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ORDINARY DIFFERENTIAL EQUATIONS

Generalized-Periodic Motions of Nonautonomous Systems

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Abstract—The definition and existence criterion are given for the generalized-periodic motions of a certain wide class of systems. The class contains all the systems that can be characterized by the classical periodic operator of displacement, the systems generated by the Volterra integral equations, and some others. A relationship is established between generalized-periodic motions and integral invariant sets.

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1. INTRODUCTION

We consider the system of differential equations with the vector notation in the form

$$\dot{x} = L(t, x), \tag{1}$$

where $x = (x^1, \dots, x^n)$ is a vector function of a real variable t and $L = (L^1, \dots, L^n)$ is a vector function that is defined and continuous, together with its partial derivatives $\partial L^i/\partial x^j$, $i, j = 1, \dots, n$, on the direct product $\mathbb{R} \times \mathbb{R}^n$ of the real axis $\mathbb{R}$ and the Euclidean vector space $\mathbb{R}^n$. We additionally assume that L is a T-periodic function of t.

The problem on the existence of periodic solutions of system (1) is known to be of great interest both from the viewpoint of the theory of differential equations and practically. The following Massera theorem (see [1]) is one of the most important results: let $n = 2$ and let every solution of system (1) be defined for all $t \geq t_0$; if there exists a solution $x = x(t, t_0, x_0)$ of this system that is bounded for these values of t, then system (1) has a T-periodic motion $x = x(t, t_0, x_p)$. According to Massera [1], this assertion remains valid for linear systems (1) of an arbitrary order. However, in the nonlinear case the Massera theorem turns out to fail even for $n = 3$ (see [2, p. 70]).

By virtue of the Birkhhoff theorems, for autonomous systems of arbitrary order the existence of a bounded solution leads to the existence of a compact minimal set that consists of recurrent solutions, and vice versa (see, e.g., [3, Chap. 5]). Of essential importance here is the known fact that many of the properties of the solutions of autonomous systems cannot be carried over to the solutions of nonautonomous systems. For example, the trajectories of system (1) may intersect each other in the nonautonomous case (see, e.g., [4, p. 115]). Therefore, the definition of the recurrence property and the Birkhoff theorem cannot be straightforwardly carried over to the nonautonomous case. However, asymptotics solutions of the recurrent type have been studied in some detail for nonautonomous systems (see, e.g., [5–8; 9, Chap. 2]).

Note that in the second half of the last century, methods of the theory of discrete dynamical systems began to be used extensively for studying nonautonomous systems (see, e.g., [10, Chap. 4]).

Based on these methods, it was established that the existence of a bounded solution for a nonautonomous periodic system entails the existence of an integral invariant set [10, p. 105]. Moreover, taking advantage of the results in the monograph [10, Chap. 4] made it possible to introduce the notion of a generalized-periodic solution of a nonautonomous system as a counterpart of the recurrent solution [11, 12].

The aim of this paper is to extend the main results of [11, 12] further to include a wide class of systems that are defined below.

2. CLASS OF SYSTEMS UNDER CONSIDERATION

Let Σ be a metric space with the metric d, let $\mathbb{R}$ be the real axis $(-\infty, +\infty)$, and let $\mathbb{R}^+$ be the real semiaxis $[0, +\infty)$.

Definition 1. Let us consider some mapping $f:\ \mathbb{R} \times \mathbb{R}^+ \times \Sigma \to \Sigma$. We set

$$f(\tau, t, x) = G(\tau, t)x$$

and assume that the following conditions are satisfied:

(a) the mapping f is jointly continuous on the set $\mathbb{R} \times \mathbb{R}^+ \times \Sigma$;

(b) the relation $G(\tau, 0)x = x$ is valid for all $(\tau, x) \in \mathbb{R} \times \Sigma$;

(c) there exists such a positive number T that the relation

$$G(\tau + T, t)G(\tau, T) = G(\tau, t + T) \tag{2}$$

is valid for $(\tau, t) \in \mathbb{R} \times \mathbb{R}^+$. Then, similar to [2, p. 348], the function $t \to f(\tau, t, x)$ is said to be a *motion* if the pair $(\tau, x) \in \mathbb{R} \times \Sigma$ is fixed, with the set Σ being referred to as the *space of motions.*

It can be easily seen that the definition of a motion is close but not equivalent to the definition of a process in [11]. First of all, this is explained by the fact that condition (2) is replaced in the definition of a process by the following, more rigid condition: the relation

$$G(\tau + s, t)G(\tau, s) = G(\tau, t + s) \tag{3}$$

is valid for all $(\tau, t, s) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$.

If the operator $G(\tau, t)$ satisfies condition (3), then it is equivalent to the classical displacement operator (see, e.g., [13, p. 12]). Obviously, such an operator satisfies condition (2), but the converse is not necessarily true. Therefore, let us call an operator that satisfies condition (2) an *extended displacement operator*.

In what follows, we consider only extended T-periodic operator, that is, an operator $G(\tau, t)$ that satisfies the condition

$$G(\tau + T, t) = G(\tau, t) \tag{4}$$

for all $(\tau, t) \in \mathbb{R} \times \mathbb{R}^+$. The system that is characterized by an extended T-periodic displacement operator is referred to as an *extended T-periodic system.*

If every solution $x = x(t, t_0, x_0)$ of system (1) is defined for all $t \geq t_0$, then the simplest example of an extended T-periodic system is the system generated by Eq. (1). This holds true for systems of functional-differential equations of retarded type with the right-hand sides that are T-periodic with respect to t (see [10, p. 99]). Let us consider some other examples.

Example 1. We consider some mapping $f:\ \mathbb{R} \times \Sigma \to \Sigma$ and set

$$f(t, x) = g^t x.$$

We assume the following:

(d) the mapping f is jointly continuous on the set $\mathbb{R} \times \Sigma$;

(e) the relation $g^0 x = x$ is valid for all $x \in \Sigma$;

(f) the group property $g^{t+s} = g^t g^s$ holds for all $t, s \in \mathbb{R}$.

Then the transformation group g^t is a dynamical system (see [3, p. 347]).

One can readily see that for each $T > 0$ the system g^t is an extended T-periodic system. Moreover, for any fixed $x \in \Sigma$ the function $t \to f(t, x)$ is a motion.

Example 2. We consider an integral equation of the Volterra type

$$y(t) = y_0 + h(t) + \int_{t_0}^{t} L(t, s, y(s))\, ds, \tag{5}$$

where $y_0 \in \mathbb{R}^n$ is the vector of parameters, $h: \mathbb{R} \to \mathbb{R}^n$ and $L: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. Suppose that the solution $y = y(t, t_0, y_0)$ of Eq. (5) is defined and jointly continuous with respect to the variables t, t_0, and y_0 in their domains for all $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$. If the functions h and L are T-periodic (the latter with respect to t and s), then Eq. (5) defines a T-extended periodic system in which

$$G(t_0, t - t_0)y_0 = y(t, t_0, y_0).$$

Moreover, it can be easily noticed that, unlike in the above examples, the displacement operator generated by Eq. (5) does not necessarily satisfy condition (3) (see [14]).

3. GENERALIZED-PERIODIC MOTIONS

Let us introduce the following notion.

Definition 2. We consider a motion $f(\tau, t, x)$. We assume that for each positive number ε there exists such a positive integer N_ε that

$$d(f(\tau, t, x), f(\tau, t + N_\varepsilon T, x)) < \varepsilon$$

for all $t \in \mathbb{R}^+$. Then we say that $f(\tau, t, x)$ is a *generalized-periodic solution.*

A T-periodic motion is an example of generalized-periodic motion. Of course, the set of generalized-periodic motions is not exhausted by T-periodic motions.

Indeed, the definition of a generalized-periodic motion is formally close to the definition of an almost periodic motion, and, in the case of dynamical systems, every almost periodic motion is a recurrent motion, but the converse is not true (see, e.g., [3, pp. 411–418]). Thus, let us establish the relationship between generalized-periodic and recurrent motions.

Example 3. We consider a dynamical system g^t (see Example 1). Let $f(t, x)$ be a motion, and let

$$\gamma(x) = \{f(t, x) : t \in \mathbb{R}\}$$

be its trajectory. Recall that the motion $f(t, x)$ of the system g^t is said to be recurrent if for each $\varepsilon > 0$ there exists such $T_\varepsilon > 0$ that the arc

$$\gamma_{s,T_\varepsilon}(x) = \{f(t, x) : t \in [s, s + T_\varepsilon]\}$$

of the trajectory $\gamma(x)$ approximates the entire trajectory $\gamma(x)$ to within ε for all $s \in \mathbb{R}$ (see [3, p. 402]). Let us also recall that a set M is said to be minimal if it is nonempty, closed, and invariant and contains no proper subset that possesses these three properties [3, p. 400].

For simplicity, suppose that the space of motions Σ is compact. Then a necessary and sufficient condition for a motion $f(t, x)$ to be recurrent consists in the closure $\bar{\gamma}(x)$ of the trajectory $\gamma(x)$ being a compact minimal set [3, pp. 402–404]. Therefore, as shown below (see Remark 3), for all $T > 0$ the recurrent motion $f(t, x)$ is generalized-periodic, and vice versa (see also [15]).

Remark 1. In the general case, for extended periodic systems the trajectory

$$\gamma^+(\tau, x) = \{f(\tau, t, x) : t \in \mathbb{R}^+\}$$

of a motion $f(\tau, t, x)$ depends not only on x but also on τ. It is the latter case that defines nonautonomous extended system, a system that is the main subject of research in this article.

The fact that the trajectories may intersect is the most important specific feature of nonautonomous systems. Consequently, straightforward transfer of the definition of a recurrent motion to nonautonomous extended periodic systems is impossible. Thus, generalized-periodic motion is an independent mathematical object that is tantamount to the classical recurrent motion for dynamical systems.

The following analog of the Massera theorem established the existence of generalized-periodic motions.

Theorem 1. *Let a motion $f(\tau,t,x)$ belong to some compact set $\Sigma_0 \subset \Sigma$. Then, from any sequence of positive integers $(N_k)_{k\in\mathbb{N}} \uparrow +\infty$ one can select such a subsequence $(N_{k_l})_{l\in\mathbb{N}} \uparrow +\infty$ that there exists a generalized-periodic motion $f(\tau,t,y)$ that belongs to Σ_0 and satisfies the conditions*

(i) *the relation*

$$\lim_{l\to+\infty} f(\tau, t+(N_{k_l}-1)T, x) = f(\tau,t,y)$$

is valid uniformly on each closed interval $[a,b] \subset \mathbb{R}^+$;

(ii) *the relation*

$$\lim_{l\to+\infty} f(\tau, t+(N_{k_{l+1}}-N_{k_l})T, y) = f(\tau,t,y)$$

is valid uniformly on the entire semiaxis $\mathbb{R}^+$.

The proof of Theorem 1 almost literally repeats the proof of Theorem 1 in [12] is therefore omitted (see also [16]).

Remark 2. It should be noted that for a dynamical system g^t under the assumptions of Theorem 1, the choice of the number T is independent of the choice of the sequence $(N_k)_{k\in\mathbb{N}}$ and vice versa (see [15, 16]).

Let us now establish the relationship between generalized-periodic motions and integral invariant sets. To this end, we first note that the minimal set is the most important invariant set. So, let us start with the relationship between generalized-periodic motions and minimal sets.

Theorem 2. *Let a motion $f(\tau,t,x)$ be situated in some compact set $\Sigma_0 \subset \Sigma$. Then the set*

$$\Omega(\tau,x) = \bigcap_{k\ge 0} \overline{\bigcup_{l\ge k} f(\tau, lT, x)}$$

is compact and invariant and is the union of compact minimal sets,[1] *with every point $(\tau,y) \in \mathbb{R}\times\Omega(\tau,x)$ being the starting point of a generalized-periodic motion $f(\tau,t,y)$ that belongs to Σ_0.*

Proof. Since Σ_0 is a compact set, the set $\Omega(\tau,x)$ is compact and invariant [10, p. 105]. Next, by Theorem 1, each point $y \in \Omega(\tau,x)$ is the starting point of a generalized-periodic motion $f(\tau,t,y)$ in Σ_0. Therefore, to prove the theorem it remains to show that $\Omega(\tau,x)$ is the union of compact minimal sets. Let us prove it.

For all $N = 0,1,\ldots$, by E_N we denote the set of points

$$f(\tau,NT,y),\ f(\tau,(N+1)T,y),\ \ldots,\ f(\tau,(N+l)T,y),\ \ldots$$

In addition, let $\bar{E}_N$ be the closure of the set E_N. Then, by virtue of Definition 2, one can readily see that each set $\bar{E}_N$ is compact and invariant. Moreover, by the construction, we have the inclusions

$$\Omega(\tau,x) \supset \bar{E}_0 \supset \bar{E}_1 \supset \cdots \supset \bar{E}_N \supset \cdots$$

[1] In the assumptions of the theorem, the invariance and minimality of sets are understood in the sense of how the operator $G(\tau,T)$ acts upon them.

Consequently, there exists a compact minimal set

$$M \subset \bigcap_{N\geq 0} \bar{E}_N$$

contained in the set $\Omega(\tau, x)$ [3, p. 401]. However, according to Definition 2, we have $\bar{E}_0 = \bar{E}_1 = \cdots = \bar{E}_N = \cdots$ Therefore, $\bar{E}_0 = M$.

Note now that the specific choice of the point $y \in \Omega(\tau, x)$ played no role in the above. The proof of the theorem is complete.

For extended periodic systems, we will use the following definitions from the monograph [10, p. 102].

Definition 3. Let $h:\ \mathbb{R} \to \Sigma$ be a continuous mapping satisfying the condition

$$h(\tau + t) = G(\tau, t)h(\tau)$$

for all $(\tau, t) \in \mathbb{R} \times \mathbb{R}^+$. Then we say that h is an *integral on* $\mathbb{R}$. If, in addition, $h(\tau) = x$, then we say that h is an *integral passing through the point* $(\tau, x) \in \mathbb{R} \times \Sigma$. However, if $\mathcal{M}$ is a set in the space $\mathbb{R} \times \Sigma$ and for all $(\tau, x) \in \mathcal{M}$ there exists such an integral h passing through the point (τ, x) that $(s, h(s)) \in \mathcal{M}$ for $s \in \mathbb{R}$, then we say that $\mathcal{M}$ is an *integral set on* $\mathbb{R}$.

Definition 4. Let $\mathcal{M}$ be an integral set on $\mathbb{R}$. For each $\tau \in \mathbb{R}$ we set

$$\mathcal{M}_\tau = \{x \in \Sigma :\ (\tau, x) \in \mathcal{M}\}.$$

An integral set $\mathcal{M}$ is said to be *invariant* if the relation

$$G(\tau, T)\mathcal{M}_\tau = \mathcal{M}_\tau$$

is valid for all $\tau \in \mathbb{R}$, i.e., $\mathcal{M}_{\tau+T} = \mathcal{M}_\tau$.

Let us now find out the meaning of Definition 4. To this end, we consider a dynamical system g^t described in Example 1.

Since the operator g^t is independent of τ, Definition 4 transforms into the following: a set $Q \subset \Sigma$ is said to be invariant if $\mathbb{R} \times Q$ is an integral set for the system g^t. Obviously, this implies that for each point $x \in Q$ there exists such an integral h passing through the point $(0, x)$ that $h(t) \in Q$ for any $t \in \mathbb{R}$ [10, p. 103]. In this case, one can readily see that the relation

$$g^t Q = Q$$

is valid for all $t \in \mathbb{R}$, which is in agreement with the classical definition of an invariant set (see, e.g., [3, p. 349]).

Therefore, the role and place of integral invariant sets is difficult to be overemphasized since it is an obvious extension of the notion of an invariant set, which was in wide use as early as at the beginning of the last century, to extended periodic systems. The importance of this extension is explained by the fact that, as mentioned above, in the nonautonomous case the trajectories of motions may intersect each other and, consequently, the classical definition of invariance cannot be straightforwardly carried over to extended periodic systems.

The simplest example of an integral invariant set is given by the set

$$\mathcal{M} = \mathbb{R} \times \Sigma.$$

The following assertion much more fully characterizes integral invariant set as a very important mathematical object.

Theorem 3. *Let $\Gamma \subset \Sigma$ be a set such that for $x \in \Gamma$ and for all $\tau \in \mathbb{R}$ the motion $f(\tau, t, x)$ lies in some compact set $\Sigma_0(x) \subset \Sigma$. In addition, let*

$$\Omega(\tau, x) = \bigcap_{k\geq 0} \overline{\bigcup_{l\geq k} f(\tau, lT, x)}, \tag{6}$$

where $(\tau, x) \in \mathbb{R} \times \Gamma$ and

$$\Omega_\Gamma = \{y \in \Omega(\tau, x) : \ (\tau, x) \in \mathbb{R} \times \Gamma\}. \tag{7}$$

In this case, if Γ is a nonempty set and condition (3) *is satisfied, then*

$$\mathcal{M} = \mathbb{R} \times \Omega_\Gamma \tag{8}$$

is an integral invariant set such that each point $(\tau, y) \in \mathcal{M}$ is the starting point of a generalized-periodic motion $f(\tau, t, y)$ in the set Ω_Γ.

Proof. According to Theorem 2, for all $(\tau, x) \in \mathbb{R} \times \Gamma$ the set $\Omega(\tau, x)$ is invariant under the action of the operator $G(\tau, T)$. In addition, by Theorem 2, each point (τ, y) of the set $\mathbb{R} \times \Omega(\tau, x)$ is the starting point of a generalized-periodic motion $f(\tau, t, y)$. Therefore, each point (τ, y) of the set $\mathcal{M}$ defined by relation (8) is also the starting point of a generalized-periodic motion $f(\tau, t, y)$. Moreover, by virtue of relations (6) and (7), every such generalized-periodic motion $f(\tau, t, y)$ belongs to the set Ω_Γ. Hence it follows that for any fixed $y \in \Omega_\Gamma$, such a continuous function $h : \ \mathbb{R} \to \Omega_\Gamma$ is defined that satisfies the condition

$$h(\tau + t) = G(\tau, t)y \tag{9}$$

for all $(\tau, t) \in \mathbb{R} \times \mathbb{R}^+$, i.e., h is an integral passing through the point (τ, y).

One can readily see that, by virtue of relation (9), for all $s \in \mathbb{R}$ the point $(s, h(s))$ lies in the set $\mathcal{M}$. Therefore, $\mathcal{M}$ is an integral set.

Since the set $\Omega(\tau, x)$ is invariant under the action of the operator $G(\tau, T)$ for $(\tau, x) \in \mathbb{R} \times \Gamma$, we have

$$G(\tau, T)\Omega_\Gamma = \Omega_\Gamma$$

for all $\tau \in \mathbb{R}$. Consequently, $\mathcal{M}$ is an integral invariant set.

This completes the proof of the theorem.

4. APPLICATION TO DYNAMICAL SYSTEMS

Let us apply the results in Section 3 to the study of the dynamical system g^t (see Example 1) and note that the following analog of Theorem 3 holds true for such a system.

Theorem 4. *Let $\Gamma \subset \Sigma$ be such a set that the motion $f(t, x)$ lies in some compact set $\Sigma_0(x) \subset \Sigma$ for all $x \in \Gamma$. In addition, let us set*

$$\Omega_T(x) = \bigcap_{k\geq 0} \overline{\bigcup_{l\geq k} f(lT, x)} \tag{10}$$

for some positive integer T and some point $x \in \Gamma$. If Γ is a nonempty set, then for all $T > 0$ the set

$$\mathfrak{M}_T = \{y \in \Omega_T(x) : \ x \in \Gamma\}$$

is the union of all minimal sets of the system g^t, with any minimal set $M \subset \mathfrak{M}_T$ being compact.

Proof. Let us fix some positive number T and a point $x \in \Gamma$. Then it follows from relation (10) that the set

$$M = \overline{\bigcup_{t\geq 0} f(t, y)}$$

is a compact minimal set of the system g^t for each point $y \in \Omega_T(x)$ [16]. However, the choice of the number T played no role, which, by virtue of the definition of the set $\mathfrak{M}_T$, proves the theorem.

Remark 3. As mentioned above, the closure of the trajectory of a recurrent motion is a compact minimal set in a compact metric space of motions. Therefore, by Theorems 3 and 4, the recurrent motion $f(t,x)$ of the dynamical system g^t is generalized-periodic in the compact metric space of motions for all $T>0$ and vice versa.

5. CONCLUSIONS

Theorem 1 is a simple criterion for the existence of generalized-periodic motions in extended periodic systems. The class of such systems is rather wide and includes the systems generated by Eq. (1), functional-differential equations of the retarded type, integral equations (5), differential equations in a Banach space, and some other systems.

Theorems 2–4 establish a relationship between generalized-periodic motions and integral invariant and minimal sets. In this connection, the role and place of integral invariant set as a very important mathematical object needs to be specially object since it is an extension of the classical invariant set to the nonautonomous case.

By virtue of Theorems 3 and 4, one can readily see that, in the case of dynamical systems, generalized-periodic motion is equivalent to recurrent one (see Remark 3). This implies that generalized-periodic motion is a generalization of recurrent motion to the case of extended periodic systems.

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