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ORDINARY DIFFERENTIAL EQUATIONS

# Linear Differential Operators and Operator Matrices of the Second Order

**A. G. Baskakov*, L. Yu. Kabantsova**, I. D. Kostrub***, and T. I. Smagina******

*Voronezh State University, Voronezh, 394006 Russia*
*e-mail: *anatbaskakov@yandex.ru, **dlju@yandex.ru,*
****ikostrub@yandex.ru, ****smagin@math.vsu.ru*

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**Abstract**—Linear differential operators (equations) of the second order in Banach spaces of vector functions defined on the entire real axis are studied. Conditions of their invertibility are given. The main results are based on putting a differential operator in correspondence with a second-order operator matrix and further use of the theory of first-order differential operators that are defined by the operator matrix. A general scheme is presented for studying the solvability conditions for different classes of second-order equations using second-order operator matrices. The scheme includes the studied problem as a special case.

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## 1. INTRODUCTION. MAIN RESULTS

Let $X$ be a complex Banach space with a norm $\|\cdot\|$, and let $\operatorname{End} X$ be the Banach algebra of linear bounded operators acting in $X$.

In the present paper, we consider the following function spaces: $C_b = C_b(\mathbb{R}, X)$ is the Banach space of continuous functions bounded on the real axis $\mathbb{R}$ and taking values in the space $X$, which is equipped with a norm that is defined by the relation $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|$; $C_0 = C_0(\mathbb{R}, X)$ is a closed subspace of functions from $C_b(\mathbb{R}, X)$ that tend to zero at infinity; $C_{b,u} = C_{b,u}(\mathbb{R}, X)$ is a closed subspace of uniformly continuous functions from $C_b(\mathbb{R}, X)$; $L^p = L^p(\mathbb{R}, X)$, $p \in [1, \infty]$, is the Banach space of Bochner measurable (classes of) functions that are defined on $\mathbb{R}$ with values in $X$ and have finite value of the quantity (which is used as the norm in the relevant space) $\|x\|_p = (\int_{\mathbb{R}} \|x(\tau)\|^p\, d\tau)^{1/p}$ for $p \neq \infty$, $\|x\|_\infty = \operatorname{ess\,sup}_{\tau\in\mathbb{R}} \|x(\tau)\|$ for $p = \infty$; $S^p(\mathbb{R}, X)$, $p \in [1, \infty)$, is the Stepanov space of functions that are locally $p$-integrable and measurable on $\mathbb{R}$, take values in $X$, and have finite value of the quantity $\|x\|_{S^p} = \sup_{t\in\mathbb{R}}(\int_0^1 \|x(t+s)\|^p\, ds)^{1/p}$.

By $L^1(\mathbb{R}, \mathcal{B})$ [respectively, $L^1(\mathbb{R})$ if $\mathcal{B} = \mathbb{C}$] we denote the Banach space of functions integrable on $\mathbb{R}$ with values in the complex Banach algebra $\mathcal{B}$ (respectively, with convolution of the functions standing for multiplication), while by $\widehat{f} : \mathbb{R} \to \mathcal{B}$ we denote the Fourier transform of a function $f \in L^1(\mathbb{R}, \mathcal{B})$, i.e.,

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} f(t) \exp(-i\lambda t)\, dt, \qquad \lambda \in \mathbb{R}.$$

By $\mathcal{F} = \mathcal{F}(\mathbb{R}, X)$ we denote one of the above-listed spaces.

In the Banach space $\mathcal{F} = \mathcal{F}(\mathbb{R}, X)$, we consider a second-order differential operator of the form

$$x''(t) + B_1(t)x'(t) + B_2(t)x(t) = g(t), \qquad t \in \mathbb{R}, \tag{1}$$

where $g \in \mathcal{F}$ and $B_1, B_2 : \mathbb{R} \to \operatorname{End} X$ are operator-valued functions from the space $L^\infty(\mathbb{R}, \operatorname{End} X)$.

We put any function $x \in \mathcal{F}$ in correspondence with a function $y:\ \mathbb{R} \to X^2 = X \times X$ (the norm $\|(x_1, x_2)\| = \max\{\|x_1\|, \|x_2\|\}$, $(x_1, x_2) \in X^2$ is considered in the Cartesian product $X^2$) of the form

$$y(t) = (x_1(t), x_2(t)), \qquad t \in \mathbb{R}, \qquad x_1 = x, \qquad x_2 = x'.$$

It follows from the very definition of the function $y$ that the function $x \in \mathcal{F}(\mathbb{R}, X)$ is a solution of Eq. (1) if and only if the function $y \in \mathcal{F}(\mathbb{R}, X^2)$ satisfies the equation [considered in $\mathcal{F}(\mathbb{R}, X^2)$]

$$y'(t) + \mathbb{B}(t)y(t) = f(t), \qquad t \in \mathbb{R}. \tag{2}$$

Here $f \in \mathcal{F}(\mathbb{R}, X^2)$, $f(t) = (0, g(t))$, $t \in \mathbb{R}$, and the operator-valued function $\mathbb{B}:\ \mathbb{R} \to \operatorname{End} X^2$ has the following form: each operator $\mathbb{B}(t) \in \operatorname{End} X^2$, $t \in \mathbb{R}$, is defined in $X \times X$ by the matrix $\begin{pmatrix} 0 & -I \\ B_2(t) & B_1(t) \end{pmatrix}$. By $I$ we denote the identity operator in any of the Banach spaces.

We sometimes identify the matrix of an operator acting in the Cartesian product of Banach spaces with an operator that is defined by this matrix. In addition, we use the canonical isomorphism of the Banach spaces $\mathcal{F}(\mathbb{R}, X^2)$ and $\mathcal{F}(\mathbb{R}, X) \times \mathcal{F}(\mathbb{R}, X)$.

Let us introduce some operators. By $D$ we denote an differentiation operator that acts by the rule $Dx = x'$ and has the domain

$$\mathcal{F}^{(1)} = \{x \in \mathcal{F}:\ x \text{ is an absolutely continuous function, } x' \in \mathcal{F}\},$$

while by $D_2$ we denote an operator that is defined by the relation

$$D_2 = D^2 + B_1 D + B_2. \tag{3}$$

The domain of this operator is given by the linear subspace

$$\mathcal{F}^{(2)} = \mathcal{F}^{(2)}(\mathbb{R}, X) = \{x \in \mathcal{F}^{(1)}:\ x \text{ is an absolutely continuous function, } x' \in \mathcal{F}^{(1)}\}.$$

We consider the differentiation operator

$$\mathbb{D}:\ \mathcal{F}^{(1)}(\mathbb{R}, X^2) \subset \mathcal{F}(\mathbb{R}, X^2) \to \mathcal{F}(\mathbb{R}, X^2)$$

defined by the matrix $\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$ and the operator

$$\mathbb{D}_1:\ \mathcal{F}^{(1)}(\mathbb{R}, X^2) \subset \mathcal{F}(\mathbb{R}, X^2) \to \mathcal{F}(\mathbb{R}, X^2)$$

that acts by the rule $\mathbb{D}_1 = \mathbb{D} + \mathbb{B}$. The operator $\mathbb{D}_1$ is thus defined by the matrix

$$\begin{pmatrix} D & -I \\ B_2 & D + B_1 \end{pmatrix}. \tag{4}$$

We write Eqs. (1) and (2) in the operator form as

$$D_2 x = g, \qquad g \in \mathcal{F}(\mathbb{R}, X); \qquad \mathbb{D}_1 y = f, \qquad f \in \mathcal{F}(\mathbb{R}, X^2).$$

Reducing higher-order differential equations to the corresponding first-order differential equations is widely used in the theory of differential equations.

Note that the simultaneous solvability of Eqs. (1) and (2) is obvious for the right-hand sides $f \in \mathcal{F}(\mathbb{R}, X^2)$ of a special form, namely, $f = (0, g)$. In this case, a solution $x \in \mathcal{F}(\mathbb{R}, X)$ of Eq. (1) and a solution $y = (y_1, y_2) \in \mathcal{F}(\mathbb{R}, X^2)$ of Eq. (2) are related by the formula $x = y_1$, $x' = y_2$.

A natural question arises of whether the operators

$$D_2:\ \mathcal{F}^{(2)}(\mathbb{R}, X) \subset \mathcal{F}(\mathbb{R}, X) \to \mathcal{F}(\mathbb{R}, X), \qquad \mathbb{D}_1:\ \mathcal{F}^{(1)}(\mathbb{R}, X^2) \subset \mathcal{F}(\mathbb{R}, X^2) \to \mathcal{F}(\mathbb{R}, X^2)$$

are simultaneously invertible. With the positive answer to this question, analysis of the invertibility of the second-order operator $D_2$ reduces to analyzing the invertibility conditions for the first-order differential operator $\mathbb{D}_1$. Hence we can use the results of [1–8].

The main results of the present paper are formulated in the theorems below.

**Theorem 1.** *The operator* $D_2 : \mathcal{F}^{(2)}(\mathbb{R}, X) \subset \mathcal{F}(\mathbb{R}, X) \to \mathcal{F}(\mathbb{R}, X)$ *is invertible if and only if the operator* $\mathbb{D}_1 : \mathcal{F}^{(1)}(\mathbb{R}, X^2) \subset \mathcal{F}(\mathbb{R}, X^2) \to \mathcal{F}(\mathbb{R}, X^2)$ *is invertible. If the operator* $D_2$ *is invertible, then an operator* $\mathbb{D}_1^{-1} \in \operatorname{End}\mathcal{F}(\mathbb{R}, X^2)$ *that is inverse to* $\mathbb{D}_1$ *is defined by (has the form of) the matrix*

$$\begin{pmatrix} (D-\lambda_0 I)^{-1} - D_2^{-1}((B_2+\lambda_0 B_1+\lambda_0^2 I)(D-\lambda_0 I)^{-1}+\lambda_0 I) & D_2^{-1} \\ \lambda_0(D-\lambda_0 I)^{-1} - DD_2^{-1}((B_2+\lambda_0 B_1+\lambda_0^2 I)(D-\lambda_0 I)^{-1}+\lambda_0 I) & DD_2^{-1} \end{pmatrix}, \tag{5}$$

*where* $\lambda_0$ *is an arbitrary number from* $\mathbb{C}\backslash(i\mathbb{R})$.

**Theorem 2.** *The spectrum of the operator* $D_2 : \mathcal{F}^{(2)} \subset \mathcal{F} \to \mathcal{F}$ *is independent of the choice of the function space* $\mathcal{F} = \mathcal{F}(\mathbb{R}, X)$. *In particular, the operator* $D_2$ *is simultaneously invertible in any of the considered function spaces* $\mathcal{F}(\mathbb{R}, X)$.

The assertion of Theorem 2 follows immediately from Theorem 1 and from results in [1, 7, 9] that contain the counterpart assertion for the first-order differential operators (in particular, $\mathbb{D}_1$). Estimates for the norms of the inverse operators were obtained in [9].

Results obtained in the present paper (for example, Theorem 1) are based on the following approach to studying abstract operators acting in Banach spaces. A similar scheme was used in [10] for studying the invertibility conditions of bounded operators. However, this scheme required some modification to be used for studying differential operators.

Let $\mathcal{X}$ be a complex Banach space, $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ be a linear operator with a nonempty resolvent set $\varrho(A)$, and $B_1$ and $B_2$ be operators from the algebra $\operatorname{End}\mathcal{X}$. We consider the linear operator

$$\mathcal{A} = A^2 + B_1 A + B_2 : D(A^2) \subset \mathcal{X} \to \mathcal{X} \tag{6}$$

with the domain $D(\mathcal{A}) = D(A^2) = \{x \in D(A) : Ax \in D(A)\}$.

Along with the operator $\mathcal{A}$, we define an operator $\mathbb{A} : D(\mathbb{A}) \subset \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ using the matrix $\begin{pmatrix} A & -I \\ B_2 & A+B_1 \end{pmatrix}$, i.e.,

$$\mathbb{A}x = (Ax_1 - x_2, B_2 x_1 + (A+B_1)x_2), \tag{7}$$

where $x = (x_1, x_2) \in D(\mathbb{A}) = D(A) \times D(A) \subset \mathcal{X} \times \mathcal{X}$.

**Theorem 3.** *Let* $A$ *be an invertible operator. The operator* $\mathcal{A}$ *is invertible if and only if* $\mathbb{A}$ *is an invertible operator. If* $\mathcal{A}$ *is an invertible operator, then the operator* $\mathbb{A}^{-1} \in \operatorname{End}\mathcal{X}^2$ *inverse to* $\mathbb{A}$ *is defined by the matrix*

$$\begin{pmatrix} A^{-1} - \mathcal{A}^{-1}B_2A^{-1} & \mathcal{A}^{-1} \\ -A\mathcal{A}^{-1}B_2A^{-1} & A\mathcal{A}^{-1} \end{pmatrix}, \tag{8}$$

*i.e.,*

$$\mathbb{A}^{-1}x = ((A^{-1} - \mathcal{A}^{-1}B_2A^{-1})x_1 + \mathcal{A}^{-1}x_2, -A\mathcal{A}^{-1}B_2A^{-1}x_1 + A\mathcal{A}^{-1}x_2)$$

*for any vector* $x = (x_1, x_2) \in \mathcal{X}^2$.

If we set $\mathcal{X} = \mathcal{F}(\mathbb{R}, X)$, $A = D$ is the operator of differentiation in $\mathcal{F}(\mathbb{R}, X)$, then $\mathcal{A} = D_2$ and $\mathbb{A} = \mathbb{D}_1$, where the operators $D_2$ and $\mathbb{D}_1$ have been introduced with formulas (3) and (4), respectively. However, such an operator $A = D$ is not invertible. Consequently, Theorem 3 cannot be used directly applied to proving the simultaneous invertibility of the operators $D_2$ and $\mathbb{D}_1$. Note that the spectrum of the differentiation operator $D : \mathcal{F}^{(1)} \subset \mathcal{F} \to \mathcal{F}$ is pure imaginary (see [1, Chap. 2, Sec. 4] for $\mathcal{F} = C_b$ and [3] for other spaces mentioned above). Therefore, any number $\lambda_0 \notin i\mathbb{R}$ is a point in the resolvent set of the differentiation operator $A = D$, and $A - \lambda_0 I$ with $\lambda_0 \notin i\mathbb{R}$ is an invertible operator. This makes it possible to establish the simultaneous invertibility of the operators $D_2$ and $\mathbb{D}_1$ in the following manner.

Let us return to considering operators defined by formulas (7) and (8). Let a number $\lambda_0 \in \mathbb{C}$ be a point in the resolvent set of the operator $A$. Then the operator $\mathcal{A}$ admits the representation

$$\mathcal{A} = \widetilde{A}^2 + \widetilde{B}_1\widetilde{A} + \widetilde{B}_2,$$

where

$$\widetilde{A} = A - \lambda_0 I, \qquad \widetilde{B}_1 = B_1 + 2\lambda_0 I, \qquad \widetilde{B}_2 = B_2 + \lambda_0 B_1 + \lambda_0^2 I \tag{9}$$

and $\widetilde{A}$ is an invertible operator. The corresponding operator $\widetilde{\mathbb{A}}$ is defined by the matrix

$$\begin{pmatrix} \widetilde{A} & -I \\ \widetilde{B}_2 & \widetilde{A} + \widetilde{B}_1 \end{pmatrix},$$

i.e.,

$$\widetilde{\mathbb{A}}x = (\widetilde{A}x_1 - x_2, \widetilde{B}_2 x_1 + (\widetilde{A} + \widetilde{B}_1)x_2), \tag{10}$$

where $x = (x_1, x_2) \in D(\widetilde{\mathbb{A}}) = D(\widetilde{A}) \times D(\widetilde{A}) \subset \mathcal{X} \times \mathcal{X}$.

**Lemma 1.** *Let $\lambda_0$ be a point in the resolvent set of the operator $A$. Then the operator $\widetilde{\mathbb{A}}$ is similar to the operator $\mathbb{A}$ with the transformation operator $\mathcal{U} \in \operatorname{End}\mathcal{X}^2$ defined by the matrix $\mathcal{U} = \begin{pmatrix} I & 0 \\ -\lambda_0 I & I \end{pmatrix}$, i.e., we have*

$$\widetilde{\mathbb{A}} = \mathcal{U}\mathbb{A}\mathcal{U}^{-1}. \tag{11}$$

Therefore, the operators $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ are simultaneously invertible, and Theorem 3 can be applied to the operator $\widetilde{\mathbb{A}}$.

**Theorem 4.** *The operator $\mathcal{A}: D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is invertible if and only if $\mathbb{A}: D(A) \times D(A) \subset \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is an invertible operator. If $\mathcal{A}$ is an invertible operator, then the operator $\mathbb{A}^{-1} \in \operatorname{End}(\mathcal{X} \times \mathcal{X})$ inverse to $\mathbb{A}$ is defined by the matrix*

$$\begin{pmatrix} (A - \lambda_0 I)^{-1} - \mathcal{A}^{-1}((B_2 + \lambda_0 B_1 + \lambda_0^2 I)(A - \lambda_0 I)^{-1} + \lambda_0 I) & \mathcal{A}^{-1} \\ \lambda_0 (A - \lambda_0 I)^{-1} - A\,\mathcal{A}^{-1}((B_2 + \lambda_0 B_1 + \lambda_0^2 I)(A - \lambda_0 I)^{-1} + \lambda_0 I) & A\,\mathcal{A}^{-1} \end{pmatrix},$$

*where $\lambda_0$ is any number from $\varrho(A)$.*

**Remark 1.** Let the Banach space $\mathcal{X}$ coincide with one of the Banach spaces $l^p = l^p(\mathbb{Z}, X)$, $p \in [1, \infty]$, of two-sided sequences $x: \mathbb{Z} \to X$ of vectors from the Banach space $X$ with the norm

$$\|x\| = \|x\|_p = \Big(\sum_{n\in\mathbb{Z}} \|x(n)\|^p\Big)^{1/p}, \quad x \in l^p, \quad p \in [1, \infty); \quad \|x\| = \|x\|_\infty = \sup_{n\in\mathbb{Z}} \|x(n)\|, \quad x \in l^\infty.$$

By $A \in \operatorname{End} l^p$ we denote the displacement operator in the space $l^p$ acting by the rule $(Ax)(k) = x(k+1)$, $k \in \mathbb{Z}$, $x \in l^p$.

Let us consider a second-order finite-difference operator $\mathcal{A}$ of the form (6), where $A$ is a displacement operator in $l^p$, and let $B_1, B_2 \in \operatorname{End} l^p$ be operators of multiplication by operator functions in $l^p$ that act by the rule $(B_k x)(n) = B_k(n)x(n)$, $n \in \mathbb{Z}$, $x \in l^p$, $k = 1, 2$.

The spectrum $\sigma(A)$ of the operator $A$ coincides with the circle $\mathbb{T} = \{\lambda \in \mathbb{C}: |\lambda| = 1\}$ (see [9]) and, consequently, $\varrho(A) \neq \varnothing$. Therefore, Theorem 3 holds true for the finite-difference operator $\mathcal{A}$. Since the displacement operator $A$ is invertible, it follows that the inverse operator $\mathbb{A}^{-1}$ belongs to $\operatorname{End} l^p$ and is defined by the matrix (8). Thus, we can now use the results of [7, 10, 11].

**Remark 2.** Operators of the form (6) include the second-order integro-differential operator

$$\mathcal{A}:\ C_b^{(2)}(\mathbb{R},X)\subset C_b(\mathbb{R},X)\to C_b(\mathbb{R},X),\qquad \mathcal{A}=D^2+B_1D+B_2, \tag{12}$$

where $D$ is the differentiation operator and $B_1$ and $B_2$ are the convolution operators

$$(B_kx)(t)=(\mu_k*x)(t)=\int\limits_{\mathbb{R}} d\mu_k(\tau)x(t-\tau),\qquad t\in\mathbb{R},\qquad x\in C_b(\mathbb{R},X),$$

with operator-valued Borel measures $\mu_k$, $k=1,2$ of bounded variation. In particular, such this class of operators includes finite-difference operators $B_k$, $k=1,2$ of the form

$$(B_kx)(t)=\sum_{j=1}^{\infty}C_{j,k}(t)x(t+h_{j,k}),\qquad t\in\mathbb{R},\qquad x\in C_b(\mathbb{R},X);$$
$$C_{j,k}\in C_b(\mathbb{R},\operatorname{End}X),\qquad j\in\mathbb{N},\qquad k=1,2,$$

that satisfy the condition

$$\sum_{j=1}^{\infty}\|C_{j,k}\|_\infty<\infty,\qquad k=1,2.$$

If $h_{j,k}\le 0$ for all $j\in\mathbb{N}$, $k=1,2$, then the operator $\mathcal{A}$ is a differential operator with retarded argument. The first-order differential-difference operator

$$\mathbb{A}:\ C_b^{(1)}(\mathbb{R},X^2)\subset C_b(\mathbb{R},X^2)\to C_b(\mathbb{R},X^2)$$

corresponding to the operator $\mathcal{A}$ and acting by the rule $\mathbb{A}x=x'+\mathbb{B}x$, $x\in C_b^{(1)}(\mathbb{R},X^2)$, will be similar. Here the finite-difference operator $\mathbb{B}$ is defined by the matrix $\begin{pmatrix}0 & -I\\ B_2 & B_1\end{pmatrix}$, i.e.,

$$(\mathbb{B}x)(t)=\Bigl(-x_2(t),\sum_{j=1}^{\infty}C_{j,2}(t)x_1(t+h_{j,2})+\sum_{j=1}^{\infty}C_{j,1}(t)x_2(t+h_{j,1})\Bigr),\qquad t\in\mathbb{R},$$

for any function pair $x=(x_1,x_2)\in C_b\times C_b$.

It follows from Theorem 3 that the operators $\mathcal{A}$ and $\mathbb{A}$ are simultaneously invertible. Therefore, we can study the operator (12) with finite-difference operators $B_1$ and $B_2$ using a number of well-known results [12] on the conditions of the invertibility of first-order differential-difference operators, including differential operators of retarded type.

**Remark 3.** For constant operator coefficients $B_1$ and $B_2$, the criteria of the almost periodicity of solutions of first-order differential equations obtained in [13–16] can be carried over to second-order differential equations.

Let us return to the differential operator $D_2:\ \mathcal{F}^{(2)}(\mathbb{R},X)\subset\mathcal{F}(\mathbb{R},X)\to\mathcal{F}(\mathbb{R},X)$ and assume that the coefficients $B_1(t)=B_1$ and $B_2(t)=B_2$, $t\in\mathbb{R}$, are constant operators, i.e., $D_2=D^2+B_1D+B_2$. We consider the corresponding pencil of operators (the characteristic polynomial) $H:\ \mathbb{C}\to\operatorname{End}X$, $H(\lambda)=\lambda^2I+B_1\lambda+B_2$, $\lambda\in\mathbb{C}$.

The *spectrum* $\sigma(H)$ *of the operator pencil* $H:\ \mathbb{C}\to\operatorname{End}X$ is defined as the set of all complex numbers $\lambda$ such that the operator $H(\lambda)$ has no inverse in $\operatorname{End}X$. The set $\varrho(H)=\mathbb{C}\backslash\sigma(H)$ is referred to as the *resolvent set of the pencil*.

**Theorem 5.** *The following conditions are equivalent*:

1. $D_2:\ \mathcal{F}^{(2)}(\mathbb{R},X)\subset\mathcal{F}(\mathbb{R},X)\to\mathcal{F}(\mathbb{R},X)$ *is an invertible operator*;
2. *the spectrum* $\sigma(H)$ *of the pencil* $H:\ \mathbb{C}\to\operatorname{End}X$ *does not intersect the imaginary axis*, *i.e.*,

$$\sigma(H)\cap(i\mathbb{R})=\varnothing;$$

3. $\mathbb{D}_1 = \mathbb{D} + \mathbb{B} : \mathcal{F}^{(1)}(\mathbb{R}, X^2) \subset \mathcal{F}(\mathbb{R}, X^2) \to \mathcal{F}(\mathbb{R}, X^2)$ *is an invertible operator*;
4. *the spectrum* $\sigma(\mathbb{B})$ *of the operator* $\mathbb{B} \in \operatorname{End} X^2$ *does not intersect the imaginary axis*, *i.e.*,

$$\sigma(\mathbb{B}) \cap (i\mathbb{R}) = \varnothing;$$

5. *the operator* $\mathcal{D} : l^p(\mathbb{Z}, X) \to l^p(\mathbb{Z}, X)$, $p \in [1, \infty]$, *defined by the relation*

$$(\mathcal{D}x)(n) = x(n+1) - e^{\mathbb{B}}x(n), \qquad n \in \mathbb{Z}, \qquad x \in l^p(\mathbb{Z}, X),$$

*is invertible in* $l^p$ *for at least one value* $p \in [1, \infty]$ (*and*, *consequently*, *is invertible for all* $p \in [1, \infty]$).

It follows from the condition 2 of Theorem 5 that all operators $H(i\lambda)$, $\lambda \in \mathbb{R}$, are invertible in the algebra $\operatorname{End} X$. Obviously, the function $\lambda \mapsto (H(i\lambda))^{-1} : \mathbb{R} \to \operatorname{End} X$ is infinitely differentiable; moreover, the estimate $\|(H(i\lambda))^{-1}\| \le c/(1 + |\lambda|^2)$, $\lambda \in \mathbb{R}$, is valid with some constant $c > 0$. We have similar inequalities for the first and second derivatives of the function $H(i\lambda)^{-1}$. The above-mentioned estimate of the function $(H(i\lambda))^{-1}$ makes it possible to define a continuous summable function $G_2 : \mathbb{R} \to \operatorname{End} X$ by the relation

$$G_2(t) = \frac{1}{2\pi} \int_{\mathbb{R}} (H(i\lambda))^{-1} \exp(i\lambda t)\, d\lambda, \qquad t \in \mathbb{R}.$$

Therefore, the Fourier transform $\widehat{G}_2 : \mathbb{R} \to \operatorname{End} X$ of the function $G_2(t)$ has the form

$$\widehat{G}_2(\lambda) = (H(i\lambda))^{-1}, \qquad \lambda \in \mathbb{R}.$$

**Theorem 6.** *If the condition* 2 *of Theorem* 5 *is valid*, *then*

$$D_2 : \mathcal{F}^{(2)}(\mathbb{R}, X) \subset \mathcal{F}(\mathbb{R}, X) \to \mathcal{F}(\mathbb{R}, X)$$

*is an invertible operator*, *and the inverse operator* $D_2^{-1} \in \operatorname{End} \mathcal{F}$ *is the operator of convolution*

$$(D_2^{-1}g)(t) = (G_2 * g)(t) = \int_{\mathbb{R}} G_2(t-s)g(s)\, ds, \qquad t \in \mathbb{R}, \qquad g \in \mathcal{F}(\mathbb{R}, X),$$

*with a summable operator function* $G_2 \in L^1(\mathbb{R}, \operatorname{End} X)$.

## 2. PROOF OF THEOREM 3

Let $\mathcal{A} : D(A^2) \subset \mathcal{X} \to \mathcal{X}$ be an invertible operator. Then it is injective, and it follows from the relation $\mathcal{A}x = 0$ that $x = 0$. Let us show that the operator $\mathbb{A}$ is also injective. It follows from the relation $\mathbb{A}x = 0$ that $Ax_1 = x_2$ and $B_2x_1 + (A + B_1)x_2 = 0$ if $x = (x_1, x_2)$. Since $A$ is an invertible operator, we have $x_1 = A^{-1}x_2$. Then $(B_2A^{-1} + A + B_1)x_2 = 0$ or $\mathcal{A}A^{-1}x_2 = 0$. Since $\mathcal{A}$ is an injective operator, it follows that $A^{-1}x_2 = 0$. Consequently, $x_2 = 0$ and, therefore, $x_1 = A^{-1}x_2 = 0$, which implies that $\mathbb{A}$ is an injective operator.

Let us show that $\mathbb{A}$ is a surjective operator. We consider the equation $\mathbb{A}x = y$, where $x = (x_1, x_2)$, while $y = (y_1, y_2)$ is an arbitrary element from $\mathcal{X}^2$. This equation is equivalent to the system of equations $Ax_1 - x_2 = y_1$ and $B_2x_1 + (A + B_1)x_2 = y_2$. One can readily see that its solution has the form

$$x_1 = (A^{-1} - \mathcal{A}^{-1}B_2A^{-1})y_1 + \mathcal{A}^{-1}y_2 \in D(A), \qquad x_2 = A\mathcal{A}^{-1}(y_2 - B_2A^{-1}y_1) \in D(A).$$

Indeed, by substituting those values in relation (7), we obtain the relations

$$\begin{aligned}
&\mathbb{A}x = (A((A^{-1} - \mathcal{A}^{-1}B_2A^{-1})y_1 + \mathcal{A}^{-1}y_2) - A\mathcal{A}^{-1}(y_2 - B_2A^{-1}y_1),\\
&B_2((A^{-1} - \mathcal{A}^{-1}B_2A^{-1})y_1 + \mathcal{A}^{-1}y_2) + (A + B_1)A\mathcal{A}^{-1}(y_2 - B_2A^{-1}y_1))\\
&\quad = (y_1, (I - (A^2 + B_1A + B_2)\mathcal{A}^{-1})B_2A^{-1}y_1 + (A^2 + B_1A + B_2)\mathcal{A}^{-1}y_2) = (y_1, y_2).
\end{aligned}$$

This proves the surjectivity, and, in view of the injectivity of the operator $\mathbb{A}$, its invertibility, too. Moreover, it follows from the representation of a solution $x = (x_1, x_2)$ that the inverse operator $\mathbb{A}^{-1}$ is defined by the matrix (8).

Now let $\mathbb{A}$ be an invertible operator. Let us check that $\mathcal{A}$ is an injective operator. Let $x \in D(\mathcal{A}) = D(A^2)$ be a solution of the equation $\mathcal{A}x = 0$. We will show that $x = 0$. Note that $(x, Ax) \in D(\mathbb{A}) = D(A) \times D(A)$ and $(x, Ax) \in \operatorname{Ker} \mathbb{A}$ since

$$\mathbb{A}(x, Ax) = (Ax - Ax, (B_2 + (A + B_1)A)x) = (0, \mathcal{A}x) = (0, 0).$$

As $\mathbb{A}$ is a surjective operator, we have $x = 0$.

Let us show that $\mathcal{A}$ is a surjective operator. We consider the equation $\mathcal{A}x = g$, where $g \in \mathcal{X}$ is an arbitrary element. We show that there exists a solution $x \in D(A^2)$ of the equation. Since $\mathbb{A}$ is an invertible operator, it follows that the equation $\mathbb{A}y = f$ with any $f \in \mathcal{X}^2$ has the solution

$$y = (x_1, x_2) \in D(A) \times D(A).$$

We set $f = (0, g)$. Then the system of equations

$$Ax_1 - x_2 = 0, \qquad B_2x_1 + (A + B_1)x_2 = g$$

is solvable. Hence we obtain $x_1 = A^{-1}x_2$, and from the second equation of the system we find that the equation $(B_2A^{-1} + A + B_1)x_2 = g$ is solvable; and, consequently, so is the equation $\mathcal{A}A^{-1}x_2 = g$. We have thereby found a solution $x_1 = A^{-1}x_2 \in D(A^2)$ of the considered equation. This implies that $\mathcal{A}$ is a surjective operator. Consequently (by virtue of the injectivity), $\mathcal{A}$ is an invertible operator. The proof of the theorem is complete.

## 3. PROOF OF LEMMA 1

Let us establish the relation $\widetilde{\mathbb{A}}\mathcal{U} = \mathcal{U}\mathbb{A}$. Taking relations (9) and (10) into account, for any vector $x = (x_1, x_2) \in D(A) \times D(A) = D(\mathbb{A})$ we obtain

$$\begin{aligned}
\widetilde{\mathbb{A}}\mathcal{U}x &= \widetilde{\mathbb{A}}(x_1, -\lambda_0 x_1 x_2) = (\widetilde{A}x_1 + \lambda_0 x_1 - x_2, \widetilde{B}_2 x_1 + (\widetilde{A} + \widetilde{B}_1)(x_2 - \lambda_0 x_1)) \\
&= (Ax_1 - x_2, (B_2 + \lambda_0^2 I + \lambda_0 B_1)x_1 - (A - \lambda_0 I)\lambda_0 x_1 \\
&\quad - (B_1 + 2\lambda_0 I)\lambda_0 x_1 + (A - \lambda_0 I + B_1 + 2\lambda_0 I)x_2) \\
&= (Ax_1 - x_2, (B_2 - \lambda_0 A)x_1 + (A + B_1 + \lambda_0 I)x_2), \\
\mathcal{U}\mathbb{A}x &= \mathcal{U}(Ax_1 - x_2, B_2x_1 + (A + B_1)x_2) = (Ax_1 - x_2, -\lambda_0(Ax_1 - x_2) + B_2x_1 + (A + B_1)x_2) \\
&= (Ax_1 - x_2, (B_2 - \lambda_0 A)x_1 + (A + B_1 + \lambda_0 I)x_2).
\end{aligned}$$

Relation (11) follows from the established relations, which completes the proof of the lemma.

## 4. PROOF OF THEOREMS 1 AND 4

Let us now prove Theorem 4. We apply Theorem 3 to the operator $\widetilde{\mathbb{A}}$ defined by formula (10), where the operator coefficients $\widetilde{A}$, $\widetilde{B}_1$, and $\widetilde{B}_2$ are defined by relation (9) and $\lambda$ belongs to $\varrho(A)$. In this case, the inverse operator $\widetilde{\mathbb{A}}^{-1} \in \operatorname{End} \mathcal{X}^2$ admits the representation

$$\widetilde{\mathbb{A}}^{-1}x = ((\widetilde{A}^{-1} - \mathcal{A}^{-1}\widetilde{B}_2\widetilde{A}^{-1})x_1 + \mathcal{A}^{-1}x_2, -\widetilde{A}\mathcal{A}^{-1}\widetilde{B}_2\widetilde{A}^{-1}x_1 + \widetilde{A}\mathcal{A}^{-1}x_2)$$

for any vector $x = (x_1, x_2) \in \mathcal{X}^2$.

By Lemma 1, the operators $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ are simultaneously invertible, and from relation (11) we obtain

$$\begin{aligned}
\mathbb{A}^{-1} = \mathcal{U}^{-1}\widetilde{\mathbb{A}}^{-1}\mathcal{U} &= \begin{pmatrix} I & 0 \\ \lambda_0 I & I \end{pmatrix} \begin{pmatrix} \widetilde{A}^{-1} - \mathcal{A}^{-1}\widetilde{B}_2\widetilde{A}^{-1} & \mathcal{A}^{-1} \\ -\widetilde{A}\mathcal{A}^{-1}\widetilde{B}_2\widetilde{A}^{-1} & \widetilde{A}\mathcal{A}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\lambda_0 I & I \end{pmatrix} \\
&= \begin{pmatrix} \widetilde{A}^{-1} - \mathcal{A}^{-1}(\widetilde{B}_2\widetilde{A}^{-1} + \lambda_0 I) & \mathcal{A}^{-1} \\ \lambda_0\widetilde{A}^{-1} - A\ \ c A^{-1}(\widetilde{B}_2\widetilde{A}^{-1} + \lambda_0 I) & A\,\mathcal{A}^{-1} \end{pmatrix}.
\end{aligned}$$

It follows from Theorem 3 that $\mathcal{A}$ is an invertible operator if and only if $\widetilde{\mathbb{A}}$ is an invertible operator, and, consequently, so is the operator $\mathbb{A}$. The proof of Theorem 4 is complete.

Theorem 1 is a straightforward corollary to Theorem 4 if $A = D$ is the operator of differentiation in $\mathcal{F}(\mathbb{R}, X)$; $\mathcal{A} = D_2$, and $\mathbb{A} = \mathbb{D}_1$.

## 5. PROOF OF THEOREM 5

The equivalence of the conditions 1 and 3 was established in Theorem 1. The equivalence of the conditions 2 and 4 follows from Theorem 4 when applied to the operators $A = (i\lambda)I$ and $\mathcal{A} = -\lambda^2 I + iB_1\lambda + B_2$, $\lambda \in \mathbb{R}$, from the algebra $\operatorname{End} X$ and to the operator $\mathbb{B} + i\lambda I \in \operatorname{End} X^2$ defined by the matrix $\begin{pmatrix} i\lambda I & -I \\ B_2 & B_1 + i\lambda I \end{pmatrix}$.

The equivalence of the conditions 3 and 4 was proved in [1, p. 119; 3; 7], and the equivalence of the conditions 3 and 5 was proved in [3, 7, 8] (see also [17], where the equivalence of the conditions 4 and 5 was proved). Thus, the conditions 1–5 of the theorem are equivalent. The proof of the theorem is complete.

## 6. PROOF OF THEOREM 6

Let us show that the operator

$$D_2 : \ \mathcal{F}^{(2)}(\mathbb{R}, X) \subset \mathcal{F}(\mathbb{R}, X) \to \mathcal{F}(\mathbb{R}, X)$$

is invertible, and the inverse has the form

$$(D_2^{-1} g)(t) = (G_2 * g)(t) = \int\limits_{\mathbb{R}} G_2(t-s) g(s)\, ds. \tag{13}$$

To prove the representation (13), we consider the auxiliary operator

$$(D^2 - \alpha^2 I) = (D - \alpha I)(D + \alpha I), \qquad \alpha > 0.$$

This operator is invertible, and the image $\operatorname{Im}(D^2 - \alpha^2 I)^{-1}$ of the operator $(D^2 - \alpha^2 I)^{-1}$ coincides with $\mathcal{F}^{(2)}$. Therefore, for any function $v \in \mathcal{F}^{(2)}$ there exists a function $u \in \mathcal{F}$ such that $v = (D^2 - \alpha^2 I)^{-1} u = f_\alpha * u$. Note that the function $f_\alpha$ has the Fourier image $\widehat{f_\alpha}(\lambda) = -(\lambda^2 + \alpha^2)^{-1}$, $\lambda \in \mathbb{R}$. Since the Fourier image of the derivative is obtained by multiplication of the Fourier image of a function by the multiplier $i\lambda$, we have

$$-(\alpha^2 I + B_2)(\lambda^2 + \alpha^2)^{-1} - i\lambda(\lambda^2 + \alpha^2)^{-1} B_1 = \widehat{f_\alpha}(\alpha^2 I + B_2) + \widehat{f'}_\alpha B_1 = \widehat{\Phi}.$$

Here the function $\widehat{\Phi} : \ \mathbb{R} \to \operatorname{End} X$ is the Fourier image of the summable function

$$\Phi = f_\alpha(\alpha^2 I + B_2) + f'_\alpha B_1 \in L^1(\mathbb{R}, \operatorname{End} X).$$

Let us return to the function $(H(i\lambda))^{-1} = \widehat{G}_2(\lambda)$, $\lambda \in \mathbb{R}$, and represent it in the form

$$(H(i\lambda))^{-1} = \frac{-1}{\lambda^2 + \alpha^2} \left( I - \frac{\alpha^2 I + B_2}{\lambda^2 + \alpha^2} - \frac{i\lambda}{\lambda^2 + \alpha^2} B_1 \right)^{-1} = \widehat{f_\alpha}(\lambda)(I + \widehat{\Phi}(\lambda))^{-1}. \tag{14}$$

According to the Bochner–Fillips theorem [18], there exists such a summable function $\Phi_1 \in L^1(\mathbb{R}, \operatorname{End} X)$ that $(I + \widehat{\Phi}(\lambda))^{-1} = I + \widehat{\Phi}_1(\lambda)$, $\lambda \in \mathbb{R}$. The Bochner–Fillips theorem can also be obtained from the results of [19–21]. A constructive proof of the Bochner–Fillips theorem was given in Theorem 10.3 in [21].

We write out the representation (14) in the form of the relation

$$(H(i\lambda))^{-1} = \widehat{G}_2(\lambda) = \widehat{f}_\alpha(\lambda)(I + \widehat{\Phi}_1(\lambda)), \qquad \lambda \in \mathbb{R},$$

therefore, $G_2 = f_\alpha * I + f_\alpha * \Phi_1$, $\lambda \in \mathbb{R}$. Hence we find that the operator $A_2 : \mathcal{F} \to \mathcal{F}$ of convolution of the function $G_2$ with functions from $\mathcal{F}$ that acts by the rule $A_2 u = G_2 * u$, $u \in \mathcal{F} = \mathcal{F}(\mathbb{R}, X)$, can be represented in the form

$$A_2 u = G_2 * u = f_\alpha * (u + \Phi_1 * u), \qquad u \in \mathcal{F}.$$

Consequently, the image $\operatorname{Im} A_2$ of the operator $A_2$ belongs to $\mathcal{F}^{(2)} = D(D_2)$.

Let us check that the relation

$$D_2 A_2 u = u \tag{15}$$

is valid for any function $u \in \mathcal{F}$. Let $v = D_2 A_2 u$ and let $f$ be any function from the algebra $L^1(\mathbb{R})$ for which the support $\operatorname{supp} \widehat{f}$ of its Fourier image $\widehat{f}$ is compact. We take the Fourier transform of both sides in the relation $f * v = f * D_2(G_2 * u)$. By allowing for the fact that the Fourier transform of a convolution is equal to the product of the Fourier transforms of its multipliers, the form of the operator $D_2 = D^2 + B_1 D + B_2$, and the rule of evaluation of the Fourier transform of derivatives, we obtain

$$\widehat{(f * v)}(\lambda) = \widehat{f}(\lambda)(D_2\widehat{(G_2 * u)})(\lambda) = \widehat{f}(\lambda)(-\lambda^2 I + i\lambda B_1 + B_2)\widehat{(G_2 * u)}(\lambda) = \widehat{f}(\lambda) H(i\lambda)\widehat{G}_2(\lambda)\widehat{u}(\lambda).$$

However, since $\widehat{G}_2(\lambda) = (H(i\lambda))^{-1}$, $\lambda \in \mathbb{R}$, it follows that

$$\widehat{(f * v)}(\lambda) = \widehat{f}(\lambda)\widehat{u}(\lambda) = \widehat{(f * u)}(\lambda), \qquad \lambda \in \mathbb{R}.$$

Consequently, $f * v = f * u$. Since functions with compact Fourier transform are dense in the algebra $L^1(\mathbb{R})$, we have $g * v = g * u$ for any function $g \in L^1(\mathbb{R})$. To prove this fact, we use the estimate $\|f * x\|_{\mathcal{F}} \le \|f\|_{L^1}\|x\|_{\mathcal{F}}$, which is valid for arbitrary functions $f \in L^1(\mathbb{R})$, $x \in \mathcal{F}$ (see [7]). Consequently, $v = u$. We have thereby proved relation (15), which implies that the operator $A_2$ is the right inverse of the operator $D_2$. By Theorem 5, the operator $D_2 : \mathcal{F}^{(2)}(\mathbb{R}, X) \subset \mathcal{F}(\mathbb{R}, X) \to \mathcal{F}(\mathbb{R}, X)$ is invertible; and, consequently, $D_2^{-1} = A_2$. The proof of the theorem is complete.

**Remark 4.** The results obtained can be used for studying the solvability of nonlinear equations of the second order [22–24].

**Remark 5.** The general theorem 4 can be applied to the differential operator $D_2$ considered in the Banach space $\mathcal{F}(\mathbb{R}_+, X)$ of vector functions on a semiaxis with a domain that is given by some closed subspace $E$ of $X$. In this case, one should use the results of [4, 5, 7], where the theory of first-order differential operators was constructed in the space of vector functions defined on the semiaxis $\mathbb{R}_+ = [0, \infty)$.

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