) and (14), we can assume without loss of generality that the sequence $(v_k)_{k=1}^{+\infty}$ converges in the norm of the space $C^{n-1}([0, b])$ and the sequences $(\overline{p}_{ik})_{k=1}^{+\infty}$ $(i=1,\ldots,n)$ converge uniformly on $[0,b]$. Without loss of generality, the sequence $(x_k)_{k=1}^{+\infty}$ is assumed to converge as well.

Let

$$v(y)=\lim_{k\to+\infty}v_k(y),\quad \overline{p}_i(y)=\lim_{k\to+\infty}\overline{p}_{ik}(y)\quad (i=1,\ldots,n),\quad x=\lim_{k\to+\infty}x_k.$$

Then, in view of relations (10) and (13), we have

$$\|v\|_{C^{n-1}([0,b])}=1, \tag{15}$$

$$\int\limits_{y_1}^{y_2}p_{1i}(x,t)\,dt\le\overline{p}_i(y_2)-\overline{p}_i(y_1)\le\int\limits_{y_1}^{y_2}p_{2i}(x,t)\,dt\quad\text{for}\quad 0\le y_1\le y_2\le b. \tag{16}$$

It follows from inequalities (16) that the functions $\overline{p}_i$ $(i=1,\ldots,n)$ are absolutely continuous and admit the representations

$$\overline{p}_i(y)=\int\limits_0^y p_i(t)\,dt\qquad (i=1,\ldots,n),$$

where the p_i $(i=1,\ldots,n)$ are measurable functions satisfying inequalities (4) almost everywhere on $[0,b]$.

By Lemma 1.1 in [3],

$$\lim_{k\to+\infty}\int\limits_0^y p_{ik}(t)v_k^{(i-1)}(t)\,dt=\int\limits_0^y p_i(t)v^{(i-1)}(t)\,dt\qquad (i=1,\ldots,n) \tag{17}$$

uniformly on $[0,b]$.

Now if we pass to the limit as $k\to+\infty$ not only in Eq. (11) but also in the relation

$$v_k^{(n-1)}(y)=v_k^{(n-1)}(0)+\sum_{i=1}^n\int\limits_0^y p_{ik}(t)v_k^{(i-1)}(t)\,dt+\int\limits_0^y\varepsilon_k(t)\,dt,$$

then, in view of conditions (12) and (17), we obtain

$$v^{(n-1)}(y)=v^{(n-1)}(0)+\sum_{i=1}^n\int\limits_0^y p_i(t)v^{(i-1)}(t)\,dt\quad\text{for}\quad 0\le y\le b,\quad \sum_{i=1}^n|h_i(v)(x)|=0.$$

Consequently, the function v is a solution of problem (5), (6) and satisfies (15). But this is impossible, because problem (5), (6) has only the trivial solution by virtue of conditions (4) and (8). The contradiction thus obtained proves the lemma.

Along with Eq. (1), consider the auxiliary equation

$$\frac{\partial^n u}{\partial y^n}=(1-\lambda)\sum_{i=1}^n l_i(x,y)\,\frac{\partial^{i-1}u}{\partial y^{i-1}}+\lambda\left[p\left(x,y,u,\frac{\partial u}{\partial y},\ldots,\frac{\partial^{n-1}u}{\partial y^{n-1}}\right)+q(x,y)\right] \tag{18}$$

depending on the parameter $\lambda\in[0,1]$, where

$$l_i(x,\cdot)\in C([0,b])\quad\text{for}\quad 0\le x\le a\quad (i=1,\ldots,n).$$

A function $u:\Omega\to\mathbb{R}$ will be called a *quasi-solution* of Eq. (18), if, for arbitrary $x\in[0,a]$, it has continuous partial derivatives $\partial^i u(x,y)/\partial y^i$ $(i=1,\ldots,n)$ on the interval $[0,b]$ and satisfies Eq. (18) on this interval.

Obviously, a quasi-solution of Eq. (18) is a solution of this equation if and only if the functions $(x, y) \longmapsto \partial^{i-1}u(x, y)/\partial y^{i-1}$ $(i = 1, \dots, n)$ are uniformly continuous in the first argument on the interval $[0, a]$.

Corollary 2 in [7] implies the following assertion.

Lemma 2. *Assume that the boundary value problem*

$$v^{(n)} = \sum_{i=1}^{n} l_i(x, y)v^{(i-1)}, \tag{19}$$

$$h_i(v)(x) = 0 \qquad (i = 1, \dots, n) \tag{20}$$

has only the trivial solution for each $x \in [0, a]$ and there exists a positive number r_0 such that for each $\lambda \in [0, 1]$ every quasi-solution of problem (18), (2) *admits the estimate*

$$\|u(x, \cdot)\|_{C^{n-1}([0,b])} \le r_0 \quad \text{for} \quad 0 \le x \le a. \tag{21}$$

Then problem (1), (2) *has at least one quasi-solution satisfying the estimate* (21).

Proof of Theorem 1. Throughout the following, r is the positive constant occurring in Lemma 1 and

$$r_0 = r \max\left\{ \sum_{i=1}^{n} |c_i(x)| + \int_0^b |q(x, t)|\, dt : \ 0 \le x \le a \right\}.$$

Let $l_i(x, y) = p_{1i}(x, y)$ for $(x, y) \in \Omega$ $(i = 1, \dots, n)$. Then problem (19), (20) has only the trivial solution for any $x \in [0, a]$. Let us show that each quasi-solution u of problem (18), (2) admits the estimate (21) for each $\lambda \in [0, 1]$.

By conditions (3) and (7), there exist functions $p_{0i} : \Omega \to \mathbb{R}$ $(i = 1, \dots, n)$ such that $p_{0i}(x, \cdot) \in C([0, b])$ for $x \in [0, a]$, $p_{1i}(x, y) \le p_{0i}(x, y) \le p_{2i}(x, y)$ for $(x, y) \in \Omega$ $(i = 1, \dots, n)$, and

$$p\left(x, y, u(x, y), \frac{\partial u(x, y)}{\partial y}, \dots, \frac{\partial^{n-1}u(x, y)}{\partial y^{n-1}}\right) = \sum_{i=1}^{n} p_{0i}(x, y)\frac{\partial^{i-1}u(x, y)}{\partial y^{i-1}} \quad \text{for} \quad (x, y) \in \Omega.$$

Hence

$$\frac{\partial^n u(x, y)}{\partial y^n} = \sum_{i=1}^{n} p_i(x, y)\frac{\partial^{i-1}u(x, y)}{\partial y^{i-1}} + \lambda q(x, y),$$

where $p_i(x, y) = (1 - \lambda)p_{1i}(x, y) + \lambda p_{0i}(x, y)$ $(i = 1, \dots, n)$ and

$$p_{1i}(x, y) \le p_i(x, y) \le p_{2i}(x, y) \quad \text{for} \quad (x, y) \in \Omega \quad (i = 1, \dots, n). \tag{22}$$

Hence we obtain the estimate (21) by Lemma 1.

Now we apply Lemma 2 and see that there obviously exists a quasi-solution u of problem (1), (2) admitting the estimate (21).

Let us prove that u is a solution of problem (1), (2). To this end, we must establish that the functions $(x, y) \mapsto \partial^{i-1}u(x, y)/\partial x^{i-1}$ $(i = 1, \dots, n)$ are uniformly continuous in the first argument.

First, note that the estimate (21) implies the estimate

$$\|u(x, \cdot)\|_{C^{n}([0,b])} \le r_1 \quad \text{for} \quad 0 \le x \le a, \tag{23}$$

where

$$r_1 = r_0 + \max\left\{ |p(x, y, z_1, \dots, z_n) + q(x, y)| : (x, y) \in \Omega, \ \sum_{i=1}^{n} |z_i| \le r_0 \right\}.$$

For arbitrary given $x, x_0 \in [0, a]$, set

$$w(x,y) = u(x,y) - u(x_0,y).$$

By condition (7), there exist functions $p_i : \Omega \to \mathbb{R}$ ($i = 1, \dots, n$) such that $p_i(x,\cdot) \in C([0,b])$ ($i = 1, \dots, n$),

$$p\left(x, y, u(x,y), \frac{\partial u(x,y)}{\partial y}, \dots, \frac{\partial^{n-1} u(x,y)}{\partial y^{n-1}}\right) \\ - p\left(x, y, u(x_0,y), \frac{\partial u(x_0,y)}{\partial y}, \dots, \frac{\partial^{n-1} u(x_0,y)}{\partial y^{n-1}}\right) = \sum_{i=1}^{n} p_i(x,y) \frac{\partial^{i-1} w(x,y)}{\partial y^{i-1}},$$

and inequalities (22) hold. Hence

$$\frac{\partial^n w(x,y)}{\partial y^n} = \sum_{i=1}^{n} p_i(x,y)\, \frac{\partial^{i-1} w(x,y)}{\partial y^{i-1}} + q_1(x, x_0, y), \tag{24}$$

where

$$q_1(x,x_0,y) = p\left(x, y, u(x_0,y), \frac{\partial u(x_0,y)}{\partial y}, \dots, \frac{\partial^{n-1} u(x_0,y)}{\partial y^{n-1}}\right) \\ - p\left(x_0, y, u(x_0,y), \frac{\partial u(x_0,y)}{\partial y}, \dots, \frac{\partial^{n-1} u(x_0,y)}{\partial y^{n-1}}\right) + q(x,y) - q(x_0,y).$$

We define functions ω_0 and ω_1 on the half-line $[0, +\infty)$ by setting

$$\omega_0(t) = \max\left\{ |p(x_1, y, z_1, \dots, z_n) - p(x_2, y, z_1, \dots, z_n)| : |x_1 - x_2| \le t, \quad 0 \le y \le b, \quad \sum_{i=1}^{n} |z_i| \le r_0 \right\} \\ + \max\{ |q(x_1,y) - q(x_2,y)| : |x_1 - x_2| \le t, \quad 0 \le y \le b \}, \qquad t \ge 0,$$

$$\omega_1(t) = \max\left\{ \sum_{i=1}^{n} |c_i(x_1) - c_i(x_2)| : |x_1 - x_2| \le t \right\} \\ + \max\left\{ \sum_{i=1}^{n} |h_i(v)(x_1) - h_i(v)(x_2)| : \|v\|_{C^n([0,b])} \le r_1, \quad |x_1 - x_2| \le t \right\}, \qquad t \ge 0.$$

Since the functions p, q, and c_i ($i = 1, \dots, n$) and the operators $h_i : C^{n-1}([0,b]) \to C([0,a])$ ($i = 1, \dots, n$) are continuous, it is obvious that the functions $\omega_i : [0, +\infty) \to [0, +\infty)$ ($i = 0, 1$) are continuous and nondecreasing and $\omega_0(0) = \omega_1(0) = 0$. On the other hand, in view of the estimates (21) and (23), it follows from Eqs. (2) and (24) that

$$\left| \frac{\partial^n w(x,y)}{\partial y^n} - \sum_{i=1}^{n} p_i(x,y) \frac{\partial^{i-1} w(x,y)}{\partial y^{i-1}} \right| \le \omega_0(|x - x_0|) \quad \text{for} \quad 0 \le y \le b,$$
$$|h_i(w(x,\cdot))(x)| \le \omega_1(|x - x_0|) \qquad (i = 1, \dots, n).$$

Hence, by Lemma 1 and condition (8), we obtain the estimate

$$\|w(x,\cdot)\|_{C^{n-1}([0,b])} \le \omega(|x - x_0|),$$

where

$$\omega(t) = r(b\omega_0(t) + \omega_1(t)).$$

Consequently,

$$\|u(x,\cdot) - u(x_0,\cdot)\|_{C^{n-1}([0,b])} \le \omega(|x - x_0|) \quad \text{for} \quad 0 \le x, x_0 \le a.$$

It is obvious from this estimate that the functions $(x,y) \longmapsto \partial^{i-1}u(x,y)/\partial y^{i-1}$ $(i = 1,\ldots,n)$ are uniformly continuous in the first argument.

To complete the proof of the theorem, it remains to show that problem (1), (2) has at most one solution. Let u_1 and u_2 be arbitrary solutions of this problem, and let $u_0(x,y) = u_1(x,y) - u_2(x,y)$. By condition (7), there exist functions $p_i \in C(\Omega)$ $(i = 1,\ldots,n)$ such that

$$\begin{aligned}\frac{\partial^n u_0(x,y)}{\partial y^n} &= p\left(x,y,u_1(x,y),\frac{\partial u_1(x,y)}{\partial y},\ldots,\frac{\partial^{n-1}u_1(x,y)}{\partial y^{n-1}}\right)\\ &\quad - p\left(x,y,u_2(x,y),\frac{\partial u_2(x,y)}{\partial y},\ldots,\frac{\partial^{n-1}u_2(x,y)}{\partial y^{n-1}}\right)\\ &= \sum_{i=1}^{n} p_i(x,y)\frac{\partial^{i-1}u_0(x,y)}{\partial y^{i-1}} \quad \text{for} \quad (x,y) \in \Omega \end{aligned} \tag{25}$$

and inequalities (22) hold. Further,

$$h_i(u_0(x,\cdot))(x) = 0 \quad \text{for} \quad x \in [0,a] \quad (i = 1,\ldots,n). \tag{26}$$

By Lemma 1 and conditions (8) and (22), it follows from identities (25) and (26) that $u_0(x,y) \equiv 0$; i.e., $u_1(x,y) \equiv u_2(x,y)$. The proof of the theorem is complete.

Proof of Theorem 2. Let

$$u_j(x,y) = g(c_{j1},\ldots,c_{jn},q_j)(x,y) \qquad (j = 1,2), \qquad u(x,y) = u_1(x,y) - u_2(x,y).$$

Then, by condition (7), there exist functions $p_i \in C(\Omega)$ $(i = 1,\ldots,n)$ such that

$$\begin{aligned}\frac{\partial^n u(x,y)}{\partial y^n} &= p\left(x,y,u_1(x,y),\frac{\partial u_1(x,y)}{\partial y},\ldots,\frac{\partial^{n-1}u_1(x,y)}{\partial y^{n-1}}\right)\\ &\quad - p\left(x,y,u_2(x,y),\frac{\partial u_2(x,y)}{\partial y},\ldots,\frac{\partial^{n-1}u_2(x,y)}{\partial y^{n-1}}\right) + q_1(x,y) - q_2(x,y)\\ &= \sum_{i=1}^{n} p_i(x,y)\frac{\partial^{i-1}u(x,y)}{\partial y^{i-1}} + q_1(x,y) - q_2(x,y) \quad \text{for} \quad (x,y) \in \Omega \end{aligned} \tag{27}$$

and inequalities (22) hold. Further,

$$h_i(u(x,\cdot))(x) = c_{1i}(x) - c_{2i}(x) \quad \text{for} \quad x \in [0,a] \quad (i = 1,\ldots,n). \tag{28}$$

By Lemma 1 and conditions (8) and (22), it follows from identities (27) and (28) that the estimate (9) holds, where r is a positive constant independent of the functions q_j and c_{ji} $(j = 1,2;$ $i = 1,\ldots,n)$. The proof of the theorem is complete.

The boundary conditions

$$u^{(0,i-1)}(x,0) = c_i(x) \quad (i = 1,\ldots,n-1), \quad h(u(x,\cdot))(x) = c_n(x) \quad \text{for} \quad 0 \le x \le a, \tag{29}$$

where

$$u^{(0,i-1)}(x,y) = \frac{\partial^{i-1}u(x,y)}{\partial y^{i-1}} \qquad (i = 1,\ldots,n),$$

$c_i \in C([0,a])$ $(i = 1,\ldots,n)$, and $h : C^{n-1}([0,b]) \to C([0,a])$ is a bounded linear operator, are a special case of conditions (2).

We say that an operator $h : C^{n-1}([0,b]) \to C([0,a])$ is *positive* if for each function $v \in C^{n-1}([0,b])$ satisfying the inequalities

$$v^{(i-1)}(y) > 0 \quad \text{for} \quad 0 < y \le b \quad (i = 1,\ldots,n) \tag{30}$$

one has the inequality

$$h(v)(x) > 0 \quad \text{for} \quad 0 \le x \le a. \tag{31}$$

For example, if $h(v)(x) = v^{(m)}(b) - v^{(m)}(y_0(x))$ or $h(v)(x) = \sum_{k=1}^{n} \alpha_k(x) v^{(k-1)}(y_k(x))$, where $m \in \{0, \dots, n-2\}$, $y_0 : [0,a] \to [0,b)$, $y_k : [0,a] \to (0,b]$ $(k = 1, \dots, n)$, and $\alpha_k : [0,a] \to [0,+\infty)$ are continuous functions, and

$$\sum_{k=1}^{n} \alpha_k(x) > 0 \quad \text{for} \quad 0 \le x \le a,$$

then h is a positive operator.

Corollary 1. *Let $n \ge 2$, and let the inequalities*

$$-l_i(x,y) \le \frac{\partial p(x,y,z_1,\dots,z_n)}{\partial z_i} \le l_0 \qquad (i = 1, \dots, n) \tag{32}$$

hold on the set $\Omega \times \mathbb{R}^n$, where l_0 is a positive constant and $l_i : \Omega \to [0,+\infty)$ $(i = 1, \dots, n)$ are continuous functions such that

$$\sum_{i=1}^{n-1} \frac{1}{(n-i)!} \int_0^b y^{n-i} l_i(x,y) \exp\left(\int_0^y l_n(x,t)\, dt \right) dy \le 1 \quad \textit{for} \quad 0 \le x \le a. \tag{33}$$

If, moreover, h is a positive operator, then there exists a unique solution of problem (1), (29).

Proof. Let $x \in [0,a]$ be an arbitrary given number, and let $p_i : [0,b] \to \mathbb{R}$ $(i = 1, \dots, n)$ be arbitrary measurable functions satisfying the inequalities

$$-l_i(x,y) \le p_i(y) \le l_0 \qquad (i = 1, \dots, n) \tag{34}$$

almost everywhere on $[0,b]$.

By Theorem 1 and inequalities (32), to prove Corollary 1, it suffices to show that the differential equation (5) with the boundary conditions

$$v^{(i-1)}(0) = 0 \qquad (i = 1, \dots, n-1), \qquad h(v)(x) = 0 \tag{35}$$

has only the trivial solution for any $x \in [0,a]$.

Assume the contrary: problem (5), (35) has a nontrivial solution v. Without loss of generality, we assume that $v^{(n-1)}(0) > 0$. Then either inequalities (30) hold or there exists a $b_1 \in (0,b]$ such that

$$v^{(i-1)}(y) > 0 \quad \text{for} \quad 0 < y < b_1 \quad (i = 1, \dots, n) \tag{36}$$

and

$$v^{(n-1)}(b_1) = 0. \tag{37}$$

The operator h is positive, and hence inequalities (30) imply inequality (31), which contradicts conditions (35).

Consequently, it remains to consider the case in which conditions (36) and (37) are satisfied. Then there exists a number $b_0 \in [0, b_1)$ such that

$$\varrho = \max\{v^{(n-1)}(t) : a \le t \le b_1\} = v^{(n-1)}(b_0) \tag{38}$$

and

$$0 < v^{(i-1)}(t) \le \frac{\varrho}{(n-i)!} t^{n-i} \quad \text{for} \quad b_0 \le t < b_1, \quad v^{(i-1)}(b_1) < \frac{\varrho}{(r-i)!} b_1^{n-i} \quad (i = 1, \dots, n-1). \tag{39}$$

By inequalities (34) and (36), the inequality

$$v^{(n)}(y) \geq -\sum_{i=1}^{n} l_i(x,y) v^{(n-i)}(y)$$

holds almost everywhere on $[b_0, b_1]$. In view of (37) and (38), we obtain

$$\varrho \leq \sum_{i=1}^{n-1} \int\limits_{b_0}^{b_1} l_i(x,y) \exp\left(\int\limits_{b_0}^{y} l_n(x,t)\,dt\right) v^{(i-1)}(y)\,dy.$$

On the other hand, it follows from inequalities (33) and (39) that

$$\sum_{i=1}^{n-1} \int\limits_{b_0}^{b_1} l_i(x,y) \exp\left(\int\limits_{b_0}^{y} l_n(x,t)\,dt\right) v^{(i-1)}(y)\,dy < \varrho.$$

The contradiction thus obtained proves the corollary.

By way of example, consider the linear differential equation

$$\frac{\partial^n u}{\partial y^n} = \sum_{i=1}^{n} p_i(x,y) \frac{\partial^{i-1} u}{\partial y^{i-1}} + q(x,y), \tag{40}$$

where $p_i \in C(\Omega)$ $(i = 1, \ldots, n)$ and $q \in C(\Omega)$.

For an arbitrary real number t, set $[t]_- = (|t| - t)/2$. Corollary 1 readily implies the following assertion.

Corollary 2. *If the operator h is positive and*

$$\sum_{i=1}^{n-1} \frac{1}{(n-i)!} \int\limits_{0}^{b} y^{n-i} [p_i(x,y)]_- \exp\left(\int\limits_{0}^{y} [p_n(x,t)]_-\,dt\right) \leq 1 \quad \textit{for} \quad 0 \leq x \leq a,$$

then there exists a unique solution of problem (40), (29).

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