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CONTROL THEORY

Notions of Equilibrium for Differential Games on Intersecting Game Sets

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Abstract—We suggest new notions of conflict equilibrium and demonstrate a technique of their use for finding a solution in arbitrary game problems on a game set common for all players and especially in problems with side interests of players in the static and dynamic settings.

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1. INTRODUCTION

The solution of arbitrary conflict problems is based on the notion of conflict equilibria without which it is impossible to define any notion of solution, because the possibility of resolving any conflict implies the existence of some stable state, i.e., the presence of some equilibrium (either static or dynamic) stable under arbitrary possible deviations of arbitrary conflict participants from it. Unfortunately, all relatively "strong" equilibria known so far do not necessarily exist in any problem for which they have been developed.

The almost centenary history of the development of game theory represented by numerous publications (for references, see [1–15]) showed that the construction of notions of equilibria whose definition does not contain any artificially imposed behavior rules for players is a very complicated problem. In addition, it was shown that all already-found notions of equilibrium are not sufficient for solving problems of an arbitrary class (two-person zero-sum games as well as coalition-free, coalitional, cooperative, static, and dynamic problems). Note that the "strongest" equilibrium existing in a problem is most interesting in general. However, strong equilibria do not necessarily exist in an arbitrary problem; moreover, a strong equilibrium can prove to be undesirable for all players, as it often turns out for the classical Nash equilibrium (e.g., see [12, p. 9]).

It turns out that, in any class of problems, one can find numerous examples in which the strongest equilibrium is not unique. Obviously, the nonuniqueness of some equilibrium can be a consequence of the existence of some (explicit or hidden) form of symmetry in a game. However, in the general case, regardless of the existence of a symmetry in a particular game, the existence of two or more situations satisfying some definition of a game equilibrium implies that such a notion admits some strengthening, which would permit one to single out some strongest equilibrium from these equivalent equilibria.

Regardless of causes of the nonuniqueness of a strongest equilibrium, the knowledge of as many notions of equilibrium as possible is very useful, because it permits one to select, from the entire set of almost equivalent equilibria, an equilibrium that turns out to be an equilibrium simultaneously from the viewpoint of the maximum notions of equilibrium existing in the problem. Thus, finding new notions of equilibrium is most important for the construction of a theory of conflicts capable of solving arbitrary practical problems.

Note that in the classical game theory [1–8], which deals only with games on a common game set for all players, an equilibrium existing in all games for which it is stated was not found. Only the theory [9–15] suggesting to construct hierarchically related chains of embedded equilibria has permitted one to claim that any game has a solution, because the weakest always existing A-equilibrium has been found. However, the problem on the nonuniqueness of the strongest equilibrium also persists in the latter theory.

It turns out for problems with side interests of the players [9–15], which are not considered in the classical theory of conflicts even though they are very important for practical applications, that first, even the A-equilibrium can be empty, and second, the problem of nonuniqueness of a strongest equilibrium is even more topical. The notion of side interests necessitates introducing a number of new notions of equilibrium and an additional notion of strong threats [9, 10, 13, 14]. In the present paper, we suggest new notions of equilibria very important in applications.

2. STATEMENTS OF CONFLICT EQUILIBRIA

Let us present statements of new notions of equilibria, suggest a method for solving problems with side interests first for static problems with an arbitrary number of players, and then show how to modify that method for dynamic conflict problems.

Assumption 1. Let Q_i, $i = 1, \dots, N$, be metric spaces, let $Q = Q_1 \times \cdots \times Q_N$, let G_i, $i = 1, \dots, N$, be the compact game set in Q of the ith player trying to maximize his continuous payoff function (functional) $J_i(q)$, $i = 1, \dots, N$, defined on the set G_i, let q_i be the strategy of the ith player, and let $q = q_1 \dots q_N$ be the vector of strategies of all players. Next, let P_k be an arbitrary coalition of k players trying to maximize its coalitional payoff function $J_{P_k} = \sum_{i \in P_k} J_i$ defined on the game set $G_{P_k} = \bigcup_{i \in P_k} G_i$ (on which each of the functions J_i, $i = 1, \dots, N$, is defined only on its own game set G_i), let $G_{P_N} = \bigcup_{i=1}^{N} G_i$ be the union of all game sets, and let $G \triangleq \bigcap_{i=1}^{N} G_i$ be their intersection (which can be empty, but we assume that each of the sets G_i has a nonempty intersection with at least one of other game sets).

The coalition P_k can choose its strategy (state) q_{P_k} from the projection $\mathrm{Pr}_{Q_{P_k}} G_{P_N}$ of the set G_{P_N} onto the space Q_{P_k} [or the cross-section $G_{P_N}(q_{P_{N-k}})$ of the set G_{P_N}].

This means that we consider a conflict problem in which the ith player tries to achieve the maximum of his payoff function J_i on its individual game set $G_i \subseteq Q$, which has a nonempty intersection with some game sets of $N-1$ remaining players; moreover, players can form arbitrary coalitions P_k and a cooperation P_N; in this case, the interests of all players simultaneously clash explicitly only on the set G.

One can assume that the coalition P_k gets side profit on the set

$$G_{P_k} \backslash (G_{P_k} \cap G_{P_{N-k}});$$

moreover, the side profit can be specified for the common game set G and for various parts G_{P_N} of the set.

Note that the number of all possible coalitions consisting of only k players is given by the binomial coefficient $\binom{N}{k} = N!/((N-k)!k!)$, and the number of all possible coalitions with the number of players from 1 to $N-1$ is $\sum_{k=1}^{N-1} \binom{N}{k} = 2^N - 2$. Below, in the study of coalitions consisting of one player, we use the following notation:

$$q^i = q_1 \dots q_{i-1} q_{i+1} \dots q_N, \qquad J^i = \sum_{k \neq i} J_k, \qquad i = 1, \dots, N, \qquad k = 1, \dots, N.$$

Definition 1. A situation $q^* = (q^*_{P_k}, q^*_{P_{N-k}}) \in G_{P_k}$, where the index P_k is treated as any particular coalition of k players, is said to be A_{P_k}-*extremal* if $G_{P_k}(q^*_{P_{N-k}}) = q^*_{P_k}$, or to each state $q_{P_k} \in G_{P_k}(q^*_{P_{N-k}}) \backslash q^*_{P_k}$ of that particular coalition P_k, one can assign at least one state $\hat{q}_{P_{N-k}} \in G_{P_k}(q_{P_k})$ of the remaining $N-k$ players such that

$$J_{P_k}(q_{P_k}, \hat{q}_{P_{N-k}}) \le J_{P_k}(q^*), \qquad \hat{q}_{P_{N-k}} \in G_{P_k}(q_{P_k}) \tag{1}$$

(this case is referred to as a *problem of the first type*), or such that

$$J_{P_k}(q_{P_k}, \hat{q}_{P_{N-k}}) \le J_{P_k}(q^*), \qquad \hat{q}_{P_{N-k}} \in G_{P_N}(q_{P_k}) \tag{2}$$

[this case is referred to as a *problem of the second type*, which differs from a problem of the first type by the fact that, in addition to threats $\hat{q}_{P_{N-k}}$ used in inequality (1), there may be ("strong") threats on the set $G_{P_N}(q_{P_k})\backslash G_{P_k}(q_{P_k})$ on which the remaining threatening players get some collective profit, and the coalition P_k gets nothing], or if to any strategy $q_{P_k} \in G(q^*_{P_{N-k}})\backslash q^*_{P_k}$ of the coalition P_k, one can assign at least one admissible strategy $\hat{q}_{P_{N-k}}$ of the remaining players such that

$$J_{P_k}(q_{P_k}, \hat{q}_{P_{N-k}}) \le J_{P_k}(q^*), \qquad \hat{q}_{P_{N-k}} \in G(q_{P_k}) \tag{3}$$

(this case is referred to as a *problem of the third type*, which is characterized by the fact that an auxiliary game is considered only on the intersection $G = \bigcap_{i=1}^N G_i$).

A situation $q^* \in \bigcap_{i=1}^N A_i$ is called an *A'_{P_k}-equilibrium in the problem of the first, second, or third type* if conditions (1), (2), or (3), respectively, are satisfied at the point q^* for all possible coalitions [their number is $\binom{N}{k} = N!/((N-k)!k!)$] that can be formed of k players. [Thus, in a sense, each of inequalities (1), (2), and (3) is a "vector" $\binom{N}{k}$-dimensional inequality.] In addition, the situation q^* is called an *A'-equilibrium* (in each of the above-listed three types of problems) if it is coalition extremal for any coalition P_k, $1 \le k < N$, from all $2^N - 2$ possible coalitions; i.e., $A' = \bigcap_{P_k} A'_{P_k}$, $1 \le k \le N-1$.

It follows from Definition 1 that, in problems of the first and third type, threats are natural (weak), and in a problem of the second type, they are "strong," where "strength" implies that, in game problems on partially intersecting game sets, for example, a threatening coalition P_{N-1} can implement its threats not only in the cross-sections $G_i(q_i)$ of the game set G_i of the ith player but also in the wider cross-sections $G_{P_N}(q_i) \supseteq G_i(q_i)$ in which it gets profit and the penalized ith player gets nothing.

In problems with side interests of the players, both the set of A'-equilibria and the basic A-equilibrium (which is not empty in arbitrary problems on a common game set for all players), which is obtained from Definition 1 if coalitions P_1 and P_{N-1} consisting of one and $N-1$ players, respectively, [9–15] are considered in it, can be empty.

However, in problems with side interests, one can replace the set of A-equilibria by the following notion of P-equilibrium and a new suggested strengthening, that is, a D^P-equilibrium.

Definition 2. A situation $q^* \in G_{P_k}$, where P_k is some particular coalition from the set $\binom{N}{k}$ of all possible coalitions consisting of k players, is said to be *P_{P_k}-extremal* if, for any attempt $q_{P_k} \in G_{P_k}(q^*_{P_{N-k}})$ of the players from that coalition to increase their profit in the situation q^* by the passage into a more preferable situation q_{P_k}, it turns out that the situation q^* is a Pareto point [on the set $J_1(G_1) \times \cdots \times J_N(G_N)$] with respect to all points $(q_{P_{N-k}}, q_{P_k})$ from the cross-section $G_{P_N}(q_{P_k})$. In particular, a situation q^* is considered to be P_{P_k}-extremal if, in the cross-section $G_{P_k}(q^*_{P_{N-k}})$, the coalition P_k cannot increase its profit (since this is a point maximizing the functional J_{P_k} in this cross-section). A situation q^* is said to be *P_{P_k}-equilibrium* (*-optimal*) if it is simultaneously P_{P_k}-equilibrium for each particular coalition of k players the number of which is equal to $\binom{N}{k}$. The intersection of all possible P_{P_k}-equilibrium situations is referred to as the *set of P-equilibria*.

The following suggested new notion of D^P-equilibrium is based on a complicated specific strengthening of the notion of P-equilibrium and is some weakening of the most interesting and useful below-represented equilibrium (6), which unfortunately does not necessarily exist in arbitrary problems. The D^P-equilibrium is useful both in problems with side interests of players in which the A-equilibrium is empty and, in general, in arbitrary conflict problems in which the notion of P-equilibrium can always be used in addition to a nonempty A-equilibrium.

Definition 3. A situation $q^* \in P_{P_k}$ is referred to as a *$D^P_{P_k}$-equilibrium* if it satisfies the "vector" $\binom{N}{k}$-dimensional relation

$$D^P_{P_k} = \operatorname{Arg} \max_{q_{P_k} \in P_{P_k}(q^*_{P_{N-k}})} J_{P_k}(\operatorname{Arg} \max_{q_{P_{N-k}} \in P_{P_k}(q_{P_k})} J_{P_{N-k}}(q)), \tag{4}$$

and the intersection of all $D^P_{P_k}$-equilibrium situations is referred to as the *D^P-equilibrium*.

To clarify the solution of game problems considered below, we represent two more equilibria [9–15], which, if nonempty, permit one to find the strongest equilibria in game problems.

Definition 4. A situation $q^* \in A_{P_k}$, where the index P_k stands for an arbitrary particular coalition of k players, is said to be B_{P_k}-*extremal* if this particular coalition satisfies the relation

$$\max_{q_{P_{N-K}} \in A_{P_k}(q^*_{P_k})} J_{P_{N-k}}(q^*_{P_k}, q_{P_{N-k}}) = J_{P_{N-k}}(q^*); \tag{5}$$

and this situation is referred to as a B_{P_k}-*equilibrium* if it satisfies all possible $\binom{N}{k}$ relations (5) corresponding to all possible coalitions of k players. Consequently, the set of all B_{P_k}-equilibria is the intersection of all sets of situations that satisfy $\binom{N}{k}$ relations (5), and the latter can be treated as some $\binom{N}{k}$-vector relation. The intersection of all B_{P_k}-equilibrium situations is referred to as the B^P-*equilibrium*.

Definition 5. A situation $q^* \in B_{P_k}$ is said to be $\bar{D}_{P_k}$-*extremal*, where the index P_k stands for an arbitrary particular coalition of k players, if this particular coalition satisfies the relation

$$\max_{q \in B_{P_k}} J_{P_k}(q) = J_{P_k}(q^*), \tag{6}$$

and this situation is referred to as $\bar{D}_{P_k}$-*equilibrium* if it satisfies all $\binom{N}{k}$ relations (6). The intersection of all $\bar{D}_{P_k}$-equilibrium situations is referred to as the $\bar{D}$-*equilibrium*.

3. SOLUTION METHOD FOR GAME PROBLEMS

The following assertion is quite useful for the solution of most conflict problems with side interests of players.

Proposition 1. *In the general case, in a game problem with noncoinciding intersecting game sets G_i, the following assertions hold.*

1. *If the strong threats* (2) *are not admitted* (*by agreement of all players*), *and the set A in the natural class of weak threats* (1) (*i.e., in a problem of the first type*) *is empty, then the strongest equilibria* (*and a solution of the problem in some sense*) *can be found by the simultaneous solution of a problem of the first type, where the A-equilibrium is replaced by the P-equilibrium, and a problem of the third type.*

2. *If strong threats* (2) *are not admitted and the set A in a problem of the first type is nonempty, then, as a basic solution, one should use a solution of a problem of the first type* (*with all of its possible iterations* [9–15]) *and use a solution of a problem of the third type* (3) *and the D^P-equilibrium to estimate the influence of side interests on the solution of the original game; moreover, if the solutions of problems of the first and third type do not coincide, then one should take the solution of a problem of the first type as a basic solution and consider a solution of a problem of the third type and a D^P-equilibrium only as a possible correction of that solution; if the solutions of the problems of the first and third type coincide, then side profits have no influence on the solution of the game* (*which is very favorable for players*).

3. *If the strong threats* (2) *are admitted and the set A in problem* (1) *is empty, then one should take a solution of the problem of the second type* (2) *as a basic solution of the original problem and consider the D^P-equilibrium and a solution of the problem of the third type* (3), *if it differs from a solution of problem* (2), *only as a possible correction of a solution of the problem of the second type; in addition, if the strongest equilibria in problems of the second and third type coincide, then side profits do not affect the solution of the game.*

4. *If the strong threats* (2) *are admitted, and the set A in a problem of the first type is nonempty, then one should find solutions of problems of the first, second, and third types; and if the strongest equilibria in them coincide, then this implies that neither types of threats nor side interest of players affect the solution of the original problem, and this is most favorable for them; but in the case of*

different equilibria in problems of the first, second, and third type one should take the solution of a problem of the second type as a basic solution and use solutions of problems of the first and third types only as corrections; moreover, in this case, it is unreasonable to construct P- and D^P-equilibria with complicated definitions.

Example 1. Consider a game with two players, where each player maximizes his (matrix) payoff function

$$J_1 = \begin{bmatrix} 1 & \cdot & 4 \\ \cdot & 5 & \cdot \\ 6 & \cdot & 3 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} 3 & \cdot & 2 \\ 4 & \cdot & \cdot \\ \cdot & 9 & 1 \end{bmatrix}.$$

Both players have three strategies: the first player chooses one of the three rows, and the second one chooses one of the three columns. In this problem, we have

$$G_1 = (a_{11}, a_{13}, a_{22}, a_{31}, a_{33}), \qquad G_2 = (a_{11}, a_{13}, a_{21}, a_{32}, a_{33}), \qquad G = (a_{11}, a_{13}, a_{33}),$$

and the game set $W' = W_1 \cup W_2$ consists of seven situations a_{ij} in the above-represented matrices such that the corresponding entries in at least one of them contain the values of the payoff functions.

First, we find the matrices A_1, A_2, and A in the class of weak threats, i.e., in case (1),

$$A_1 = \begin{bmatrix} \cdot & \cdot & + \\ \cdot & + & \cdot \\ + & \cdot & + \end{bmatrix}, \qquad A_2 = \begin{bmatrix} + & \cdot & \cdot \\ + & \cdot & \cdot \\ \cdot & + & \cdot \end{bmatrix}.$$

Obviously, in the considered game on partially intersecting sets G_1 and G_2 in the class of weak threats (1), the set A has the form $A = A_1 \cap A_2 = \varnothing$. Since the weakest notion of equilibrium is empty, we consider even weaker notions of equilibria mentioned in Definitions 2 and 3. Since it is almost impossible to find these equilibria on the basis of their definitions without a geometric representation of the matrices J_1 and J_2 on the plane, we represent the values of the payoff functions J_1 and J_2 in Fig. 1. In the first quadrant [in the coordinate system (J_1, J_2)], we represent the mapping $(J_1(G), J_2(G))$ only of the set G on which both functions J_1 and J_2 are defined, and the mapping $J_1(G_1 \backslash G)$ is conditionally represented to the left from the axis J_1, because the function J_2 is undefined on the set $(G_1 \backslash G)$. Similarly, the mapping $J_2(G_2 \backslash G)$ is conditionally shown below the axis J_2.

We construct the sets P_1 and P_2 described by Definition 2 and their intersection P,

$$P_1 = \begin{bmatrix} + & \cdot & + \\ \cdot & + & \cdot \\ + & \cdot & \cdot \end{bmatrix}, \qquad P_2 = \begin{bmatrix} + & \cdot & + \\ + & \cdot & \cdot \\ \cdot & + & + \end{bmatrix}, \qquad P = \begin{bmatrix} + & \cdot & + \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}.$$

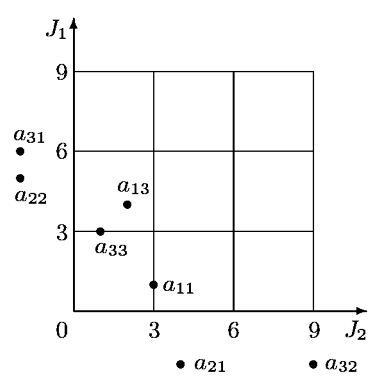

Fig. 1.

By evaluating the D^P-equilibrium by formulas (4), we obtain

$$D_1^P(a_{11}) = \operatorname{Arg} \max_{q_1 \in P_1(q_2^*)} J_1(\operatorname{Arg} \max_{q_2 \in P_1(q_1)} J_2(q_1, q_2)) = \operatorname{Arg} \max_{q_1 \in P_1(q_2^*)} J_1(a_{11}, \varnothing) = a_{11},$$
$$D_2^P(a_{21}) = \operatorname{Arg} \max_{q_2 \in P_2(q_1^*)} J_2(\operatorname{Arg} \max_{q_1 \in P_2(q_2)} J_1(q_1, q_2)) = \operatorname{Arg} \max_{q_2 \in P_2(q_1^*)} J_2(a_{11}, a_{13}) = a_{11};$$

i.e., only the situation a_{11} is a D^P-equilibrium.

Therefore, only by using Definitions 2 and 3, we find the strongest equilibrium in the problem of the first type with the notion of A_i-extremality replaced by the notion of the P_i-extremality. Obviously, this is only a preliminary result.

To evaluate the influence of side profits on the solution of the considered problem and find the solution itself, one should consider other types of auxiliary problems as well. First, we find the set of equilibria for the auxiliary problem of the second type, i.e., in the case of the use of strong threats (2) by players. In this case, we obtain

$$A_1^{(s)} = \begin{bmatrix} + & \cdot & + \\ \cdot & + & \cdot \\ + & \cdot & + \end{bmatrix}, \qquad A_2^{(s)} = \begin{bmatrix} + & \cdot & + \\ + & \cdot & \cdot \\ \cdot & + & + \end{bmatrix}, \qquad A^{(s)} = \begin{bmatrix} + & \cdot & + \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & + \end{bmatrix},$$
$$B_1^{(s)} = (a_{11}, a_{33}), \qquad B_2^{(s)} = (a_{11}, a_{13}), \qquad B^{(s)} = a_{11}.$$

It turns out that the situation a_{11} is the strongest equilibrium in the original game in the class of strong threats as well.

By noting that, obviously, the cooperative profit in this auxiliary game is achieved in the situation a_{33} and is equal to 9, we find the optimal (fair) sharing of that cooperative profit depending fully on the situation of the strongest equilibrium and defined by the formulas [13, p. 174]

$$y_1 = \frac{1}{4}9, \qquad y_2 = \frac{3}{4}9.$$

A solution in the class of strong threats should be considered to be dominating, because if it is favorable for at least one player, then he can use these threats although they are considered to be prohibited. However, in spite of the coincidence of solutions of problems of the first and second types, players surely prefer to take into account the solution obtained in a problem of the third type, i.e., a game on the intersection $G_1 \cap G_2 = G$ that is a common game set for players on which they explicitly conflict with each other [obviously, strong threats do not act on the common set, and only weak threats (1) should be considered]:

$$J_1^G = \begin{bmatrix} 1 & \cdot & 4 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 3 \end{bmatrix}, \qquad J_2^G = \begin{bmatrix} 3 & \cdot & 2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}.$$

For this auxiliary game on the set G, we obtain

$$A_1^G = \begin{bmatrix} + & \cdot & + \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & + \end{bmatrix}, \qquad A_2^G = \begin{bmatrix} + & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & + \end{bmatrix}, \qquad A^G = \begin{bmatrix} + & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & + \end{bmatrix}.$$

Hence we obtain $B_1 = B_2 = B = (a_{11}, a_{33})$; moreover, all stronger notions of equilibrium ($\bar{D}$, D', and others [10, 15]) do not single out the strongest equilibrium in this pair. Since the pair of strongest equilibria (a_{11}, a_{33}) in the auxiliary game on the set G is "indistinguishable," it follows that a fair sharing of the cooperative profit in this auxiliary game is given by the relations [13, p. 175]

$$y_1 = 9\frac{1+3}{4+4} = 4.5, \qquad y_2 = 9\frac{1+3}{4+4} = 4.5;$$

i.e., the share y_1 of the first player is given by the product of the cooperative profit equal to 9 by the ratio, whose numerator is equal to the sum of profits of the first player in the strongest equivalent equilibrium situations, and the denominator is equal to the sum of profits of both players in these situations. The fair share y_2 of the second player is defined in a similar way.

Obviously, cooperative sharing in this auxiliary game differs essentially from the solution of auxiliary problems of the first and second type. It follows that side profits in the original game affect the result of the game. Players should take the sharing found from the problem of the second type as a basis and correct it with regard of the sharing found from a problem of the third type.

4. DIFFERENTIAL GAMES WITH SIDE INTERESTS OF PLAYERS

Unlike static problems, for dynamic problems, the notion of A^c-equilibrium [9–15] is more fruitful than the notion of A-equilibrium; unlike the A-equilibrium, for it one can obtain necessary equilibrium conditions similar to necessary optimality conditions in variational problems, which, in turn, permit one to reduce the original differential game to some set of static "local" game problems (where Hamiltonians are used as payoff functions); their solution is much simpler than the solution of the original differential game.

Note that a formal generalization of the notion of A-equilibrium to differential games would imply that, for any attempt of one conflict participant to deviate at some time t' from some $A(t')$-equilibrium situation [i.e., from a point $x(t')$ lying on the trajectory $x(t)$], there always exists a penalizing strategy (i.e., by default it is assumed that such a penalizing exists somewhere on the remaining part of the trajectory). But in this case, on the one hand, the evaluation of $A(t')$ at time t' depends on infinitely many subsequent points on that trajectory, which is extremely complicated; consequently, it seems to be essentially impossible to find usable necessary optimality conditions for the set $A(t)$ in a similar case of the penalty implementation.

However, these disadvantages can be removed for a wide class of differential games by introducing (instead of A) the notion of an A^c-equilibrium [9–15], which implies that, for any admissible deviation of any ith player from a situation of equilibrium on some time interval, other players can use penalty strategies only on the same time interval rather than at some other times; in this connection, such an equilibrium can said to be "coordinated." Note that this constraint (introduced in the definition of the A^c-equilibrium) imposed on the character of penalty strategies is not important for a wide class of problems. From the practical viewpoint, the above-mentioned "defect" of the use of very simple necessary conditions of the A^c-equilibrium in comparison with the case of very unusable necessary conditions of the A-equilibrium is not very important, because first, the simplicity of finding a solution in the case of the use of necessary conditions for the existence of an A^c-equilibrium dominates much the threat that a found solution turns out to be not a solution, and second, in the general case, one can always verify whether a solution found with the use of necessary conditions of an A^c-equilibrium is really a solution. Thus, as a practical basis of the study of differential games, one can use the below-represented conditions [9–15] of an equilibrium in the class of pure and mixed strategies, which are based on the notion of an A^c-equilibrium and its strengthening similar to those used for the A-equilibrium.

Let us illustrate the possibility of the use of suggested new notions of equilibrium for the solution of differential games.

Consider conflicting dynamical systems described by differential equations in which the ith player ($i = 1, \dots, N$) uses pure strategies $u_i(t)$ or mixed strategies $q_i(u_i, t)$ to maximize his payoff functional (criterion)

$$J_i(q) = \int\limits_T dt \int\limits_{W_i(t)} f_0^i(u, x, t)\, dq, \qquad i = 1, \dots, N, \tag{7}$$

under the constraints

$$\dot{x} = \int\limits_{W'(t)} f(u, x, t)\, dq, \qquad t \in T = [t_0, t_1] \subset E^1, \tag{8}$$

$$(u, t) \in W' \subset U \times T, \tag{9}$$

$$x_j(t_0) = x_j^0, \qquad j = 1, \ldots, n, \qquad x_k(t_1) = x_k^1, \qquad k \in K \subset \{1, \ldots, n\}, \tag{10}$$

where $x = (x_1, \ldots, x_n) \in E^n$ is the n-dimensional Euclidean space; $U = \bigcup_{k=1}^N U_k$; U_k is a finite-dimensional space, $k = 1, \ldots, N$; $u = (u_1, \ldots, u_N)$; $W' = \bigcup_{k=1}^N W_k$; $W = \bigcap_{k=1}^N W_k$; W_i are compact sets in U; $q_i(u_i, t)$ is a mixed strategy of the ith player; $q(u, t) = q_1(u_1, t) \cdots q_N(u_N, t)$; $W(t)$ and $W'(t)$ are cross-sections of the sets W and W' at time $t \in T = [t_0, t_1]$; $\hat{U}_i = \operatorname{Pr}_{U_i} W$ is the projection of the set W onto U_i; Q_i is the set of mixed strategies $q_i(u_i, t)$ of the ith player in problem (7)–(9) with the initial condition $x(t_0) = x^0$ and with the set W replaced by the set $\hat{U} = \hat{U}_1 \times \cdots \times \hat{U}_N$; $q^i = q_1 \cdots q_{i-1} q_{i+1} \cdots q_N$; and $J^i = \sum_{k=1}^N J_k - J_i$.

Let G' be a subset of the set $Q = \prod_{i=1}^N Q_i$ formed only by strategies $q(u, t)$ that can provide the validity of all constraints of the problem, where condition (9) introduces a close dependence between strategies of players into problem (7)–(10), and the constraints (10) introduce an implicit dependence; in this connection, at Lebesgue almost every time $t \in T$, the set G' contains only measures $q(\cdot, t)$ whose supports lie in $W'(t)$. The sets G_i and G are defined in a similar way.

Assumption 2. Let $T = [t_0, t_1]$ be a bounded fixed segment of the real line E^1, let W be a compact set in $U \times T$; let

$$\hat{f} = (f_0^1, \ldots, f_0^N, f_1, \ldots, f_n) : \; U \times E^n \times T \to E^{n+N}$$

be a mapping such that the function $\hat{f}(u, x, \cdot)$ is Lebesgue measurable for all $u \in U$ and $x \in E^n$, the function $\hat{f}(\cdot, \cdot, t)$ is continuous for each $t \in T$, and the function $|\hat{f}|$ is majorized on T by the function $s(t)(|x| + 1)$, where $s(t)$ is some integrable function; let $x(t) : \; T \to E^n$ be an absolutely continuous function satisfying Eq. (8); in addition, let the function $\hat{f}$ satisfy the Lipschitz condition

$$|\hat{f}(u, \bar{x}, t) - \hat{f}(u, x, t)| \le b(t)|\bar{x} - x|$$

with an integrable function $b(t)$ for all $u \in U$, $x, \bar{x} \in E^n$, and $t \in T$.

Note that the A^c-equilibrium is obtained from Definition 1 of an A-equilibrium for game problems on intersecting sets by an additional requirement [after the list of requirements (1)–(3)] represented in the following definition.

Definition 6. A situation q^* in the differential game (7)–(10) is A_i^c-*extremal* if each of relations (1)–(3) holds under the condition that the nonzero (in the Lebesgue sense) set in T on which $\hat{q}^i(t) \ne q^{*i}(t)$ is a subset of a set in T on which $q_i(t) \ne q_i^*(t)$. A situation $q^* \in \bigcap_{i=1}^N A_i^c$ is an A^c-*equilibrium in a problem of the first*, *second*, *and third types*, respectively, if conditions (1)–(3), respectively, hold at the point q^* for all $i = 1, \ldots, N$; i.e., $A^c = A_1^c \cap \cdots \cap A_N$.

The following necessary conditions for an A^c-equilibrium permit one to reduce the solution of the original differential game to the solution of some auxiliary ("local") static games in which the Hamiltonians of players of the original differential game are used as payoff functions [9–15]. In general, in practice the assumption $A^c = A$ can be used even in the case of an arbitrarily complicated nonlinear problem (7)–(10), because a solution found with the use of below-represented necessary optimality conditions can approximately be estimated from the viewpoint of its optimality. In any case, it is almost impossible to find a solution of a complicated differential game without the below-represented theorem. Its use for the solution of very complicated essentially nonlinear differential games was illustrated by an example in [10, pp. 138–146].

To find solutions of differential games with side interests of players, one can use some modifications of necessary conditions of the existence of equilibria obtained in [10–12, 14], in particular, Theorem 3 in [9].

Theorem. *Let q^* be an A^c-equilibrium in problem* (7)–(10) *with N players*, *which satisfies Assumption* 2. *Then there exist N nonzero absolutely continuous vector functions $p^i(t) = (p_0^i, p_1^i(t), \ldots, p_n^i(t))$, $p_0^i = 1$, $i = 1, \ldots, N$, satisfying the equations*

$$\dot{p}_k^i = -\int\!\!\int_{W_i(t)} p^i \frac{\partial f^i}{\partial x_k} \, dq^*, \quad k = 1, \ldots, n, \quad i = 1, \ldots, N, \quad p_j^i(t_1) = 0, \quad j \notin K, \tag{11}$$

almost everywhere in T, *where* $f^i = (f_0^i, f_1, \ldots, f_n)$, *the Hamiltonians* $H^i = \int_{W_i(t)} p^i f^i \, dq^*$ *are continuous in* T, *and the* A^c-*equilibrium situation* q^* *satisfies the inequalities*

$$[H^i](\hat{q}^i, q_i) \le [H^i](q^*), \qquad q_i \in G(q^{*i}), \qquad \hat{q}^i \in G(q_i), \qquad i = 1, \ldots, N. \tag{12}$$

Example 2. Let us present a much more detailed study (with the use of the above-suggested new notions of equilibrium) of a rather complicated differential game with two players on intersecting game sets, which was posed in [9]. The players choose their strategies $u_1(t)$ and $u_2(t)$ so as to maximize their payoff functionals

$$J_1 = \int_0^1 x_1 \, dt, \qquad J_2 = \int_0^1 x_2 \, dt \tag{13}$$

under the constraints

$$\dot{x}_1 = f_1(u_1, u_2) = (u_1 - u_2)^2, \tag{14}$$

$$\dot{x}_2 = f_2(u_1, u_2) = (u_1 + u_2)^2, \tag{15}$$

$$x_1(0) = 0, \qquad x_2(0) = 0, \tag{16}$$

where the game set W_1 of the first player is given by the interior of the figure $OKFHO$ (Fig. 2) including its boundary, and the set W_2 is the interior of the figure $OEFLO$ together with its boundary. Thus the total game set is $W' = W_1 \cup W_2$, and the game set $W = W_1 \cap W_2 = OKFLO$ on which the players have a close conflict is given by the intersection of their game sets W_1 and W_2 on which their payoff functionals J_i are defined. Figure 2 represents some specific level lines of the functions $f_1 = \text{const}$ and $f_2 = \text{const}$.

First, let us find the strongest equilibrium in the natural class of weak threats (1).

Since the statement of problem (13)–(16) does not contain products of state coordinates and controls, and the problem is linear in the state coordinates, it follows that one can set $A^c = A$ and, by using the necessary optimality (equilibrium) conditions (14) and (15), reduce the solution of the considered differential game to the solution of only one auxiliary ("local") static game in which the Hamiltonians of players are payoff functions.

First, let us find a solution of Eq. (11) and use it to reduce the Hamiltonians of the game (13)–(16) to a form convenient for the statement of a "local" game. Since the Hamiltonians have the form

$$H^1 = p_0^1 x_1 + p_1^1 (u_1 - u_2)^2 + p_2^1 (u_1 + u_2)^2, \qquad H^2 = p_0^2 x_2 + p_1^2 (u_1 - u_2)^2 + p_2^2 (u_1 + u_2)^2,$$

it follows that Eqs. (11) can be reduced to the form

$$\dot{p}_1^1 = -p_0^1, \qquad p_1^1(1) = 0, \qquad \dot{p}_2^1 = 0, \qquad p_2^1(1) = 0,$$
$$\dot{p}_2^2 = -p_0^2, \qquad p_2^2(1) = 0, \qquad \dot{p}_1^2 = 0, \qquad p_1^2(0) = 0.$$

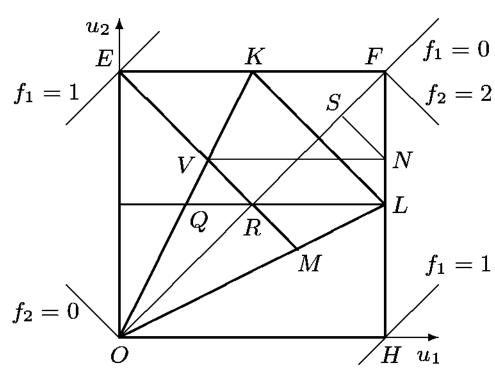

Fig. 2.

These equations have the following obvious solutions:

$$p_1^1 = p_0^1(1-t) = (1-t), \qquad p_2^1 = 0, \qquad p_2^2 = p_0^2(1-t) = (1-t), \qquad p_1^2 = 0;$$

their substitution into the Hamiltonians reduces them to the form

$$H^1 = x_1 + (1-t)(u_1 - u_2)^2, \qquad H^2 = x_2 + (1-t)(u_1 + u_2)^2.$$

Since $(1-t) > 0$ on the entire trajectory (outside the inessential point $t = 1$), it follows that finding a solution of the original differential game (13)–(16) can be reduced to finding a solution of the following auxiliary static "local" game (at each time t) with the payoff functions

$$f_1 = (u_1 - u_2)^2, \qquad f_2 = (u_1 + u_2)^2. \tag{17}$$

For the "local" game (17), we first find new equilibria $P = P_1 \cap P_2$ and D^P, which requires the simultaneous analysis of Fig. 2 and its mapping onto the plane (f_1, f_2) shown in Fig. 3. Note that the segment between the points O and (K, L) shown in Fig. 3 is described [in view of Eqs. (14) and (15)] by the equation $f_2 = 9f_1$, and the curve between the points F and (EH) is described by the equation

$$f_2 = 4 - 4\sqrt{f_1} + f_1.$$

The search of the equilibria $P = P_1 \cap P_2$ and D^P mentioned in Definitions 2 and 3 is very difficult even with the use of Fig. 3 :

$$P_1 = [VKF] \cup [NH], \qquad P_2 = [OK] \cup [KF],$$
$$P = [VK] \cup [KF], \qquad D^P = [VK].$$

Thus, as a preliminary result, in the auxiliary "local" game with side interests of players, we find that all situations from the segment $[VK]$ are equivalent to the strongest equilibria in Fig. 2.

Next, we find the strongest equilibria in the auxiliary problem of the type (1) with weak threats:

$$\begin{aligned} &A_1 = W_1, \qquad A_2 = EFLME, \qquad A = KFLMVK, \\ &B_1 = [OK] \cup [KF], \qquad B_2 = [ML] \cup [LN] \cup [VE], \qquad B = V. \end{aligned} \tag{18}$$

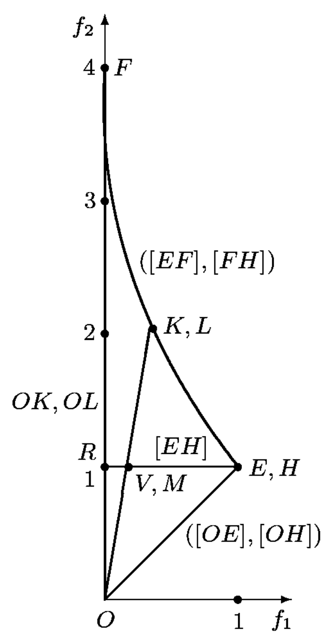

Fig. 3.

It follows from Fig. 2 that the set A_2 is the closed polygon $[EFLME]$, the set A is the closed polygon $[KFLMVK]$, the set B_1 consists of two closed straight-line segments $[OK]$ and $[KF]$, the set B_2 consists of three closed straight-line segments $[ML]$, $[LN]$, and $[VE]$, and the set B is reduced to the singleton V in Fig. 2, which provides the strongest equilibrium in the "local" game.

The analysis of the auxiliary "local" game (17) in the class of strong threats (2) lead to the same equilibria (18).

However, an auxiliary game on the intersection W of game sets of players leads to a set of equilibria that contains both the strongest equilibrium found above in local problems and some additional set $[OQ)$. Indeed, we have

$$A_1^W = W, \qquad A_2^W = KFLK \cup OK = A^W,$$
$$B_1^W = [OK] \cup [KF], \qquad B_2^W = [OQ) \cup [LN] \cup [VK], \qquad B^W = [VK] \cup [OQ).$$

The intersection of all types of strongest equilibria in the above-considered auxiliary local problems provides the unique point V, which should be chosen as the strongest equilibrium. In this case, it is natural to consider that all points of the half-open interval $(VK]$ are weaker equilibria, and all point of the half-open interval $[OQ)$ are even weaker equilibria.

The point V of the strongest equilibrium in auxiliary "local" games defines the pair of constant equilibrium strategies of players $(u_1^*, u_2^*) = (1/3, 2/3)$ in the original differential game. By substituting this pair into the original differential game, we obtain the equations

$$\dot{x}_1 = (u_1^* - u_2^*)^2 = 1/9, \qquad \dot{x}_2 = (u_1^* + u_2^*)^2 = 1.$$

By integrating those equations, we get $x_1 = t/9$ and $x_2 = t$. After the substitution of these solutions into the payoff functionals, we have the following profits of players in the equilibrium situation:

$$J_1 = \int_0^1 x_1\, dt = \frac{1}{18}, \qquad J_2 = \int_0^1 x_2\, dt = \frac{1}{2}.$$

If the players cooperate, then they can get much more than in the equilibrium situation. Indeed, a cooperative solution is attained at the point F in Fig. 2 at which their strategies are equal to $u_1 = u_2 = 1$. The equations of motion in the case of a cooperative solution can be reduced to the form $\dot{x}_1 = 0$, $\dot{x}_2 = 4$, and their solutions, to the form $x_1 = 0$, $x_2 = 4t$. By evaluating the cooperative profit of players, we obtain $J_1 + J_2 = J_2 = \int_0^1 4t\, dt = 2$, which is much more than the sum of their profits in the strongest equilibrium situation. The fair sharing of the cooperative profit is given by formulas (4) in [9] (or (4.2) in [13, p. 174]): $y_1 = 2/10$, $y_2 = 18/10$, where y_i is the fair share of the ith player in the cooperative profit, which is equal to 2.

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