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ORDINARY DIFFERENTIAL EQUATIONS

Inverse Sturm–Liouville Problem with Spectral Polynomials in Nonsplitting Boundary Conditions

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Abstract—Theorems on the unique reconstruction of a Sturm–Liouville problem with spectral polynomials in nonsplitting boundary conditions are proved. Two spectra and finitely many eigenvalues (one spectrum and finitely many eigenvalues for a symmetric potential) of the problem itself are used as the spectral data. The results generalize the Levinson uniqueness theorem to the case of nonsplitting boundary conditions containing polynomials in the spectral parameter. Algorithms and examples of solving relevant inverse problems are also presented.

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1. INTRODUCTION

By L we denote the Sturm–Liouville problem

$$-y'' + q(x)y = \lambda y, \tag{1}$$

$$U_1(y) = y'(0) - hy(0) + a(\lambda)y(\pi) = 0, \tag{2}$$

$$U_2(y) = y'(\pi) + H(\lambda)y(\pi) + b(\lambda)y(0) = 0, \tag{3}$$

where $q(x) \in \mathcal{L}_2(0,\pi)$, $H(\lambda)$, $a(\lambda)$, and $b(\lambda)$ are complex-coefficient polynomials in the spectral parameter.

Such problems arise in technical physics, mechanics [1, pp. 18–53], flow-duct acoustics [2], electrodynamics [3], and other fields. These problems were studied in detail in [4–6].

The inverse problem for L that contains arbitrary degree polynomials $a(\lambda)$ and $b(\lambda)$ has not been considered before. Special cases of the problem L were predominantly studied for splitting and nonsplitting boundary conditions of the form

$$V_1(y) = a_{11}y(0) + y'(0) + a_{13}y(\pi) = 0, \tag{4}$$

$$V_2(y) = a_{21}y(0) + a_{23}y(\pi) + y'(\pi) = 0. \tag{5}$$

The Sturm–Liouville inverse problem for L for $a(\lambda) \equiv b(\lambda) \equiv 0$ and $H(\lambda) = H = \text{const}$ was considered for the first time in works by Ambartsumyan and Borg and Levinson [7]. It has been comprehensively investigated by now (see [8–10]).

Very few works were devoted to the inverse problem for L with splitting boundary conditions $[a(\lambda) \equiv b(\lambda) \equiv 0]$ and with a polynomial $H(\lambda)$ (see [11]). Freiling and Yurko [11] studied a non-self-adjoint problem with splitting boundary conditions in which the polynomials in the spectral parameter are the coefficients multiplying $y(0)$, $y'(0)$, $y(\pi)$, and $y'(\pi)$. It was shown that this problem can be uniquely reconstructed based on one of the following three sets.

1. The Weil function and the zeros of the polynomials in one of the boundary conditions.

2. The spectrum and residues of the Weil function at spectrum points and the zeros of the polynomials in one of the boundary conditions.

3. Two spectra and the zeros of the polynomials in one of the boundary conditions.

The inverse problem in which nonsplitting boundary conditions do not contain polynomials in the spectral parameter was studied by Stankevich, Sadovnichii, Yurko, Marchenko, Plaksina, Gasymov, Guseinov, Nabiev, and others. A self-adjoint problem with periodic [$y(0) = y'(\pi) = 0$, $y'(0) = y'(\pi) = 0$] or antiperiodic [$y(0) = -y'(\pi) = 0$, $y'(0) = -y'(\pi) = 0$] boundary conditions was originally considered. Sadovnichii [12] started studying an inverse nonself-adjoint Sturm–Liouville problem (1), (4), (5) with nonsplitting boundary conditions. He showed that using three spectra as well as two sequences of weight numbers and two sequences of residues of specific functions suffices for the unique reconstruction of the nonself-adjoint Sturm–Liouville problem with nonsplitting boundary conditions, with these spectral data being used essentially. In [13], an example is provided that shows that if auxiliary problems are selected as in [12], the spectrum of the original problem, the spectra of two auxiliary problems, and two sequences of weight numbers may not be sufficient for the reconstruction of the boundary value problem. Therefore, the requirement on using the residues of specific functions is essential, and the theorem proved in [12] is, in a sense, unimprovable.

Attempts were made later to choose reconstructed or auxiliary problems so as to decrease the amount of spectral data needed for the reconstruction (see [14]). In particular, it was shown that the auxiliary problems can be selected so that to find the unique solution of the inverse problem (1), (4), (5) it suffices to use only the spectra of two auxiliary problems and two eigenvalues of the original Sturm–Liouville problem. This set is minimum among the previously used sets of spectral data. In addition, the obtained theorems have generalized the Borg uniqueness theorem and the Gasymov and Levitan solvability theorem [14] to the case of nonself-adjoint problems with nonsplitting boundary conditions.

In the present paper, we will obtain theorems on the unique reconstruction of the problem L equipped with symmetric and nonsymmetric potentials and polynomials $a(\lambda)$ and $b(\lambda)$ that enter the boundary conditions, with the problem L not necessarily being self-adjoint. We use only two spectra and finitely many eigenvalues of the problem L in the case of an arbitrary potential and one spectrum and finitely many eigenvalues of the problem L in the case of a symmetric potential [for $H(\lambda) \equiv h$] as the spectral data for the reconstruction of the problem L. The results generalize relevant theorems on the unique reconstruction of the Sturm–Liouville inverse problem with splitting boundary conditions.

2. SPECTRUM OF THE PROBLEM

Theorem 1. *If $h + H(\lambda) \neq 0$, then the spectrum of problem* (1)–(3) *is a countable set.*

Proof. Let A denote the matrix of the boundary conditions (2), (3)

$$A = \begin{Vmatrix} -h & 1 & a(\lambda) & 0 \\ b(\lambda) & 0 & H(\lambda) & 1 \end{Vmatrix},$$

and M_{ij} stand for its minors consisting of the ith and jth columns.

Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be linearly independent solutions of Eq. (1) satisfying the conditions

$$y_1(0,\lambda) = 1, \qquad y_1'(0,\lambda) = 0, \qquad y_2(0,\lambda) = 0, \qquad y_2'(0,\lambda) = 1. \tag{6}$$

The eigenvalues of problem (1)–(3) are the roots of the entire function (see [8, pp. 33–36])

$$\begin{aligned}\Delta(\lambda) &= M_{12} + M_{34} + M_{32}y_1(\pi,\lambda) + M_{42}y_1'(\pi,\lambda) + M_{13}y_2(\pi,\lambda) + M_{14}y_2'(\pi,\lambda) \\ &= a(\lambda) - b(\lambda) - H(\lambda)y_1(\pi,\lambda) - y_1'(\pi,\lambda) - (hH(\lambda) + a(\lambda)b(\lambda))y_2(\pi,\lambda) - hy_2'(\pi,\lambda).\end{aligned} \tag{7}$$

For the determinant $\Delta(\lambda)$, we have the following two possible cases [9, p. 27] :

1. $\Delta(\lambda) \equiv 0$, then each number λ is an eigenvalue;

2. $\Delta(\lambda) \not\equiv 0$, then there are at most countable many eigenvalues without limit points.

We have the asymptotic formulas [9, pp. 62–65]

$$\begin{aligned} y_1(x,\lambda) &= \cos(sx) + s^{-1}u(x)\sin(sx) + O(s^{-2}),\\ y_2(x,\lambda) &= s^{-1}\sin(sx) - s^{-2}u(x)\cos(sx) + O(s^{-3}),\\ y_1'(x,\lambda) &= -s\sin(sx) + u(x)\cos(sx) + O(s^{-1}),\\ y_2'(x,\lambda) &= \cos(sx) + s^{-1}u(x)\sin(sx) + O(s^{-2}), \end{aligned} \tag{8}$$

where

$$u(x) = \frac{1}{2}\int\limits_0^x q(t)\,dt$$

for a sufficiently large $\lambda = s^2 \in \mathbb{R}$.

It follows from (8) that the spectrum of problem (1)–(3) is the empty set only if $M_{12} + M_{34} \equiv C =$ const $\neq 0$ and

$$M_{32}y_1(\pi,\lambda) + M_{42}y_1'(\pi,\lambda) + M_{13}y_2(\pi,\lambda) + M_{14}y_2'(\pi,\lambda) \equiv 0. \tag{9}$$

It follows from relations (7) and (9) that $h + H(\lambda) \equiv 0$. However, by the assumption of the theorem, $h + H(\lambda) \neq 0$. Consequently, the spectrum is nonempty.

The characteristic determinant $\Delta(\lambda)$ of problem (1)–(3) cannot be the identical zero as otherwise relations (7) and (9) would imply that $h + H(\lambda) \equiv 0$, which contradicts the assumptions of the theorem.

It follows from (8) that the spectrum of problem (1)–(3) can be finite if and only if $M_{12} + M_{34}$ is a polynomial in the spectral parameter and condition (9) is satisfied. From relations (7) and (9), we obtain the identity $h + H(\lambda) \equiv 0$, which contradicts the assumption of the theorem.

These contradictions demonstrate that the spectrum is countable. The proof of the theorem is complete.

The following assertion can be proved similarly.

Theorem 2. *If $H(\lambda) \equiv h$ and the degree of the polynomial $a(\lambda)b(\lambda)$ is greater than unity, the spectrum of problem* (1)–(3) *is a countable set.*

3. STATEMENT OF INVERSE PROBLEMS

Let functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be solutions of Eq. (1) that satisfy the initial conditions

$$\varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = h, \tag{10}$$

$$\psi(\pi,\lambda) = 1, \qquad \psi'(\pi,\lambda) = -H(\lambda). \tag{11}$$

Let us introduce a function

$$\chi(\lambda) = \varphi'(x,\lambda)\psi(x,\lambda) - \varphi(x,\lambda)\psi'(x,\lambda)$$

that is independent of $x \in [0,\pi]$.

The function $\chi(\lambda)$ is an entire function and its zeros coincide with the eigenvalues μ_n of the problem L_1 for Eq. (1).

Problem $\mathbf{L_1}$.

$$-y'' + q(x)y = \lambda y, \qquad y'(0) - hy(0) = 0, \qquad y'(\pi) + H(\lambda)y(\pi) = 0.$$

By $\chi_1(\lambda)$ and $M(\lambda)$ we denote functions

$$\chi_1(\lambda) = \psi(0,\lambda), \qquad M(\lambda) = -\frac{\chi_1(\lambda)}{\chi(\lambda)}.$$

The function $M(\lambda)$ is the Weil function of the problem L_1 [11].

The function $\chi_1(\lambda)$ is a characteristic function of the problem for Eq. (1) with the boundary conditions

$$y(0) = 0, \qquad y'(\pi) + H(\lambda)y(\pi) = 0 \tag{12}$$

and with the eigenvalues ν_n. The problem (1), (12) will be denoted by L_2 in what follows.

The residues of the Weil function $M(\lambda)$ at the points $\lambda = \mu_n$ [11] will be denoted by M_n, viz.,

$$M_n = \operatorname*{Res}_{\lambda=\mu_n} M(\lambda) = -\frac{\chi_1(\lambda)}{\chi(\lambda)}.$$

The inverse problems for L are formulated as follows.

Inverse problem 1. Given the eigenvalues of the problems L, L_1, and L_2, find $q(x)$, $H(\lambda)$, $a(\lambda)$, and $b(\lambda)$ (reconstruct the problem L).

Inverse problem 2. Given the eigenvalues of the problem L and the Weil function $M(\lambda)$ of the problem L_1, find $q(x)$, $H(\lambda)$, $a(\lambda)$, and $b(\lambda)$ (reconstruct the problem L).

Inverse problem 3. Given the eigenvalues of the problems L, L_1, and M_n, find $q(x)$, $H(\lambda)$, $a(\lambda)$, and $b(\lambda)$ (reconstruct the problem L).

We will also consider an inverse problem for a special case of the problem L. We consider the following problem.

Problem $\mathbf{L}_0$.

$$-y'' + q(x)y = \lambda y, \qquad y'(0) - hy(0) = 0, \qquad y'(\pi) + hy(\pi) = 0, \qquad h \in \mathbb{R}.$$

Inverse problem 4. Let the eigenvalues of the problems L and L_0 be known, let $H(\lambda) \equiv h$, let the condition $q(x) = q(x-\pi)$ be satisfied almost everywhere, and let $q(x)$ be a real-valued function. Find $q(x)$, h, $a(\lambda)$, and $b(\lambda)$, i.e., reconstruct the problem L.

4. UNIQUENESS THEOREMS FOR THE SOLUTION OF INVERSE PROBLEMS FOR L WITH THE USE OF THE ENTIRE SPECTRUM OF THE PROBLEM L

Below we prove uniqueness theorems for the above-posed problems 1, 2, and 3.

Theorem 3. *Let the degree of the polynomial $H(\lambda)$ be not less than unity and let the polynomials $a(\lambda)$ and $b(\lambda)$ be of different evenness. Then the problems* L, L_1, *and* L_2 *can be uniquely reconstructed based on the eigenvalues of the problems* L *and* L_1 *and the residues M_n.*

Theorem 4. *Let the assumptions of Theorem* 3 *be valid. Then the problems* L, L_1, *and* L_2 *can be uniquely reconstructed based on the eigenvalues of the problem* L *and the Weil function $M(\lambda)$ of the problem* L_1.

Theorem 5. *Let the assumptions of Theorem* 3 *be satisfied. Then the problems* L, L_1, *and* L_2 *can be uniquely reconstructed based on three spectra* (*namely, based on the eigenvalues of the problems* L, L_1, *and* L_2).

Proof of Theorems 3–5. Since $h + H(\lambda) \neq 0$, it follows that the spectrum of the problem L is countable. The uniqueness of reconstruction of the function $q(x)$, the coefficient h, and the polynomial $H(\lambda)$ follows from [11], where it was shown that the problem L_1 can be uniquely reconstructed based on the following three data sets: (i) the spectra of the problems L_1 and L_2; (ii) the Weil function $M(\lambda)$ of the problem L_1; (iii) the eigenvalues of the problem L_1 and the residues M_n. It remains to prove the uniqueness of the reconstruction of the polynomials $a(\lambda)$ and $b(\lambda)$ based on all the eigenvalues of the problem L. Suppose that there exist two pairs of polynomials $(a(\lambda), b(\lambda))$ and $(\widetilde{a}(\lambda), \widetilde{b}(\lambda))$ for which the corresponding problems (1)–(3) have identical spectra. It follows from the asymptotic representations (8) that the characteristic function is an entire function of order 1/2. Then it follows from the Hadamard factorization theorem that the characteristic determinants $\Delta(\lambda)$ and $\widetilde{\Delta}(\lambda)$ for problems (1)–(3) with different pairs of polynomials $(a(\lambda), b(\lambda))$ and $(\widetilde{a}(\lambda), \widetilde{b}(\lambda))$ are related by the identity

$$\Delta(\lambda) - C\widetilde{\Delta}(\lambda) \equiv 0.$$

This, together with relations (7) and (8) and the inequality $H(\lambda) \neq 0$, implies that $C = 1$ and $a(\lambda) \equiv \widetilde{a}(\lambda)$, $b(\lambda) \equiv \widetilde{b}(\lambda)$. Consequently, the problems L, L_1, and L_2 can be uniquely reconstructed. The proof of the theorems is complete.

5. UNIQUENESS THEOREMS FOR THE SOLUTION OF INVERSE PROBLEMS FOR L WITH THE USE OF FINITELY MANY EIGENVALUES OF THE PROBLEM L

Now we show that knowing the entire countable set of eigenvalues is a redundant requirement for the unique reconstruction of the problem L. For the unique reconstruction of the problem L, it suffices to know the spectra of two problems L_1 and L_2 and finitely many eigenvalues of the problem L.

Let the problem L_1 be uniquely reconstructed. We denote the unknown coefficients of the polynomial $a(\lambda)$ by x_i, $i = 1, \dots, m_1$, the unknown coefficients of the polynomial $b(\lambda)$ by x_i, $i = 1 + m_1, \dots, m_2 + m_1$, and different products of the coefficients of the polynomial $a(\lambda)b(\lambda)$ by x_i, $i = 1 + m_1 + m_2, \dots, m_3 + m_2 + m_1$.

By substituting $m_1 + m_2 + m_3$ eigenvalues of problem (1)–(3) in the characteristic determinant (7), we obtain the system of $m_1 + m_2 + m_3$ equations for $m_1 + m_2 + m_3$ unknowns x_i. We denote the determinant of this system by D. It follows from the Kramer theorem that if $D \neq 0$, then all the values x_i and, therefore, the polynomials $a(\lambda)$ and $b(\lambda)$ are uniquely defined. Consequently, the following assertions hold true.

Theorem 6. *If the polynomials $a(\lambda)$ and $b(\lambda)$ occurring in the boundary conditions of problem* (1)–(3) *are polynomials of different evenness, then the problem* L *can be uniquely reconstructed based on $m_1 + m_2 + m_3$ eigenvalues of the problem* L, *the spectrum of the problem* L_1, *and the residues M_n.*

Theorem 7. *If the polynomials $a(\lambda)$ and $b(\lambda)$ occurring in the boundary conditions of problem* (1)–(3) *are polynomials of different evenness, then the problem* L *can be uniquely reconstructed bases on $m_1 + m_2 + m_3$ eigenvalues of the problem* L *and the Weil function $M(\lambda)$ of the problem* L_1.

Theorem 8. *If the polynomials $a(\lambda)$ and $b(\lambda)$ occurring in the boundary conditions of problem* (1)–(3) *are polynomials of different evenness, then the problem* L *can be uniquely reconstructed based on the spectra of the problems* L_1 *and* L_2 *and $m_1 + m_2 + m_3$ eigenvalues of the problem* L *such that $D \neq 0$.*

6. UNIQUENESS THEOREMS FOR THE SOLUTION OF INVERSE PROBLEMS FOR L IN THE CASE $q(x) = q(x - \pi)$; $H(\lambda) \equiv h$

Levinson [7] considered the Sturm–Liouville problem L_0 with a symmetric potential.

The following assertion was proved for this problem.

Levinson theorem. *If $q(x) = q(x - \pi)$, then the function $q(x)$ and the number h are uniquely determined based on the spectrum of the problem* L_0.

Below we generalize this theorem to the case of nonsplitting boundary conditions with spectral polynomials in the boundary conditions.

Theorem 9. *Let the conditions $q(x) = q(x - \pi)$ and $H(\lambda) \equiv h$ be satisfied almost everywhere, let $a(\lambda)$ and $b(\lambda)$ be polynomials of different evenness, with the degree of the polynomial $a(\lambda)b(\lambda)$ being known and above unity. Then the problems* L *and* L_0 *are uniquely reconstructed based on two spectra* (*namely, the eigenvalues of the problems* L *and* L_0).

Proof. It follows from Theorem 2 that the spectrum of the problem L is countable. The uniqueness of the reconstruction of the function $q(x)$ and the number h follows from the Levinson theorem [7]. The uniqueness of the reconstruction of the polynomials $a(\lambda)$ and $b(\lambda)$ on the basis of all the eigenvalues of the problem L can be proved similar to the proof of Theorems 3–5.

Now we show that if $q(x) = q(x - \pi)$ and $H(\lambda) \equiv 0$, then knowing the entire countable set of eigenvalues is a redundant requirement for the unique reconstruction of the problem L. For the unique reconstruction of the problem L, it suffices to know the spectrum of the problem L_0 and finitely many eigenvalues of the problem L.

Let the problem L_0 be uniquely reconstructed based on its spectrum. Following the lines of derivation of Theorems 6–8, we find that the polynomials are uniquely defined. Consequently, the following assertion is valid.

Theorem 10. *Let the conditions $q(x) = q(x - \pi)$ and $H(\lambda) \equiv h$ be satisfied almost everywhere, and let the polynomials $a(\lambda)$ and $b(\lambda)$ occurring in the boundary conditions of problem* (1)–(3) *be polynomials of different evenness. Then the problem* L *can be uniquely reconstructed based on the spectrum of the problem* L_0 *and $m_1 + m_2 + m_3$ eigenvalues of the problem* L *such that $D \neq 0$.*

Remark. Theorems 3–10 are valid not only for polynomials of different evenness but also when the exponents of the degrees of the polynomials $a(\lambda)$ and $b(\lambda)$ do not coincide and are known. If this condition fails, then the reconstruction problem (1)–(3) has two solutions.

7. ALGORITHMS FOR SOLVING INVERSE PROBLEMS FOR L

Theorems 6–8 and 10 make it possible to construct algorithms for solving the inverse problems 1–4.

Algorithm for Solving Inverse Problem 1

1. The problem L_1, i.e., the function $q(x)$, the coefficient h, and the polynomial $H(\lambda)$, is reconstructed using the eigenvalues of the problem L_1 and the residues M_n of the Weil function and following the lines of algorithm 1 in [11, p. 16].

2. The determined function $q(x)$ is used to write Eq. (1) and find linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of Eq. (1) that satisfy conditions (6).

3. The found solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ are substituted in the characteristic determinant (7).

4. $m_1 + m_2 + m_3$ eigenvalues of the problem L are substituted into the characteristic determinant to obtain a system of $m_1 + m_2 + m_3$ equations for $m_1 + m_2 + m_3$ unknowns.

5. If the determinant D of the resultant system is nonzero, then, by using the Kramer formulas, the coefficients x_i, $i = 1, \dots, m_1$, of the polynomial $a(\lambda)$ and the coefficients x_i, $i = m_1 + 1, \dots, m_1 + m_2$, of the polynomial $b(\lambda)$ are determined, i.e., the reconstruction of the problem L is now complete.

Algorithm for Solving Inverse Problem 2

1. The Weil function $M(\lambda)$ of the problem L_1 and algorithm 2 in [11, p. 16] are used to reconstruct the problem L_1, that is, the function $q(x)$, the coefficient h, and the polynomial $H(\lambda)$.

2. Now, the problem L can be completely reconstructed by following steps 2–5 of the algorithm for solving inverse problem 1.

Algorithm for Solving Inverse Problem 3

1. The spectra of the problems L_1 and L_2 are used to reconstruct the problem L_1, i.e., the function $q(x)$, the coefficient h, and the polynomial $H(\lambda)$, following the lines of algorithm 3 in [11, p. 16].

2. Now, the problem L can be completely reconstructed by following steps 2–5 of the algorithm for solving inverse problem 1.

Algorithm for Solving Inverse Problem 4

1. The eigenvalues of the problem L_0 are used to reconstruct the problem L_0, i.e., the function $q(x)$ and the coefficient h, following the lines of methods from [7].

2. Now, the problem L can be completely reconstructed by following steps 2–5 of the algorithm for solving inverse problem 1.

Let us exemplify the above algorithms for solving the inverse problem.

Example 1. Let the polynomials $H(\lambda)$, $a(\lambda)$, and $b(\lambda)$ have the form

$$H(\lambda) = H, \qquad a(\lambda) = x_1\lambda^2, \qquad b(\lambda) = x_2\lambda + x_3\lambda^3;$$

let five eigenvalues of the problem L have the form

$$\lambda_1 = 1.1730, \qquad \lambda_2 = 3.9281, \qquad \lambda_3 = 9.0332, \qquad \lambda_4 = 25.012, \qquad \lambda_5 = 35.991;$$

let the eigenvalues μ_k of the problem L_1 be the zeros of the equation

$$\cos\sqrt{\mu}\,\pi - \sqrt{\mu}\,\sin\sqrt{\mu}\,\pi = 0; \tag{13}$$

and let the eigenvalues ν_k of the problem L_1 be the zeros of the equation

$$\cos\sqrt{\nu}\,\pi + \frac{\sin\sqrt{\nu}\,\pi}{\sqrt{\nu}\,\pi} = 0. \tag{14}$$

Let us find a solution by the algorithm for inverse problem 3.

1. By using algorithm 3 from [11, p. 16], we obtain $q(x) \equiv 0$, $h = 0$, and $H(\lambda) \equiv 1$.

2. Since $q(x) \equiv 0$, it follows that Eq. (1) has the form $-y'' = \lambda y$, and its linearly independent solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ satisfying condition (6) have the form

$$y_1(x,\lambda) = \cos(sx), \qquad y_2(x,\lambda) = \frac{\sin(sx)}{s}. \tag{15}$$

3. After substituting $y_1(x,\lambda)$ and $y_2(x,\lambda)$ in the characteristic determinant (7), we obtain

$$\Delta(\lambda) = x_1\lambda^2 - x_2\lambda - x_3\lambda^3 - \cos(s\pi) + s\sin(s\pi) - (x_4\lambda^3 + x_5\lambda^5)\,\frac{\sin(s\pi)}{s}.$$

4. By substituting $1 + 2 + 2 = 5$ eigenvalues of the problem L in the characteristic determinant, we obtain the following system of five equations for five unknowns:

$$\begin{gathered}
1.3758x_1 - 1.6138x_3 - 1.1730x_2 + 0.38429x_4 + 0.52872x_5 = -0.68685,\\
15.430x_1 - 60.610x_3 - 3.9281x_2 + 1.7342x_4 + 26.759x_5 = 1.1108,\\
81.600x_1 - 737.11x_3 - 9.0333x_2 + 4.2675x_4 + 348.23x_5 = -0.94755,\\
625.62x_1 - 15648x_3 - 25.012x_2 + 12.256x_4 + 7667.9x_5 = -0.98040,\\
1295.4x_1 - 46622x_3 - 35.991x_2 + 17.746x_4 + 22987x_5 = 1.0137.
\end{gathered}$$

5. The determinant $D = 9.4840 \times 10^6$ of that system is nonzero. Therefore, using the Kramer formula, we find the coefficient x_1 of the polynomial $a(\lambda) = x_1\lambda^2$ and the coefficients x_2 and x_3 of the polynomial $b(\lambda) = x_2\lambda + x_3\lambda^3$, i.e., $x_1 = 2.00$, $x_2 = 3.00$, and $x_3 = 4.00$.

Consequently, the desired problem L has the form

$$-y'' = \lambda y, \qquad y'(0) + 2\lambda^2 y(\pi) = 0, \qquad y'(\pi) + y(\pi) + (3\lambda + 4\lambda^3)y(0) = 0.$$

Example 2 (with polynomials of the same evenness). Let $H(\lambda) = H$ and let $a(\lambda)$ and $b(\lambda)$ be polynomials of the same evenness

$$a(\lambda) = x_1 + x_2\lambda^2, \qquad b(\lambda) = x_3 + x_4\lambda^2.$$

Seven eigenvalues of the problem L are

$$\lambda_1 = 1.0919, \qquad \lambda_2 = 3.9591, \qquad \lambda_3 = 9.0175, \qquad \lambda_4 = 15.990,$$
$$\lambda_5 = 25.006, \qquad \lambda_6 = 35.996, \qquad \lambda_7 = 49.003.$$

Let the eigenvalues μ_k of the problem L_1 be given by the zeros of Eq. (13) and let the eigenvalues ν_k of the problem L_1 be given by the zeros of Eq. (14).

Let us find a solution by the algorithm for inverse problem 3.

1–3. Using algorithm 3 from [11, p. 16], we obtain $q(x) \equiv 0$, $h = 0$, $H(\lambda) \equiv 1$ and the same characteristic determinant as in Example 1.

4 and 5. If we substitute $2 + 2 + 3 = 7$ eigenvalues of the problem L in the characteristic determinant, we obtain a system of linear equations with proportional columns. The columns that correspond to the variables x_1 and x_3 coincide with the columns that correspond to the variables x_2 and x_4. The determinant D of the system is zero. Therefore, if $a(\lambda)$ and $b(\lambda)$ are polynomials of the same evenness, the polynomials $a(\lambda)$ and $b(\lambda)$ are sought differently as a solution of a nonlinear system of equations. As a result, we obtain two solutions:

1. $x_1 = 1.00$, $x_2 = 2.00$, $x_3 = 3.00$, $x_4 = 4.00$;
2. $x_1 = -3.00$, $x_2 = -4.00$, $x_3 = -1.00$, $x_4 = -2.00$.

Therefore, the inverse identification problem L also has two solutions:

$$-y'' = \lambda y, \qquad y'(0) + (1 + 2\lambda)y(\pi) = 0, \qquad y'(\pi) + y(\pi) + (3 + 4\lambda)y(0) = 0$$

and

$$-y'' = \lambda y, \qquad y'(0) - (3 + 4\lambda)y(\pi) = 0, \qquad y'(\pi) + y(\pi) - (1 + 2\lambda)y(0) = 0.$$

Example 3. Let $H(\lambda) = h$, let $q(x)$ be a real-valued function with the condition $q(x) = q(x-\pi)$ satisfied almost everywhere, let the functions $a(\lambda)$ and $b(\lambda)$ be polynomials of the form

$$a(\lambda) = x_1\lambda^2, \qquad b(\lambda) = x_2\lambda + x_3\lambda^3;$$

let five eigenvalues of the problem L be given by the values

$$\lambda_1 = 1.1405, \quad \lambda_2 = 3.9278, \quad \lambda_3 = 9.0333, \quad \lambda_4 = 25.012, \quad \lambda_5 = 35.991;$$

and let eigenvalues μ_k of the problem L_0 be given by the zeros of the equation

$$\left(\sqrt{\mu} - \frac{1}{\sqrt{\mu}}\right)\sin\sqrt{\mu}\,\pi - 2\cos\sqrt{\mu}\,\pi = 0.$$

Let us find a solution by the algorithm for inverse problem 4.

1. The eigenvalues of the problem L_0 are used to obtain $q(x) = 0$ and $h = 1$ following the lines of methods from [7].

2. Since $q(x) = 0$, it follows that Eq. (1) has the form $-y'' = \lambda y$, and its linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ satisfying condition (6) have the form (15).

3. Substituting $y_1(x, \lambda)$ and $y_2(x, \lambda)$ in the characteristic determinant (7), we obtain

$$\Delta(\lambda) = x_1\lambda^2 - x_2\lambda - x_3\lambda^3 - 2\cos(s\pi) + s\sin(s\pi) - (1 + x_4\lambda^3 + x_5\lambda^5)\frac{\sin(s\pi)}{s}.$$

4. Substituting $1 + 2 + 2 = 5$ eigenvalues of the problem L in the characteristic determinant, we obtain the system of five equations for five unknowns:

$$\begin{aligned}
1.3007x_1 - 1.1405x_2 - 1.4834x_3 + 0.29423x_4 + 0.38271x_5 &= -1.9268,\\
15.427x_1 - 3.9278x_2 - 60.595x_3 + 1.7416x_4 + 26.868x_5 &= 2.0809,\\
81.600x_1 - 9.0333x_2 - 737.11x_3 + 4.2659x_4 + 348.10x_5 &= -1.9532,\\
625.62x_1 - 25.012x_2 - 15648x_3 + 12.256x_4 + 7667.7x_5 &= -1.9811,\\
1295.4x_1 - 35.991x_2 - 46622x_3 + 17.746x_4 + 22988x_5 &= 2.0133.
\end{aligned}$$

5. The determinant $D = 3.9545 \times 10^6$ of this system is nonzero. Therefore, by using the Kramer formula, we find the coefficient x_1 of the polynomial $a(\lambda) = x_1\lambda^2$ and the coefficients x_2 and x_3 of the polynomial $b(\lambda) = x_2\lambda + x_3\lambda^3$, i.e., $x_1 = 2.00$, $x_2 = 3.00$, and $x_3 = 4.00$.

Consequently, the desired problem L has the form

$$-y'' = \lambda y, \qquad y'(0) + 2\lambda^2 y(\pi) = 0, \qquad y'(\pi) + y(\pi) + (3\lambda + 4\lambda^3)y(0) = 0.$$

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