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NUMERICAL METHODS

Concise Formulas for Strain Analysis of Soft Biological Tissues

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Abstract—We describe a method for the approximate solution of nonlinear elasticity problems in the framework of finite deformation for the case of hyperelastic isotropic materials. This method enables one to write the resulting equations from the finite element method in analytical form, which reduces the amount of computations and simplifies the implementation. This approach is implemented for several types of hyperelastic materials used to describe the mechanical behavior of soft biological tissues.

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1. INTRODUCTION

In the recent years, the role of mathematical modeling in the solution of various biomedical problems has been steadily increasing. Predictive modeling of various types of surgery procedures and the development of telesurgery, where surgery is carried out by robots, may serve as examples. An adequate description of the mechanical behavior of soft biological tissues by mathematical modeling methods is of key importance for successful progress in this field of medicine.

The development of minimally invasive surgery was the first impetus to the development of methods for modeling soft tissue deformations [1–3]. In particular, this was motivated by the development of surgical simulators for surgeon training [4]. Since the methods to be used were required to produce results online, simplified models were chosen such as linear models or mass-spring models. Although these models were fairly easy to implement, they failed to produce an adequate description of the mechanical behavior of soft tissues.

Experimental data show that the mechanical behavior of soft tissues is extremely nonlinear, which necessitates solving nonlinear elasticity problems with finite (large) strains taken into account. The paper [5] suggests an approach in which the strain of nonlinear membranes made of a Saint Venant–Kirchhoff material (which is one of the simplest nonlinear models) is modeled by a set of nonlinear springs, which is more efficient than the conventional approach from the viewpoint of implementation and the amount of computations. The paper [5] also suggests to use the interpolation properties of barycentric coordinates and the principle of minimum potential energy; in the case of triangular finite elements for a Saint Venant–Kirchhoff material, this provides all required formulas in concise analytical form.

The concept suggested in [5] can be applied to the whole class of isotropic hyperelastic materials that can be used to describe the nonlinear behavior of soft biological tissues. The present paper develops an algorithm for the approximate solution of nonlinear elasticity problems for the case of finite strains of hyperelastic isotropic materials. By analogy with the Saint Venant–Kirchhoff material, we obtain a concise analytical representation of all required equations, which makes it fairly easy to implement arbitrary constitutive equations for a hyperelastic isotropic material. This may become a convenient tool when developing the constitutive equations for soft tissues and solving inverse problems in the study of mechanical properties of biological tissues. Although all problems considered in the present paper are given in the two-dimensional setting, our approach can also be implemented in the three-dimensional case in a similar way.

2. CONSTITUTIVE EQUATIONS FOR SOFT TISSUES

Consider the domain $\Omega^s(t) \subset \mathbb{R}^2$ occupied by an elastic body at time t. We write $\Omega_s = \Omega^s(0)$ for the domain at the initial time.

The deformation $\mathbf{x} = \varphi(\mathbf{X}, t)$ of the elastic body is defined as a vector function

$$\varphi \; : \; \Omega_s \times [0, t] \to \Omega^s(t),$$

the corresponding displacements being $\mathbf{u}(\mathbf{X}, t) := \varphi(\mathbf{X}, t) - \mathbf{X}$. We define the deformation gradient by the formula $\mathbf{F} := \partial\varphi/\partial\mathbf{X} = \mathbb{I} + \nabla_0\mathbf{u}$, where $\mathbb{I}$ is the identity matrix and $\nabla_0 := \partial/\partial\mathbf{X}$. We also write $J := \det(\mathbf{F})$ and $\nabla := \partial/\partial\mathbf{x}$. We introduce the right Cauchy–Green strain tensor $\mathbb{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ to be used as a strain measure.

The mechanical behavior of soft biological tissues is extremely nonlinear [6]. As a rule, it is described by a hyperelastic material model in the framework of finite strains, and one often uses the assumption that the material is isotropic [7, 8]. We consider a hyperelastic isotropic material in what follows.

By the definition of hyperelastic material, there exists an elastic potential $\psi(\mathbf{F})$ such that the Cauchy stress tensor σ has the form [9, p. 117]

$$\sigma = \frac{1}{J}\frac{\partial\psi(\mathbf{F})}{\partial\mathbf{F}}\mathbf{F}^{\mathrm{T}}.$$

The potential energy U of the elastic body is expressed via the elastic potential by the formula

$$U = \int\limits_{\Omega_s} \psi(\mathbf{F})\, d\Omega = \int\limits_{\Omega^s(t)} J^{-1}\psi(\mathbf{F})\, d\Omega. \tag{2.1}$$

One often expresses the elastic potential ψ as a function of the right Cauchy–Green strain tensor $\mathbb{C}$, and then

$$\sigma = \frac{2}{J}\mathbf{F}\frac{\partial\psi(\mathbb{C})}{\partial\mathbb{C}}\mathbf{F}^{\mathrm{T}}.$$

Since the material is isotropic, it follows that the elastic potential $\psi(\mathbb{C})$ is a function of the invariants of the tensor $\mathbb{C}$; i.e., $\psi(\mathbb{C}) = W_e(I_1, J)$, where $I_1 = \operatorname{tr}(\mathbb{C})$ [10].

So far, a broad variety of elastic potentials have been suggested for the description of mechanical behavior of soft tissues. There are a number of papers studying the advantages and drawbacks of some forms of constitutive equations for specific soft biological tissues (e.g., muscular tissue [7], brain and fat tissues [11], and liver [13]).

We specify some isotropic material constitutive equations that are quite often used to describe the mechanical behavior of soft biological tissues and which will be used in the subsequent analysis. The neo-Hookean model

$$W_{\mathrm{NH}} = \frac{\mu}{2}(I_1 - 2) + \frac{\mu}{2}(d(J^2 - 1) - 2(d + 1)(J - 1)) \tag{2.2}$$

is one of the simplest, most frequently used models. For example, it was used to describe the mechanical behavior of kidney and liver for a surgical simulator. However, the neo-Hookean model does not work well at medium and large strains [7]. The Gent model

$$W_{\mathrm{Gent}} = -\frac{\mu}{2}J_m \ln\left(1 - \frac{I_1 - 2}{J_m}\right) + \frac{\mu}{2}(d(J^2 - 1) - 2(d + 1)(J - 1)) \tag{2.3}$$

allows one to describe the nonlinear behavior of the load–deformation curve at large strains and can be used for soft tissues (such as arterial walls [13, 14]) containing reinforcing fibers. The Yeoh model

$$W_{\text{Yeoh}} = \sum_{i=1}^{3} c_i \left(\frac{I_1}{J} - 2\right)^i + \frac{d}{2}(J-1)^2 \tag{2.4}$$

proved to be good for describing the behavior of various soft tissues [7].

Here and in the following, μ, d, J_m, and c_i are material constants.

3. EQUILIBRIUM EQUATIONS

The equilibrium equations of an elastic solid in differential form read

$$\operatorname{div}\sigma + \mathbf{b} = 0 \quad \text{in the domain} \ \ \Omega^s(t), \tag{3.1}$$

where $\mathbf{b}$ is the bulk force density.

Let $\partial\Omega^s(t) = \Gamma_u(t)\bigcup\Gamma_t(t)$, where $\Gamma_u(t) = \overline{\Gamma}_u(t)$. Consider the mixed boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \ \ \Gamma_u(t), \qquad \sigma\mathbf{n} = \mathbf{t_0} \quad \text{on} \ \ \Gamma_t(t), \tag{3.2}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega^s(t)$ and $\bar{\mathbf{u}}$ and $\mathbf{t_0}$ are given displacements and forces on the boundaries $\Gamma_u(t)$ and $\Gamma_t(t)$, respectively.

The finite element approach to the approximate solution of Eqs. (3.1), (3.2) is based on the following weak formulation of the problem [15, p. 47 of the Russian translation]: find $\mathbf{u} \in \tilde{H}^1(\Omega^s(t))$ such that

$$\int\limits_{\Gamma_t(t)} \mathbf{t_0}\cdot\delta\mathbf{v}\,dS + \int\limits_{\Omega^s(t)} \mathbf{b}\cdot\delta\mathbf{v}\,d\Omega - \int\limits_{\Omega^s(t)} \sigma : \nabla\delta\mathbf{v}\,d\Omega = 0 \quad \text{for any} \ \ \delta\mathbf{v} \in \tilde{H}^1_0(\Omega^s(t)), \tag{3.3}$$

where

$$\tilde{H}^1(\Omega^s(t)) = \{\mathbf{v} \in H^1(\Omega^s(t)) : \mathbf{v} = \bar{\mathbf{u}} \ \text{ on } \ \Gamma_u(t)\},$$
$$\tilde{H}^1_0(\Omega^s(t)) = \{\mathbf{v} \in H^1(\Omega^s(t)) : \mathbf{v} = \mathbf{0} \ \text{ on } \ \Gamma_u(t)\}.$$

On the other hand, since there exists an elastic potential for hyperelastic materials, we can rewrite problem (3.3) in the form of the following virtual work principle [16, p. 177 of the Russian translation]: find $\mathbf{u} \in \tilde{H}^1(\Omega^s(t))$ such that

$$\delta W - \delta U = 0, \tag{3.4}$$

where the internal energy increment

$$\delta U = \int\limits_{\Omega^s(t)} \sigma : \nabla\,\delta\mathbf{u}\,dS$$

is due to the work

$$\delta W = \int\limits_{\Gamma_t(t)} \mathbf{t_0}\cdot\delta\mathbf{u}\,dS + \int\limits_{\Omega^s(t)} \mathbf{b}\cdot\delta\mathbf{u}\,d\Omega$$

of external forces applied to the boundary and in the bulk of the body. In view of the representation (2.1), Eq. (3.4) can be written as

$$\delta W - \frac{\partial}{\partial\mathbf{u}}\left(\int\limits_{\Omega_s} \psi(\mathbf{F})\,d\Omega\right)\cdot\delta\mathbf{u} = 0. \tag{3.5}$$

Thus, each of statements (3.1), (3.2), (3.3), and (3.5) can serve as a basis for the subsequent discretization. Note that the finite element solutions of problems (3.3) and (3.5) coincide [5].

4. FINITE ELEMENT DISCRETIZATION OF THE EQUILIBRIUM EQUATIONS

Consider the simplest finite element method in which the displacement field is approximated by continuous functions linear on each triangle of a given conformal triangulation of the domain Ω_s.

When using commercial finite element packages, the standard approach is to linearize Eqs. (3.3) and compute the following integral [9, p. 178] :

$$\int\limits_{\Omega^s(t)} \nabla\delta\mathbf{v} : c : \varepsilon \, d\Omega,$$

where

$$c := \frac{4}{J}\mathbf{F}\otimes\mathbf{F} : \frac{\partial^2\psi}{\partial\mathbb{C}\partial\mathbb{C}} : \mathbf{F}^{\mathrm{T}}\otimes\mathbf{F}^{\mathrm{T}}, \qquad \varepsilon := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}).$$

The tensor c is called the elasticity tensor and must necessarily be specified when using new constitutive equations in strain problems. Many of the constitutive equations suggested for the description of soft tissue behavior require special subroutines to be written for determining the tensor c.

For a standalone implementation of the finite element method, it is convenient to use barycentric coordinates. Let a mesh triangle T_P with vertices $\mathbf{P}_1, \mathbf{P}_2$, and $\mathbf{P}_3$ be transformed by the deformation $\varphi(\mathbf{X}, t)$ into a triangle T_Q with vertices $\mathbf{Q_1}, \mathbf{Q_2}$, and $\mathbf{Q_3}$. We denote the area of the original triangle T_P by A_p and the area of the deformed triangle T_Q by A_q; then $J = A_q/A_p$.

Let $(\lambda_1(\mathbf{X}), \lambda_2(\mathbf{X}), \lambda_3(\mathbf{X}))$ be the barycentric coordinates of a point $\mathbf{X}$. Then the coordinates of each point $\mathbf{X} \in T_P$ of the undeformed triangle and the corresponding point $\mathbf{x} = \varphi(\mathbf{X}) \in T_Q$ of the deformed triangle can be represented in the form

$$\mathbf{X} = \sum_{i=1}^{3} \lambda_i(\mathbf{X})\mathbf{P}_i, \qquad \mathbf{x} = \sum_{i=1}^{3} \lambda_i(\mathbf{X})\mathbf{Q}_i, \tag{4.1}$$

and the displacement $\mathbf{u}$ of the point $\mathbf{X}$ is given by

$$\mathbf{u} := \mathbf{x} - \mathbf{X} = \sum_{i=1}^{3} \lambda_i(\mathbf{X})(\mathbf{Q}_i - \mathbf{P}_i) = \sum_{i=1}^{3} \lambda_i(\mathbf{X})\mathbf{u}_i, \tag{4.2}$$

where $\mathbf{u}_i$ is the displacement of the mesh node $\mathbf{P}_i$. The interpolation properties of the barycentric coordinates ensure the simplicity of the expressions (4.1) and (4.2).

It follows from the definition $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ of the deformation gradient and relations (4.1) that

$$\mathbf{F} = \sum_{i=1}^{3} \mathbf{Q}_i \otimes \mathbf{D}_i, \tag{4.3}$$

where $\mathbf{a}\otimes\mathbf{b} := (a_1, a_2)^{\mathrm{T}}(b_1, b_2)$ and $\mathbf{D}_i := \partial\lambda_i/\partial\mathbf{X}$; i.e., the vectors $\mathbf{D}_i$ are completely determined by the geometry of the triangleT_P,

$$\mathbf{D}_i = \frac{1}{2A_p}(\mathbf{P}_{i+1} - \mathbf{P}_{i+2})^{\perp}, \qquad i = 1, 2, 3.$$

Here and in the following, we use the notation $\mathbf{P}_4 := \mathbf{P}_1$, $\mathbf{P}_5 := \mathbf{P}_2$, and $\mathbf{X}^{\perp} := (X_2, -X_1)^{\mathrm{T}}$ if $\mathbf{X} = (X_1, X_2)^{\mathrm{T}}$.

We use Eq. (4.3) to obtain the elementary expressions

$$\mathbb{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \sum_{i=1}^{3}\sum_{j=1}^{3}(\mathbf{Q_i}\cdot\mathbf{Q_j})\mathbf{D_i}\otimes\mathbf{D_j}$$

for the right Cauchy–Green strain tensor and

$$I_1 = \mathrm{tr}(\mathbb{C}) = \sum_{i=1}^{3}\sum_{j=1}^{3}(\mathbf{Q_i}\cdot\mathbf{Q_j})(\mathbf{D_i}\cdot\mathbf{D_j}) \tag{4.4}$$

for the first invariant of the tensor $\mathbb{C}$. Since our basis functions are linear, it follows that the elastic potential $\psi(\mathbf{F})$ is constant on each triangle, and the contribution U_p of the triangle T_P to internal energy (the triangle strain energy) is given by

$$U_p = A_p\psi(\mathbf{G})$$

according to (2.1), where $\mathbf{G}$ is an arbitrary point of T_P.

Now we can use formulation (3.3) or (3.5) for the approximate solution of the strain problem. The only important difference between the finite-element solutions of problems (3.3) and (3.5) is in the computation of the interior energy integrals. The integrals related to the work of external forces are exactly the same for both formulations and hence will not be discussed here.

Consider the standard finite element approach based on statement (3.3). The domain $\Omega^s(t)$ is determined by the unknown displacement field $\mathbf{u}$, and so problem (3.3) is written in practice in the coordinates $\mathbf{X}$ of the original domain Ω_s. We are interested in the last integral in formula (3.3), which is written in the form

$$\int\limits_{\Omega^s(t)} \sigma : \nabla\delta\mathbf{v}\,d\Omega = \int\limits_{\Omega_s} J\sigma\mathbf{F}^{-T} : \nabla_0\delta\mathbf{v}\,d\Omega,$$

and then the integral over Ω_s is replaced by the sum of integrals over the triangles T_P forming the triangulation of Ω_s, the function $\delta\mathbf{v}$ being assumed to belong to the class $\tilde{H}^1_0(\Omega_s)$ with the preimage of the boundary $\Gamma_u(t)$ in the initial configuration.

Consider the last integral for the hyperelastic materials described in Section 2. Since $\delta\mathbf{v} = \sum_{i=1}^{3}\lambda_i(\mathbf{X})\delta\mathbf{v}_i$ and $\nabla_0\delta\mathbf{v} = \sum_{i=1}^{3}\delta\mathbf{v}_i\otimes\nabla_0\lambda_i(\mathbf{X})$ on the triangle T_P, it follows in view of the identity $J\sigma\mathbf{F}^{-T} : (\delta\mathbf{v}_i\otimes\nabla_0\lambda_i) = \delta\mathbf{v}_i\cdot J\sigma\mathbf{F}^{-T}\nabla_0\lambda_i$ that

$$\sum_{i=1}^{3}\delta\mathbf{v}_i\cdot\int\limits_{T_P} J\sigma\mathbf{F}^{-T}\nabla_0\lambda_i\,dT = \sum_{i=1}^{3}\delta\mathbf{v}_i\cdot\int\limits_{T_P}(\mu(\nabla_0\mathbf{u}-\hat{\nabla}_0\mathbf{u}) + \mu d(J-1)(\mathbb{I}+\hat{\nabla}_0\mathbf{u}))\nabla_0\lambda_i\,dT \tag{4.5}$$

for the neo-Hookean material (2.2),

$$\begin{aligned}&\sum_{i=1}^{3}\delta\mathbf{v}_i\cdot\int\limits_{T_P} J\sigma\mathbf{F}^{-T}\nabla_0\lambda_i\,dT\\ &\quad= \sum_{i=1}^{3}\delta\mathbf{v}_i\cdot\int\limits_{T_P}\Bigg(\mu(\nabla_0\mathbf{u}-\hat{\nabla}_0\mathbf{u}) + \mu d(J-1)(\mathbb{I}+\hat{\nabla}_0\mathbf{u}) + \mu\frac{(2\nabla_0\cdot\mathbf{u}+\nabla_0\mathbf{u}:\nabla_0\mathbf{u})(\mathbb{I}+\nabla_0\mathbf{u})}{J_m - 2\nabla_0\cdot\mathbf{u}-\nabla_0\mathbf{u}:\nabla_0\mathbf{u}}\Bigg)\nabla_0\lambda_i\,dT\end{aligned} \tag{4.6}$$

for the Gent model (2.3), and

$$\begin{aligned}&\sum_{i=1}^{3}\delta\mathbf{v}_i\cdot\int\limits_{T_P} J\sigma\mathbf{F}^{-T}\nabla_0\lambda_i\,dT\\ &\quad= \sum_{i=1}^{3}\delta\mathbf{v}_i\cdot\int\limits_{T_P}\Bigg(\bigg(\frac{2W_1}{J}-\frac{I_1W_1}{J^2}+d(J-1)\bigg)\mathbb{I} + \frac{2W_1}{J}\nabla_0\mathbf{u} - \frac{I_1W_1}{J^2}\hat{\nabla}_0\mathbf{u} + d(J-1)\hat{\nabla}_0\mathbf{u}\Bigg)\nabla_0\lambda_i\,dT\end{aligned} \tag{4.7}$$

for the Yeoh model (2.4), where we have introduced the following notation:

$$\hat{\nabla}_0\mathbf{w} := \begin{pmatrix} \partial w_2/\partial X_2 & -\partial w_2/\partial X_1 \\ -\partial w_1/\partial X_2 & \partial w_1/\partial X_1 \end{pmatrix}, \quad \mathbf{w} = (w_1, w_2)^{\mathrm{T}}, \quad W_1 = c_1 + 2c_2(I_1/J-2) + 3c_3(I_1/J-2)^2.$$

Let us apply the concept suggested for the Saint Venant–Kirchhoff material in [5] to an arbitrary hyperelastic isotropic material. One can obtain equations for the new coordinates of the mesh nodes from Eq. (3.5). Consider the contribution of each triangle containing the ith mesh node to the nodal forces. Let $\mathbf{F}_i(T_P)$ and $\mathbf{F}_{i,\mathrm{ext}}(T_P)$ be the elastic force and the external force, respectively, at the vertex of a triangle T_P at the ith mesh node; then

$$\mathbf{F}_i(T_P) = -\frac{\partial U_p}{\partial \mathbf{Q}_i}, \qquad \mathbf{F}_{i,\mathrm{ext}}(T_P) = \int\limits_{\Gamma_t^e(t)} \mathbf{t_0}\lambda_i\, dS + \int\limits_{T_Q} \mathbf{b}\lambda_i\, d\Omega.$$

We sum over neighboring triangles and obtain the static equilibrium equation for the ith mesh node in the form

$$\sum_{T_p\in S_i} (\mathbf{F}_i(T_p) + \mathbf{F}_{i,\mathrm{ext}}(T_P)) = 0, \tag{4.8}$$

where S_i is the set of triangles containing the ith mesh node. Thus, the following theorem holds.

Theorem. *For an isotropic hyperelastic material, one has* $\psi(\mathbf{G}) = W_e(I_1, J)$ *and the elastic forces at the ith vertex of the triangle are given by the expression*

$$\mathbf{F}_i = -\frac{\partial U_p}{\partial \mathbf{Q}_i} = -A_p\left(\frac{\partial W_e}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{Q}_i} + \frac{\partial W_e}{\partial J}\frac{\partial J}{\partial \mathbf{Q}_i}\right), \tag{4.9}$$

$$\frac{\partial I_1}{\partial \mathbf{Q}_i} = 2\sum_{n=1}^{3}(\mathbf{D_n}\cdot\mathbf{D}_i)\mathbf{Q}_n^{\mathrm{T}}, \tag{4.10}$$

$$\frac{\partial J}{\partial \mathbf{Q}_i} = \frac{1}{2A_p}(\mathbf{Q}_{i+1} - \mathbf{Q}_{i+2})^{\perp}, \qquad i = 1, 2, 3. \tag{4.11}$$

The derivatives $\partial W_e/\partial I_1$ and $\partial W_e/\partial J$ are completely determined by the form of the constitutive equations.

If necessary, one can obtain analytical formulas for $\partial\mathbf{F}_i/\partial\mathbf{Q}_j$. Let us find the expression for the elastic forces $\mathbf{F}_i$ in our hyperelastic models.

Corollary. *One has*

$$\mathbf{F}_i(T_p) = -A_p\frac{\mu}{2}\left(\frac{\partial I_1}{\partial \mathbf{Q}_i} + (2Jd - 2(d+1))\frac{\partial J}{\partial \mathbf{Q}_i}\right) \tag{4.12}$$

for the neo-Hookean material (2.2),

$$\mathbf{F}_i(T_p) = -A_p\frac{\mu}{2}\left(\frac{J_m}{J_m - (I_1 - 2)}\frac{\partial I_1}{\partial \mathbf{Q}_i} + (2Jd - 2(d+1))\frac{\partial J}{\partial \mathbf{Q}_i}\right) \tag{4.13}$$

for the Gent model (2.3), *and*

$$\mathbf{F}_i(T_p) = -A_p\left(\frac{W_1}{J}\frac{\partial I_1}{\partial \mathbf{Q}_i} - \frac{I_1W_1}{J^2}\frac{\partial J}{\partial \mathbf{Q}_i} + d(J-1)\frac{\partial J}{\partial \mathbf{Q}_i}\right) \tag{4.14}$$

for the Yeoh model (2.4), *where I_1, $\partial I_1/\partial \mathbf{Q}_i$, and $\partial J/\partial \mathbf{Q}_i$ are defined by Eqs.* (4.4), (4.10), *and* (4.11), *respectively.*

Note that the expressions (4.12), (4.13), and (4.14) are much more concise and convenient for computations than the corresponding expressions (4.5), (4.6), and (4.7). Thus, the mesh static equilibrium equations for deformable soft biological tissues are simpler for statement (3.5). The solution of the nonlinear system (4.8) can be obtained either by the classical Newton method (the corresponding Jacobian can be written out in both cases) or by a Jacobian-free Newton–Krylov method [17, 18].

5. NUMERICAL EXPERIMENTS

Consider the problem on the uniaxial extension of a square membrane by a force P. The deformation has the form

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2. \tag{5.1}$$

[Here $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$ and $\mathbf{X} = (X_1, X_2)^{\mathrm{T}}$.]

The numbers λ_1 and λ_2 are called principal stretch ratios.

For the material parameters, we use the values for a human artery [13], which are

$$\mu = 3 \times 10^3\,\mathrm{N/m}, \qquad J_m = 2.3, \qquad d = 10,\ 10^2,\ 10^3.$$

For the Yeoh model, one possible set of values is [19]

$$c_1 = 0.441 \times 10^3\,\mathrm{N/m}, \qquad c_2 = 0.437 \times 10^3\,\mathrm{N/m}, \qquad c_3 = 0.885 \times 10^3\,\mathrm{N/m}, \qquad d = 10^6\,\mathrm{N/m}.$$

The membrane dimensions are taken to be 1 cm × 1 cm. The membrane thickness for determining the material constants is taken to be 1 mm.

The principal stretch ratios λ_1 and λ_2 obtained by solving system (4.8) for the three materials are given in the table. They coincide with the solution obtained with the use of the weak formulation (3.3) and semi-analytical methods. Since the solution (5.1) is linear, it follows that the approximation error is zero regardless of the triangulation.

6. CONCLUSION

We have described an approach to strain analysis of hyperelastic isotropic materials. A distinguishing feature of this approach is a concise form of the equations, which makes the problem less computationally intensive and provides a relatively easy implementation of arbitrary constitutive equations for an isotropic hyperelastic model used to describe the mechanical behavior of soft biological tissues. Although we have only considered static problems in the two-dimensional setting, our approach can be applied in a similar way to dynamic problems or in the three-dimensional

The principal stretch ratios λ_1 and λ_2 obtained by the method suggested in [5] for various parameter values for the neo-Hookean material and the Gent and Yeoh models.

P, N	d	neo-Hookean material		Gent model		Yeoh model	
		λ_1	λ_2	λ_1	λ_2	λ_1	λ_2
1	10	0.99252	1.00920	0.99246	1.00912	–	–
1	100	0.99178	1.00845	0.99178	1.00845	–	–
1	1000	0.99171	1.00838	0.99171	1.00837	–	–
1	10^6 N/m	–	–	–	–	0.97226	1.02858
5	10	0.96306	1.04665	0.96298	1.04578	–	–
5	100	0.95962	1.04291	0.95975	1.04273	–	–
5	1000	0.95927	1.04254	0.95939	1.04240	–	–
5	10^6 N/m	–	–	–	–	0.88451	1.13085

setting. One restriction of our approach is that only linear finite elements are used. Note that analytical expressions similar to (4.9) for the force $\mathbf{F}_i$ can be obtained with the use of higher-order elements for the class of hyperelastic isotropic materials linearly depending on the first invariant I_1. The present paper uses slightly compressible material models; this is a conventional approach to mathematical modeling of soft tissue deformation, although all experimental data are processed under the assumption that soft tissues are incompressible. The approach can be further developed to take into account the anisotropy of tissues and study modeling of incompressible materials.

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