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NUMERICAL METHODS

Consistent Two-Sided Estimates for the Solutions of Quasilinear Parabolic Equations and Their Approximations

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Abstract—For a linearized finite-difference scheme approximating the Dirichlet problem for a multidimensional quasilinear parabolic equation with unbounded nonlinearity, we establish pointwise two-sided solution estimates consistent with similar estimates for the differential problem. These estimates are used to prove the convergence of finite-difference schemes in the grid L_2 norm.

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1. INTRODUCTION

The maximum principle allows one not only to establish the uniqueness of the solution and its continuous dependence on the input data for parabolic and elliptic equations but also, in contrast to the energy inequality method, to obtain a priori upper bounds for the solution in the uniform norm for such problems of arbitrary dimension with a nonself-adjoint elliptic operator [1, p. 500]. The maximum principle is also used in the theory of finite-difference schemes to study the stability of the finite-difference solution with respect to the input data and its convergence to the exact solution of the problem in the uniform norm. Finite-difference methods satisfying the grid maximum principle are usually said to be *monotone* [2, p. 228; 3, p. 296]. Various classes of monotone finite-difference schemes have been developed and studied for multidimensional linear convection–diffusion equations (e.g., see the monograph [4, p. 35]). Monotone schemes play an important role in computational practice in that they allow one to obtain oscillation-free numerical solutions even in the case of nonsmooth solutions [5].

Lower (or, in the general case, two-sided) estimates of solutions of differential-difference problems are of no less importance. Such estimates are especially important when studying the properties of numerical methods for problems with unbounded nonlinearity, because in this case one needs to establish that the grid solution lies in a neighborhood of the values of the exact solution [6, 7]. For linear problems, these estimates enable one to find the range of values of the desired solution in terms of the problem input data (the coefficients and right-hand side of the equation as well as the initial and boundary conditions). In the nonlinear case, such estimates permit one to prove the nonnegativity of the exact solution, which is important in physical problems, as well as to find conditions on the input data under which the problem is parabolic or elliptic.

Finding nontrivial estimates for the solutions of initial–boundary value problems is based on a special trick, originally applied by Ladyzhenskaya [8] (see also the monograph [9, p. 22]), whereby one makes a parameter-dependent change of variables and then minimizes or maximizes some functions with respect to this parameter; the resulting extremal values give the corresponding estimates for the solution. Naturally, one also needs such estimates in numerical algorithms for the approximate solution of initial–boundary value problems. The theory of finite-difference schemes [2, p. 229] includes the technique, well developed for linear problems, of the grid maximum principle, which provides two-sided estimates for the approximate solution. These estimates for the solutions of finite-difference problems are less accurate [10] than the corresponding estimates for the solutions of differential problems [9, p. 22]. Note that similar estimates for the finite element method in the case of linear problems and problems with bounded nonlinearity were obtained by Farago et al. (e.g., see [11]).

The present paper develops the technique in [8] as applied to finite-difference schemes for a quasilinear parabolic equation with unbounded nonlinearity and to the derivation of two-sided estimates, completely consistent with the estimates of solutions of the corresponding differential problem, for the solutions of these schemes. Note that the two-sided estimates proved here are independent of the values of the coefficients multiplying the second derivatives in the equation. A straightforward application of these estimates proves the convergence of the linearized finite-difference scheme in the grid L_2 norm.

2. TWO-SIDED ESTIMATES FOR SOLUTIONS OF INITIAL–BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS WITH UNBOUNDED NONLINEARITY

Consider the following problem for a quasilinear parabolic equation in the parallelepiped $\bar{Q}_T = \bar{\Omega} \times [0,T]$, where $\Omega = \{\mathbf{x} : 0 < x_\alpha < l_\alpha,\ \mathbf{x} = (x_1, x_2),\ \alpha = 1,2\}$:

$$\frac{\partial u}{\partial t} = \frac{\partial W_1}{\partial x_1} + \frac{\partial W_2}{\partial x_2} + f(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \Omega \times (0,T), \tag{1}$$

with the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{2}$$

and the Dirichlet boundary conditions

$$u(\mathbf{x}, t) = \mu(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \partial\Omega \times [0,T], \tag{3}$$

where $W_\alpha = k_\alpha(u)\dfrac{\partial u}{\partial x_\alpha}$, $\alpha = 1,2$, the functions $k_\alpha = k_\alpha(u)$, $\alpha = 1,2$, are sufficiently smooth, the functions f, u_0, and μ are continuous, and the corresponding matching conditions are satisfied.

Let $u(\mathbf{x}, t)$ be a solution of problem (1)–(3), and let $D_u = [m_1, m_2]$ be a closed interval containing the range of the solution; i.e., $m_1 \le u(\mathbf{x}, t) \le m_2$. Since the functions $k_\alpha = k_\alpha(u)$, $\alpha = 1,2$, are smooth, it follows that there exist constants $k_{\alpha,1}, k_{\alpha,2}$, and L_α such that

$$|k_\alpha'(u)| \le L_\alpha, \quad k_{\alpha,1} \le k_\alpha(u) \le k_{\alpha,2}, \quad u \in D_u, \quad (\mathbf{x}, t) \in \bar{Q}_T, \quad \alpha = 1,2. \tag{4}$$

Note that the second condition in (4) follows from the first, the constants $k_{\alpha,1}$ and $k_{\alpha,2}$ being solely introduced to make the exposition more convenient. Assume that the function $u(\mathbf{x}, t)$ is continuous in the domain $\bar{Q}_T$, its derivatives occurring in Eq. (1) are continuous in Q_T, and the function itself satisfies Eq. (1) in Q_T, the initial condition (2), and the boundary conditions (3). Set $Q_{t_1} = \{(\mathbf{x}, t) \in Q_T : t \le t_1\}$.

The following theorem, crucial for the goals of the present paper, was established in [8].

Theorem 1. *The classical solution* $u(\mathbf{x}, t)$ *of problem* (1)–(3) *satisfies the two-sided estimate*

$$u(\mathbf{x}, t_1) \ge m_1 = \sup_{\lambda > 0}\Big(e^{\lambda t_1} \min\Big\{0, \min_{(\mathbf{x},t)\in Q_{t_1}} e^{-\lambda t}\{\mu(\mathbf{x}, t), u_0(\mathbf{x})\}, \lambda^{-1} \min_{(\mathbf{x},t)\in Q_{t_1}} f(\mathbf{x}, t)e^{-\lambda t}\Big\}\Big), \tag{5}$$

$$u(\mathbf{x}, t_1) \le m_2 = \inf_{\lambda > 0}\Big(e^{\lambda t_1} \max\Big\{0, \max_{(\mathbf{x},t)\in Q_{t_1}} e^{-\lambda t}\{\mu(\mathbf{x}, t), u_0(\mathbf{x})\}, \lambda^{-1} \max_{(\mathbf{x},t)\in Q_{t_1}} f(\mathbf{x}, t)e^{-\lambda t}\Big\}\Big) \tag{6}$$

for every $t_1 \in [0,T]$.

Let us outline the proof of the upper bound (6) in a form convenient to us, because a similar argument will be used below in the finite-difference case.

Take the auxiliary function $v(\mathbf{x},t) = u(\mathbf{x},t)e^{-\lambda t}$, where λ is a positive parameter. Let $(\mathbf{x^0}, t^0)$ be a point of maximum of v in the parallelepiped $\bar{Q}_{t_1}$, and let $v^0 = v(\mathbf{x^0}, t^0)$. There exist only three possibilities.

1. The maximum v^0 is nonpositive [and then $v(\mathbf{x},t) \le 0$ for $(\mathbf{x},t) \in \bar{Q}_{t_1}$].

2. The point $(\mathbf{x^0}, t^0)$ lies on the boundary of $\bar{Q}_T$ [and then $v(\mathbf{x},t) \le \max\limits_{(\mathbf{x},t)\in Q_{t_1}} e^{-\lambda t}\{\mu(\mathbf{x},t), u_0(\mathbf{x})\}$ for $(\mathbf{x},t) \in \bar{Q}_{t_1}$].

3. The maximum v^0 is positive, and the point $(\mathbf{x^0}, t^0)$ is an interior point of Q_T.

In case 3, the relations

$$\frac{\partial v(\mathbf{x}^0,t^0)}{\partial t} \ge 0, \qquad \frac{\partial v(\mathbf{x}^0,t^0)}{\partial x_\alpha} = 0, \qquad \frac{\partial^2 v(\mathbf{x}^0,t^0)}{\partial x_\alpha^2} \le 0, \qquad \alpha = 1,2,$$

hold at the point $(\mathbf{x^0}, t^0)$. Hence it follows from the equation

$$\frac{\partial v}{\partial t}e^{\lambda t} + \lambda v e^{\lambda t} = e^{\lambda t}\sum_{i=1}^{2}\frac{\partial^2 v}{\partial x_i^2}k_i(ve^{\lambda t}) + e^{2\lambda t}\sum_{i=1}^{2}\frac{\partial k_i}{\partial u}(ve^{\lambda t})\left(\frac{\partial v}{\partial x_i}\right)^2 + f$$

that the inequality $\lambda v^0 e^{\lambda t} \le f$ is satisfied, which gives the following estimate for the auxiliary function v:

$$v(\mathbf{x},t) \le \lambda^{-1}\max_{(\mathbf{x},t)\in Q_{t_1}} f(\mathbf{x},t)e^{-\lambda t}, \qquad (\mathbf{x},t)\in Q_{t_1}.$$

We combine cases 1–3, return to the original function u, and obtain the upper bound (6). A similar argument for a point of minimum gives the lower bound (5).

3. APPLICATION OF THE MAXIMUM PRINCIPLE FOR FINITE-DIFFERENCE SCHEMES WITH INPUT DATA OF VARIABLE SIGN

To obtain the finite-difference counterpart of the differential estimates, we use the maximum principle for finite-difference schemes with input data of variable sign. Given finitely many points (i.e., a mesh Ω_h) in n-dimensional Euclidean space, we assign exactly one stencil $\mathcal{M}(x)$ to each point $x \in \Omega_h$; the stencil can be an arbitrary subset of Ω_h containing x. The set $\mathcal{M}'(x) = \mathcal{M}(x)\setminus x$ will be called the neighborhood of x. Let $A(x)$, $B(x,\xi)$, and $F(x)$ be real-valued functions defined for any $x, \xi \in \Omega_h$. For each point $x \in \Omega_h$, consider the equation [2, p. 226]

$$A(x)y(x) = \sum_{\xi\in\mathcal{M}'(x)} B(x,\xi)y(\xi) + F(x), \qquad x\in\Omega_h, \tag{7}$$

which is called the canonical form of a finite-difference scheme at the point x. Along with the mesh Ω_h, consider a subset $\overline{\omega}_h \subset \Omega_h$ and define

$$\overline{\Omega}_h = \bigcup_{x\in\omega_h}\mathcal{M}(x).$$

We assume the usual positivity conditions

$$B(x,\xi) \ge 0, \qquad \xi\in\mathcal{M}'(x), \qquad x\in\Omega_h, \tag{8}$$

$$D(x) = A(x) - \sum_{\xi\in\mathcal{M}'(x)} B(x,\xi) > 0, \qquad x\in\Omega_h, \tag{9}$$

to be satisfied for the coefficients of the finite-difference scheme (7), which guarantee its unique solvability, monotonicity, and (in the linear case) stability in the uniform norm under small perturbations in the input data. Note that if conditions (8) and (9) are satisfied, then $A(x) > 0$, $x \in \Omega_h$. The following theorem was proved in [7].

Theorem 2. *Let the coefficient positivity conditions* (8) *and* (9) *be satisfied. Then the maximum and minimum values of the solution of the finite-difference scheme* (7) *lie in the following range determined by the input data*:

$$\min_{x\in\Omega_h}\frac{F(x)}{D(x)} \le y(x) \le \max_{x\in\Omega_h}\frac{F(x)}{D(x)}. \tag{10}$$

By way of example, consider an application of Theorem 2 to the following initial–boundary value problem for a quasilinear parabolic equation in the rectangle $\bar{\Pi}=\{0\le x\le l,\ 0\le t\le T\}$:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+r(x)\frac{\partial u}{\partial x},\qquad 0<x<l,\qquad 0<t\le T, \tag{11}$$

$$u(x,0)=u_0(x),\qquad u(0,t)=\mu_1(t),\qquad u(l,t)=\mu_2(t).$$

By definition [12, p. 23], problem (11), (12) is parabolic if there exist two constants k_1 and k_2 such that

$$0<k_1\le k(u)\le k_2,\qquad u\in\bar{D}_u,\qquad k_1,k_2=\text{ const}. \tag{13}$$

Now consider a special case of Eq. (11), namely, the Gamma equation [13, 14]

$$\frac{\partial u}{\partial t}=\frac{\partial^2\beta}{\partial x^2}+r(x)\frac{\partial u}{\partial x} \tag{14}$$

obtained by a transformation of the nonlinear Black–Scholes equation to a quasilinear parabolic equation.

If $\beta=u/(1-\varrho u)^2$ and $\varrho>0$, then we obtain the coefficient

$$k(u)=\frac{1+\varrho u}{(1-\varrho u)^3}. \tag{15}$$

By virtue of inequalities (13), Eq. (14) is parabolic if the coefficient (15) satisfies

$$k(u)=(1+\varrho u)/(1-\varrho u)^3>0$$

for all $u\in\bar{D}_u$, i.e., if

$$-\varrho^{-1}<u<\varrho^{-1}. \tag{16}$$

To construct a monotone finite-difference scheme, consider the equation [2, p. 170]

$$\frac{\partial u}{\partial t}=\tilde{L}u+f,\qquad \tilde{L}u=\varkappa(x,u)\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+r(x)\frac{\partial u}{\partial x} \tag{17}$$

with a perturbed operator, where

$$\varkappa(x,u)=\frac{1}{1+R(x,u)},\qquad R(x,u)=\frac{h|r(x)|}{2k(u)}.$$

Let us represent $r(x)$ as the sum $r=r^++r^-$, where $r^+=(r+|r|)/2\ge 0$ and $r^-=(r-|r|)/2\le 0$, and approximate $r\dfrac{\partial u}{\partial x}$ by the expression

$$\left(r\frac{\partial u}{\partial x}\right)_i=\left(\frac{r}{k}\left(k\frac{\partial u}{\partial x}\right)\right)_i\sim b_i^+a_{i+1}(u)u_{x,i}+b_i^-a_i(u)u_{\bar{x},i},$$

where

$$b_i^+=\frac{r_i^+}{k(u_i)}\ge 0,\qquad b_i^-=\frac{r_i^-}{k(u_i)}\le 0,\qquad a_i(u)=\frac{1}{2}(k(u_{i-1})+k(u_i)),$$

$$u_{x,i}=(u_{i+1}-u_i)/h,\qquad u_{\bar{x},i}=(u_i-u_{i-1})/h.$$

Let us approximate the differential operator $\tilde{L}$ for given $t = t_j$ by the finite-difference operator

$$\tilde{\Lambda}\widehat{y} = \varkappa(a(y)\widehat{y}_{\bar{x}})_x + b^+ a(y)^{(+1)}\widehat{y}_x + b^- a(y)\widehat{y}_{\bar{x}}, \quad \text{where} \quad a(y)^{(+1)} = a_{i+1}(y).$$

Then to Eq. (17) on the uniform space-time grid

$$\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau, \quad \bar{\omega}_h = \{x_i = ih,\ i = 0, \ldots, N,\ hN = l\}, \quad \bar{\omega}_h = \omega_h \cup \{x_0 = 0,\ x_N = l\},$$
$$\bar{\omega}_\tau = \{t_n = n\tau,\ n = 0, \ldots, N_0,\ \tau N_0 = T\}, \quad \bar{\omega}_\tau = \omega_\tau \cup \{t_{N_0} = T\},$$

we assign the finite-difference scheme

$$\begin{aligned} \frac{y_i^{n+1} - y_i^n}{\tau} &= \frac{\varkappa_i^n}{h}\left(a_{i+1}^n(y)\frac{y_{i+1}^{n+1} - y_i^{n+1}}{h} - a_i^n(y)\frac{y_i^{n+1} - y_{i-1}^{n+1}}{h}\right) \\ &\quad + b_i^+ a_{i+1}^n(y)\frac{y_{i+1}^{n+1} - y_i^{n+1}}{h} + b_i^- a_i^n(y)\frac{y_i^{n+1} - y_{i-1}^{n+1}}{h} + f_i^{n+1}, \\ y_0^{n+1} &= \mu_1(t_{n+1}), \qquad y_N^{n+1} = \mu_2(t_{n+1}), \end{aligned} \tag{18}$$

where $\varkappa_i^n = \varkappa(x_i, y_i^n)$. By Theorem 2, conditions (16) are satisfied [5] for the finite-difference scheme (18) of the form (7) approximating the initial–boundary value problem for Eq. (14) and satisfying the coefficient positivity conditions (8) and (9), provided that

$$-\varrho^{-1} < \min_{(x,t)\in\bar{Q}_T}\{\mu_1(t), \mu_2(t), u_0(x)\} \le y_i^n \le \max_{(x,t)\in\bar{Q}_T}\{\mu_1(t), \mu_2(t), u_0(x)\} < \varrho^{-1}.$$

4. TWO-SIDED ESTIMATES OF SOLUTIONS OF FINITE-DIFFERENCE SCHEMES

To approximate problem (1)–(3) on a uniform space-time grid in the rectangle $\bar{Q}_T$,

$$\bar{\omega}_{h_\alpha} = \{x_{\alpha,i_\alpha} = i_\alpha h_\alpha,\ i_\alpha = 0, \ldots, N_\alpha;\ h_\alpha N_\alpha = l_\alpha\}, \qquad \alpha = 1, 2,$$
$$\bar{\omega}_\tau = \{t_n = n\tau,\ n = 0, \ldots, N_0;\ \tau N_0 = T\}, \qquad \bar{\omega}_\tau = \omega_\tau \cup \{t_{N_0} = T\},$$
$$\bar{\omega} = \bar{\omega}_{h_1} \times \bar{\omega}_{h_2} \times \bar{\omega}_\tau, \qquad \omega_{t_n} = \{(\mathbf{x}, t) \in \bar{\omega} : t \le t_n\},$$

we use the linearized finite-difference scheme

$$y_t = (a_1(y)\hat{y}_{\bar{x}_1})_{x_1} + (a_2(y)\hat{y}_{\bar{x}_2})_{x_2} + \hat{f}, \tag{19}$$
$$y(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \bar{\omega}_h, \tag{20}$$
$$y|_{\bar{\omega}\cap\partial Q_T} = \mu. \tag{21}$$

As usual, the stencil functionals

$$a_\alpha(y) = 0.5(k_\alpha(y_{i_\alpha - 1}) + k_\alpha(y_{i_\alpha})), \qquad \alpha = 1, 2, \tag{22}$$

are chosen from the second-order consistency condition [2, p. 140]

$$(a_\alpha(u)\hat{u}_{\bar{x}_\alpha})_{x_\alpha} - \frac{\partial}{\partial x_\alpha}\left(k_\alpha(u)\frac{\partial u}{\partial x_\alpha}\right) = O(h_\alpha^2 + \tau)$$

for the elliptic operator with respect to the spatial variables.

Here and in what follows, we use the standard notation [2, p. 65]

$$y = y_{i_1 i_2}^n = y(x_{1,i_1}, x_{2,i_2}, t_n), \quad y_t = \frac{\hat{y} - y}{\tau}, \quad \hat{y} = y_{i_1 i_2}^{n+1}, \quad v_{\bar{x}_\alpha} = \frac{v_{i_\alpha} - v_{i_\alpha - 1}}{h_\alpha}, \quad v_{x_\alpha} = \frac{v_{i_\alpha + 1} - v_{i_\alpha}}{h_\alpha}$$

of the theory of finite-difference schemes.

Theorem 3. *The solution* $y(x,t)$ *of problem* (19)–(21) *satisfies the two-sided estimate*

$$y(x,t_n) \geq m_{1,\tau} = \sup_{\lambda>0}\Big(e^{\lambda t_n}\min\Big\{0, \min_{(\mathbf{x},t)\in\omega_{t_n}} e^{-\lambda t}\{\mu(\mathbf{x},t), u_0(\mathbf{x})\}, \frac{\tau}{e^{\lambda\tau}-1}\min_{(\mathbf{x},t)\in\omega_{t_n}} f(\mathbf{x},t)e^{-\lambda t}\Big\}\Big), \quad (23)$$

$$y(x,t_n) \leq m_{2,\tau} = \inf_{\lambda>0}\Big(e^{\lambda t_n}\max\Big\{0, \max_{(\mathbf{x},t)\in\omega_{t_n}} e^{-\lambda t}\{\mu(\mathbf{x},t), u_0(\mathbf{x})\}, \frac{\tau}{e^{\lambda\tau}-1}\max_{(\mathbf{x},t)\in\omega_{t_n}} f(\mathbf{x},t)e^{-\lambda t}\Big\}\Big) \quad (24)$$

at any point $(\mathbf{x},t_n)\in\omega$.

Proof. Let us prove the upper bound (24). Take the auxiliary function $z = z(\mathbf{x},t_n) = y(\mathbf{x},t_n)e^{-\lambda t_n}$, where λ is a positive parameter. Let $(\mathbf{x^0},t^0)$ be a point of maximum of z in the grid domain ω_{t_n}, and let $z^0 = z(\mathbf{x^0},t^0)$. There exist only three possibilities:

1. The maximum z^0 is nonpositive [and then $z(\mathbf{x},t)\leq 0$, $(\mathbf{x},t)\in\omega_{t_n}$].

2. The point $(\mathbf{x^0},t^0)$ lies on the boundary of ω_{t_n} [and then $z(\mathbf{x},t) \leq \max\limits_{(\mathbf{x},t)\in\omega_{t_n}} e^{-\lambda t_n}\{\mu(\mathbf{x},t),u_0(\mathbf{x})\}$ for $(\mathbf{x},t)\in\omega_{t_n}$].

3. The maximum z^0 is positive, and the point $(\mathbf{x^0},t^0)$ is an interior point of ω_{t_n}.

In case 3, we have

$$\frac{\hat z e^{\lambda\tau} - z}{\tau} = (a_1(y)\hat z_{\bar x_1})_{x_1}e^{\lambda\tau} + (a_2(y)\hat z_{\bar x_2})_{x_2}e^{\lambda\tau} + e^{-\lambda t_n}\hat f.$$

We rewrite this equation in the canonical form

$$\begin{aligned}
C^n_{i_1i_2}z^{n+1}_{i_1i_2} &= A^n_{1,i_1i_2}z^{n+1}_{i_1-1i_2} + B^n_{1,i_1i_2}z^{n+1}_{i_1+1i_2} + A^n_{2,i_1i_2}z^{n+1}_{i_1i_2-1} + B^n_{2,i_1i_2}z^{n+1}_{i_1i_2+1} + K^n_{i_1i_2}z^n_{i_1i_2} + F^n_{i_1i_2},\\
C^n_{i_1i_2} &= \frac{e^{\lambda\tau}}{\tau} + \frac{e^{\lambda\tau}}{h_1^2}(a_{1,i_1+1i_2} + a_{1,i_1i_2}) + \frac{e^{\lambda\tau}}{h_2^2}(a_{2,i_1i_2+1} + a_{2,i_1i_2}),\\
A^n_{1,i_1i_2} &= \frac{e^{\lambda\tau}}{h_1^2}a_{1,i_1i_2}, \quad B^n_{1,i_1i_2} = \frac{e^{\lambda\tau}}{h_1^2}a_{1,i_1+1i_2}, \quad A^n_{2,i_1i_2} = \frac{e^{\lambda\tau}}{h_2^2}a_{2,i_1i_2}, \quad B^n_{2,i_1i_2} = \frac{e^{\lambda\tau}}{h_2^2}a_{2,i_1i_2+1},\\
K^n_{i_1i_2} &= \frac{1}{\tau}, \qquad F^n_{i_1i_2} = f^{n+1}_{i_1i_2}e^{-\lambda t_n},
\end{aligned}$$

and introduce the coefficients $D^n_{i_1i_2}$, $i_\alpha = 1,\dots,N_\alpha$, $\alpha = 1,2$, as follows:

$$D^n_{i_1i_2} = C^n_{i_1i_2} - A^n_{1,i_1i_2} - A^n_{2,i_1i_2} - B^n_{1,i_1i_2} - B^n_{2,i_1i_2} - K^n_{i_1i_2}, \qquad i_\alpha = 1,\dots,N_\alpha - 1, \qquad \alpha = 1,2.$$

Note that $y\in D_u$ for $t_n = 0$. We carry out the proof by induction over time layers. Since

$$D^n_{i_1i_2} = \frac{\tau}{e^{\lambda\tau}-1} > 0$$

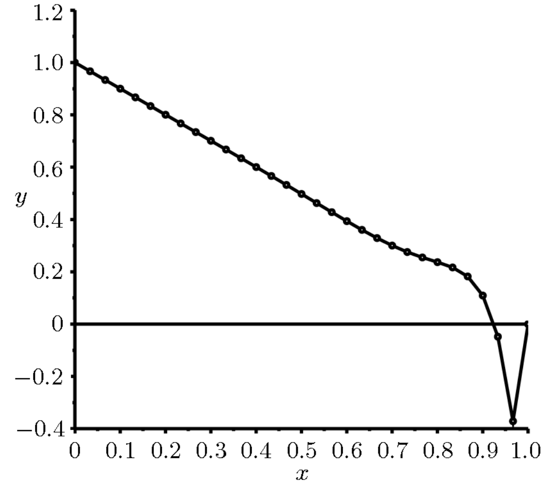
for $\lambda\tau > 0$, we see that the assumptions of Theorem 2 are satisfied for $n = 1$ and the estimate

$$z^n_{i_1i_2} \leq \frac{\tau}{e^{\lambda\tau}-1}\max_{(\mathbf{x},t)\in\omega_{t_n}} fe^{-\lambda t}$$

holds by inequality (10). We combine cases 1–3 and obtain the inequality

$$z \leq \max\Big\{0, \max_{(\mathbf{x},t)\in\omega_{t_n}} e^{-\lambda t}\{\mu(\mathbf{x},t),u_0(\mathbf{x})\}, \frac{\tau}{e^{\lambda\tau}-1}\max_{(\mathbf{x},t)\in\omega_{t_n}} f(\mathbf{x},t)e^{-\lambda t}\Big\}.$$

Now we return to the original function y and obtain the upper bound (24). Similar computations for the minimum give the lower bound (23). We have proved the induction assumption. Note that, by the results obtained above, $y^1\in D_u$ and the stencil functionals $a_1(y^1)$ and $a_2(y^1)$ satisfy the conditions of parabolicity on the solution (positivity). The argument for the inductive step differs



Solution of the finite-difference scheme (27) at time $t = 1$.

from that for the induction assumption only in the notation of indices. The proof of the theorem is complete.

Remark 1. The resulting estimates (5), (6) and (23), (24) have the form

$$m_1 \le u \le m_2, \qquad m_{1,\tau} \le y \le m_{2,\tau}.$$

Since $\tau/(e^{\lambda\tau} - 1) \le \lambda^{-1}$, it follows that $m_1 \le m_{1,\tau}$ and $m_{2,\tau} \le m_2$. In this sense, the difference estimates inherit the properties of the differential problem.

Thus, we have shown that the solution of the linearized finite-difference scheme (19)–(21) lies in the range of the exact solution of the differential problem (1)–(3) without any conditions on the grid increments. One can ask how important this is.

Consider the following initial–boundary value problem for a one-dimensional linear parabolic equation with known exact solution:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in (0,1), \qquad t \in (0,1], \qquad u(x,0) = 1, \qquad 0 < x < 1, \tag{25}$$

$$u(0,t) = 1, \qquad u(1,t) = 0, \qquad t \in [0,1]. \tag{26}$$

The two-sided estimate (5), (6) for the solution of problem (25), (26) has the form $0 \le u \le 1$. Let us approximate this equation by the Crank–Nicolson scheme [10] with the parameters indicated below:

$$y_t = y^{(0.5)}_{\bar{x}x}, \qquad y^{(0.5)} = 0.5(\hat{y} + y), \qquad h = 1/30, \qquad \tau = 1/11. \tag{27}$$

We see from the figure that the solution of the finite-difference scheme (27) does not preserve positivity, and there arise nonphysical oscillations. They are known to be caused by the violation of the sufficient conditions for the monotonicity of the Crank–Nicolson scheme, which have the form $\tau \le h^2/2$ [2, p. 269].

5. CONVERGENCE OF THE FINITE-DIFFERENCE SCHEME IN THE GRID L_2 NORM

If one manages to obtain two-sided estimates for the solutions of finite-difference schemes, then the convergence analysis for linearized numerical algorithms results in a linear problem for the error $z = y - u$ of the method. In this section, we additionally assume that the exact solution of problem (1)–(3) is sufficiently smooth; namely, $u(\mathbf{x}, t) \in C^{4,2}(Q_T)$.

Remark 2. The convergence of the solution of a linearized finite-difference scheme in the grid L_2 norm was proved for an initial–boundary value problem for a one-dimensional quasilinear parabolic equation in the class of essentially generalized solutions in [6]. Namely, it was assumed that the solution of the differential problem is continuous and its first derivative $\partial u/\partial x$ has a jump discontinuity in a neighborhood of finitely many lines of discontinuity. The existence of the time derivative in any sense was not assumed.

We rewrite the finite-difference equation (19) in the form

$$y_t^n = W_{1h,x_1}^{n+1} + W_{2h,x_2}^{n+1} + f^{n+1}, \tag{28}$$

where

$$W_{1h}^{n+1} = W_{1h,i_1-1/2i_2}^{n+1} = W_{1h}(x_{1i_1-1/2}, x_{2i_2}, t_{n+1}) = a_{1,i_1i_2}^n y_{\bar{x}_1,i_1i_2}^{n+1},$$
$$W_{2h}^{n+1} = W_{2h,i_1i_2-1/2}^{n+1} = W_2(x_{1i_1}, x_{2i_2-1/2}, t_{n+1}) = a_{2,i_1i_2}^n y_{\bar{x}_2,i_1i_2}^{n+1}$$

$i_\alpha = 1, \ldots, N_\alpha$, $\alpha = 1, 2,$. Hence the discrepancy on the exact solution satisfies the equation

$$u_t^n = W_{1,x_1}^{n+1} + W_{2,x_2}^{n+1} + f^{n+1} + \psi^{n+1}. \tag{29}$$

We subtract Eqs. (28) from the respective equations (29) and obtain the following problem for the error z of the method:

$$\begin{aligned} z_t^n &= (W_{1h}^{n+1} - W_1^{n+1})_{x_1} + (W_{2h}^{n+1} - W_2^{n+1})_{x_2} - \psi^{n+1}, \\ z_{i_1i_2}^0 &= 0, \qquad i_\alpha = 1, \ldots, N_\alpha, \qquad \alpha = 1, 2, \\ z_{0i_2}^{n+1} &= 0, \qquad z_{N_1i_2}^{n+1} = 0, \qquad i_2 = 1, \ldots, N_2, \qquad n = 0, \ldots, N_0 - 1, \\ z_{i_10}^{n+1} &= 0, \qquad z_{i_1N_2}^{n+1} = 0, \qquad i_1 = 1, \ldots, N_1, \qquad n = 0, \ldots, N_0 - 1. \end{aligned} \tag{30}$$

We define the following inner products and the corresponding norms:

$$(u, v) = \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2-1} h_1 h_2 u_{i_1i_2} v_{i_1i_2}, \qquad \|u\| = \sqrt{(u, u)},$$
$$(u, v)_\alpha = \sum_{i_\alpha=1}^{N_\alpha} \sum_{i_{3-\alpha}=1}^{N_{3-\alpha}-1} h_1 h_2 u_{i_1i_2} v_{i_1i_2}, \qquad \|u\|_\alpha = \sqrt{(u, u)_\alpha}, \qquad \alpha = 1, 2.$$

Theorem 4. *The error of the solution of the finite-difference scheme* (19)–(21) *satisfies the estimate*

$$\|z^{n+1}\| \le c(h_1^2 + h_2^2 + \tau), \qquad c = \text{ const } > 0.$$

Proof. We take the inner product of Eq. (30) by $2\tau z^{n+1}$ and obtain

$$2\tau(z_t^n, z^{n+1}) = 2\tau(z^{n+1}, \delta W_{1x_1}^{n+1}) + 2\tau(z^{n+1}, \delta W_{2x_2}^{n+1}) - 2\tau(z^{n+1}, \psi^{n+1}). \tag{31}$$

Let us use the identity $z^{n+1} = \dfrac{1}{2}(z^{n+1} + z^n) + \dfrac{\tau}{2} z_t^n$ to represent the left-hand side of Eq. (31) in the form

$$2\tau(z_t^n, z^{n+1}) = \|z^{n+1}\|^2 - \|z^n\|^2 + \tau^2 \|z_t^n\|^2.$$

We apply the summation by parts formula [2, p. 98] to the first two terms on the right-hand side in Eq. (31) and obtain

$$2\tau(z^{n+1}, \delta W_{\alpha x_\alpha}^{n+1}) = -2\tau(z_{\bar{x}_\alpha}^{n+1}, \delta W_\alpha^{n+1})_\alpha, \qquad \alpha = 1, 2.$$

The substitution of these relations into Eq. (31) gives

$$\|z^{n+1}\|^2 - \|z^n\|^2 + \tau^2 \|z_t^n\|^2 = -2\tau(z_{\bar{x}_1}^{n+1}, \delta W_1^{n+1})_1 - 2\tau(z_{\bar{x}_2}^{n+1}, \delta W_2^{n+1})_2 - 2\tau(z^{n+1}, \psi^{n+1}). \tag{32}$$

We replace the resulting values $\delta W^{n+1}_{1,i_1-1/2 i_2}$ and $\delta W^{n+1}_{2,i_1 i_2-1/2}$ in Eq. (32) and find the relation

$$\|z^{n+1}\|^2 - \|z^n\|^2 + \tau^2\|z^n_t\|^2 = -2\tau\sum_{i=1}^{2}(z^{n+1}_{\bar{x}_i}, a_i(y^n)y^{n+1}_{\bar{x}_i} - a_i(u^n)u^{n+1}_{\bar{x}_i})_i + 2\tau(z^{n+1},\psi^{n+1}). \qquad (33)$$

Since

$$a_\alpha(y^n_{i_1i_2})y^{n+1}_{\bar{x}_\alpha,i_1i_2} - a_\alpha(u^n_{i_1i_2})u^{n+1}_{\bar{x}_\alpha,i_1i_2} = a_\alpha(y^n_{i_1i_2})z^{n+1}_{\bar{x}_\alpha,i_1i_2} + (a_\alpha(y^n_{i_1i_2}) - a_\alpha(u^n_{i_1i_2}))u^{n+1}_{\bar{x}_\alpha,i_1i_2}, \quad \alpha = 1,2,$$

we obtain the representation

$$(z^{n+1}_{\bar{x}_\alpha}, a_\alpha(y^n)y^{n+1}_{\bar{x}_\alpha} - a_\alpha(u^n)u^{n+1}_{\bar{x}_\alpha})_\alpha = (z^{n+1}_{\bar{x}_\alpha}, a_\alpha(y^n)z^{n+1}_{\bar{x}_\alpha})_\alpha + (z^{n+1}_{\bar{x}_\alpha}, (a_\alpha(y^n) - a_\alpha(u^n))u^{n+1}_{\bar{x}_\alpha})_\alpha.$$

Since $a_\alpha(y) \ge k_{\alpha,1} > 0$ for any $y \in [m_{h,1}, m_{h,2}]$ by (22), we see in view of conditions (4) that

$$(z^{n+1}_{\bar{x}_\alpha}, a_\alpha(y^n)z^{n+1}_{\bar{x}_\alpha})_\alpha \ge k_{\alpha,1}\|z^{n+1}_{\bar{x}_\alpha}\|^2_\alpha.$$

For the functions k_α, $\alpha = 1,2$, there exist constants L_α, $\alpha = 1,2$, such that

$$|a_\alpha(y^n_{i_1i_2}) - a_\alpha(u^n_{i_1i_2})| \le L_\alpha|z^n_{i_1i_2}|_{\alpha,(0.5)},$$

where

$$|z_{i_1i_2}|_{1,(0.5)} = \frac{|z_{i_1i_2}| + |z_{i_1-1,i_2}|}{2}, \qquad |z_{i_1i_2}|_{2,(0.5)} = \frac{|z_{i_1i_2}| + |z_{i_1,i_2-1}|}{2}.$$

Hence we obtain the inequality

$$(z^{n+1}_{\bar{x}_\alpha}, (a_\alpha(y^n) - a_\alpha(u^n))u^{n+1}_{\bar{x}_\alpha})_\alpha \le L_\alpha(|z^n|_{\alpha,(0.5)}|z^{n+1}_{\bar{x}_\alpha}|, |u^{n+1}_{\bar{x}_\alpha}|)_\alpha.$$

The solution $u(x,t)$ of problem (1)–(3) is sufficiently smooth, and hence we have the estimate

$$|u^{n+1}_{\bar{x}_\alpha}| = \frac{1}{h_\alpha}\int\limits_{x_{\alpha i_{\alpha-1}}}^{x_{\alpha i_\alpha}}\left|\frac{\partial u^{n+1}}{\partial x_\alpha}\right| dx_\alpha \le c, \qquad \alpha = 1,2.$$

Now we apply the ε-inequality and obtain

$$L_\alpha(|z^n|_{\alpha,(0.5)}|z^{n+1}_{\bar{x}_\alpha}|, |u^{n+1}_{\bar{x}_\alpha}|)_\alpha \le L_\alpha c\varepsilon_\alpha\|z^{n+1}_{\bar{x}_\alpha}\|^2_\alpha + \frac{L_\alpha c}{4\varepsilon_\alpha}\|z^n\|^2_\alpha.$$

Here and in the following, $\varepsilon_i = \text{const} > 0$, $i = 1,2,\ldots$ Thus, for the first two terms on the right-hand side in (33) we have the estimate

$$-2\tau(z^{n+1}_{\bar{x}_\alpha}, a_\alpha(y^n)y^{n+1}_{\bar{x}_\alpha} - a_\alpha(u^n)u^{n+1}_{\bar{x}_\alpha})_\alpha \le -2\tau(k_{\alpha,1} - L_\alpha c\varepsilon_\alpha)\|z^{n+1}_{\bar{x}_\alpha}\|^2_\alpha + \frac{\tau L_\alpha c}{2\varepsilon_\alpha}\|z^n\|^2_\alpha.$$

The last term on the right-hand side in (33) satisfies the estimate

$$-2\tau(z^{n+1},\psi^{n+1}) = -2\tau(\tau z^n_t + z^n, \psi^{n+1}) \le 2\tau^2\varepsilon_3\|z^n_t\|^2 + \frac{\tau^2}{2\varepsilon_3}\|\psi^{n+1}\|^2 + 2\tau\varepsilon_4\|z^n\|^2 + \frac{\tau}{2\varepsilon_4}\|\psi^{n+1}\|^2.$$

We take into account all these inequalities and, in view of the right-hand side of (33), arrive at the estimate

$$\|z^{n+1}\|^2 + \tau^2(1-2\varepsilon_3)\|z^n_t\|^2 + 2\tau(k_{1,1} - L_1c\varepsilon_1)\|z^{n+1}_{\bar{x}_1}\|^2_1 + 2\tau(k_{2,1} - L_2c\varepsilon_2)\|z^{n+1}_{\bar{x}_2}\|^2_1$$
$$\le \left(1 + \tau\left(\frac{L_1c}{2\varepsilon_1} + \frac{L_2c}{2\varepsilon_2} + 2\varepsilon_4\right)\right)\|z^n\|^2 + \tau\left(\frac{1}{2\varepsilon_3} + \frac{1}{2\varepsilon_4}\right)\|\psi^{n+1}\|^2.$$

Consequently,

$$\|z^{n+1}\|^2 + \tau^2(1-2\varepsilon_3)\|z_t^n\|^2 + 2\tau(k_{1,1} - L_1 c\varepsilon_1)\|z_{\bar{x}_1}^{n+1}\|_1^2 + 2\tau(k_{2,1} - L_2 c\varepsilon_2)\|z_{\bar{x}_2}^{n+1}\|_2^2 \le (1+\tau c)\|z^n\|^2 + \tau c(h_1^2 + h_2^2 + \tau)^2.$$

We take ε_1, ε_2, and ε_3 small enough that the inequalities

$$1 - 2\varepsilon_3 > 0, \qquad 2(k_{1,1} - L_1 c\varepsilon_1) \ge k_{1,1}, \qquad 2(k_{2,1} - L_2 c\varepsilon_2) \ge k_{2,2}$$

be satisfied. Thus, we arrive at the definitive estimate

$$\|z^{n+1}\|^2 \le (1+\tau c)\|z^n\|^2 + \tau c(h_1^2 + h_2^2 + \tau)^2 \le e^{\tau c}\|z^n\|^2 + \tau c(h_1^2 + h_2^2 + \tau)^2.$$

We apply the finite-difference counterpart of the Gronwall lemma [2, p. 273] to the last inequality and obtain the desired estimate. The proof of the theorem is complete.

Remark 3. It is a purely editorial task to generalize the results of this paper to convection–diffusion problems of arbitrary dimension.

Remark 4. In the theory of the finite element method, estimates of solutions via functions depending on the minimization or maximization of some functionals of the input data over auxiliary functions were applied by Repin (e.g., see [15]).

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