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NUMERICAL METHODS

Estimates for the Difference between Exact and Approximate Solutions of Parabolic Equations on the Basis of Poincaré Inequalities for Traces of Functions on the Boundary

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Abstract—We study a method for the derivation of majorants for the distance between the exact solution of an initial–boundary value reaction–convection–diffusion problem of the parabolic type and an arbitrary function in the corresponding energy class. We obtain an estimate (for the deviation from the exact solution) of a new type with the use of a maximally broad set of admissible fluxes. In the definition of this set, the requirement of pointwise continuity of normal components of the dual variable (which was a necessary condition in earlier-obtained estimates) is replaced by the requirement of continuity in the weak (integral) sense. This result can be achieved with the use of the domain decomposition and special embedding inequalities for functions with zero mean on part of the boundary or for functions with the zero mean over the entire domain.

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1. INTRODUCTION

The present paper deals with guaranteed and computable estimates for the difference between generalized solutions of evolution reaction–convection–diffusion problems of the parabolic type and arbitrary functions that belong to admissible energy classes. Such estimates are often said to be a posteriori. They are required for numerical analysis, where one needs an adequate estimate for the accuracy of a particular approximate solution constructed by an arbitrary numerical method.

First functional a posteriori error estimates for linear elliptic equations were obtained independently in [1, 2]. However, they had the essential disadvantage of complicated practical implementation owing to the fact that such estimates hold only under a certain condition; more precisely, the dual variable occurring in the estimate (the flux in scalar problems and the stress in vector ones) should satisfy differential equations of the divergence type. It was shown in [3–6] that guaranteed functional estimates of the error can be derived under much less restrictive conditions for the dual variable, which permits one to derive efficient and easy-to-compute majorants for the error. Variational arguments were used in the first papers dealing with this approach (see [3] and [4]). Later, similar results were obtain by a different (nonvariational) method based on transformations of the generalized statement of the problem [5, 6]. Functional majorants do not contain local interpolation constants depending on the grid. They are *universal*; i.e., they hold for arbitrary functions in an admissible energy class. They can be computed with the use of standard well-studied finite element approximations in the space of $H(\mathrm{div})$-continuous finite elements (for example, Raviart–Thomas and Brezzi–Douglas–Marini elements). For many equations of mathematical physics, it was theoretically shown and practically justified that majorants of the functional type also generate efficient local error indicators, which can be used as a reliable criterion for adaptive algorithms. A detailed comparison of the above-described approaches can be found in the monograph [7].

In the present paper, we use a functional approach for the derivation of guaranteed and practically computable estimates for the distance between an arbitrary function (in an admissible energy class) and the exact solution of an initial–boundary value problem of the parabolic type

$$u_t - \operatorname{div} p + a \cdot \nabla u + \lambda^2 u = f, \qquad (x,t) \in Q_T, \tag{1}$$

$$p = A\nabla u, \qquad (x,t) \in Q_T, \tag{2}$$

$$u(\cdot, 0) = u_0, \qquad x \in \Omega, \tag{3}$$

$$u = g, \qquad (s,t) \in S_D, \tag{4}$$

$$p \cdot n = F, \qquad (s,t) \in S_N, \tag{5}$$

where $Q_T := \Omega \times (0,T)$ is a space-time cylinder in which $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, is a bounded domain with Lipschitz boundary $\partial\Omega$ and $(0,T)$, $T > 0$, is a time interval. Next, $S_T := \partial\Omega \times [0,T]$ stands for the cylindrical lateral surface, where $\partial\Omega$ consists of measurable nonintersecting parts Γ_D and Γ_N corresponding to the mixed Dirichlet–Neumann boundary condition,

$$S_T := \partial\Omega \times [0,T] = (\Gamma_D \cup \Gamma_N) \times [0,T] = S_D \cup S_N.$$

On the boundary part S_N, the vector n defines the unit outward normal on $\partial\Omega$, and the following conditions are satisfied:

$$f \in L^2(Q_T), \quad u_0 \in H^1_g(\Omega), \quad g \in L^2(0,T;H^1(\Gamma_D)) \quad \text{and} \quad F \in L^2(0,T;H^1(\Gamma_N)). \tag{6}$$

The convection and reaction functions a and λ satisfy the conditions

$$a \in L^\infty(\Omega, \mathbb{R}^d), \qquad \operatorname{div} a \in L^\infty(\Omega), \qquad |a| \le \overline{a}, \tag{7}$$

$$\lambda \in L^\infty(\Omega), \qquad |\lambda| \le \overline{\lambda}, \tag{8}$$

for almost all $t \in (0,T)$. Moreover, the flux a satisfies the condition

$$\varkappa(x,t) = (n \cdot a)(x,t) > 0 \quad \text{for almost all} \quad (x,t) \in S_N. \tag{9}$$

The matrix $A = \{A_{ij}\}_{i,j=1}^d$ $[A_{ij} \in L^\infty(\Omega)]$ is symmetric and satisfies the condition

$$\underline{\nu}_A |\xi|^2 \le A(x)\xi \cdot \xi \le \overline{\nu}_A |\xi|^2, \qquad \xi \in \mathbb{R}^d, \qquad 0 < \underline{\nu}_A \le \overline{\nu}_A < \infty, \tag{10}$$

where $|\xi| := \sqrt{\xi \cdot \xi}$. By virtue of condition (10), one can introduce the equivalent norms

$$\|\tau\|^2_{\Omega,A} := \int\limits_\Omega A\tau \cdot \tau \, dx, \quad \tau \in L^2(\Omega, \mathbb{R}^d), \quad \text{and} \quad \|q\|^2_{Q_T,A} := \int\limits_{Q_T} Aq \cdot q \, dx\, dt, \quad q \in L^2(Q_T, \mathbb{R}^d).$$

By multiplying (1) by a test function $\eta \in H_0^{1,1}(Q_T)$, where

$$H_0^{1,1}(Q_T) := \{u \in L^2(Q_T) |\ \nabla u \in L^2(Q_T, \mathbb{R}^d),\ \partial_t u \in L^2(Q_T),\ u|_{S_D} = 0\},$$

one can give the generalized statement of problem (1)–(5) as follows: find a function $u \in H_g^{1,1}(Q_T) := H_0^{1,1}(Q_T) + g$ satisfying the integral identity

$$\int\limits_{Q_T} (A\nabla u \cdot \nabla \eta + \lambda^2 u\eta + a \cdot \nabla u \eta - u\eta_t)\, dx\, dt + \int\limits_\Omega ((u\eta)(x,T) - (u\eta)(x,0))\, dx$$

$$= \int\limits_{Q_T} f\eta \, dx\, dt + \int\limits_{S_N} F\eta \, ds\, dt, \qquad \eta \in H_0^{1,1}(Q_T). \tag{11}$$

The generalized problem (11) is uniquely solvable provided that relations (6)–(8) hold (e.g., see [8–11]).

In the present paper, we derive a posteriori error estimates following [12], where a method for the derivation of functional error majorants for parabolic equations was suggested for the first time for the example of the thermal conduction problem. First examples of the practical use of functional a posteriori estimates for the heat equation were described in [13], and the properties of two-sided error estimates for the reaction–diffusion equation were studied in detail in [14]. A form of the error majorant for evolution convection–diffusion equations which admits a discontinuity of the arbitrary function with respect to the time variable was obtained in [15]. Error majorants for parabolic time-periodic problems (with a polyharmonic finite-element discretization) were considered in [16]. Indicators based on an estimate of the residual functional for a class of evolution equations were considered in [17–20] (see also the bibliography therein).

A majorant of the error function $e := u - v$ [where $v \in H_g^{1,1}(Q_T)$ is assumed to be an arbitrary function] is constructed for the norm

$$[e]^2_{(\nu,\theta,\chi,\zeta)} := \nu\|\nabla e\|^2_{Q_T,A} + \|\theta(\varrho)e\|^2_{Q_T} + \|\chi(\varkappa)e\|^2_{S_N} + \zeta\|e(\cdot,T)\|^2_{\Omega}, \tag{12}$$

where $\|\cdot\|^2_{\omega}$ is the L^2-norm on the set ω, ν and ζ are some positive weights, and $\theta(\varrho)$ and $\chi(\varkappa)$ are positive weight functions depending on

$$\varrho_0^2 \le \varrho^2 := \lambda^2 - \frac{1}{2}\operatorname{div} a \le \overline{\varrho}(\overline{\lambda}, \overline{a}) \tag{13}$$

as well as on $\varkappa$ [see condition (9)]. The norm (12) is the natural (energy) norm for problem (11).

The first part of the paper deals with functional a posteriori estimates for the error in the distance to the exact solution of the considered problem, which contain global constants (i.e., constants related to the entire domain) in the Friedrichs inequality and the trace inequality. For domains of complicated geometric form with mixed boundary conditions, the computation of these constants (or their upper guaranteed bounds) is a very complicated problem. These computations can be simplified if one uses the decomposition of the domain Ω into a set of sufficiently simple convex subdomains. This way, one can obtain estimates containing only the corresponding local constants (see [6, 21, 22]). In the second part of the present paper, we consider a posteriori error estimates under much weaker constraints on the continuity of the auxiliary fluxes; more precisely, the requirement of pointwise continuity on the boundaries of subdomains (elements) is replaced by the requirement of continuity of the means. The resulting set differs from the earlier used set [14, 21] and provides more freedom in the implementation of estimates.

2. ERROR ESTIMATE USING A CONTINUOUS FLUX FIELD

First, we obtain a majorant for the distance to the exact solution of a reaction–convection–diffusion equation of the parabolic type with mixed Dirichlet–Neumann boundary conditions. In the derivation of these estimates, we use transformations of integral identities and inequalities which follow from the embedding theorem for functions defined on the entire domain Ω. Note that the error function $e = u - v$ can be used as a test function in identity (11), which gives the relation

$$\int_{Q_T} (A\nabla e \cdot \nabla e + (a \cdot \nabla e)e + \lambda^2 e^2)\,dx\,dt + \frac{1}{2}\|e(\cdot,T)\|^2_{\Omega} - \frac{1}{2}\|e(\cdot,0)\|^2_{\Omega}$$
$$= \int_{Q_T} ((f - v_t - \lambda^2 v - a\cdot\nabla v)e - A\nabla v\cdot\nabla e)\,dx\,dt + \int_{S_N} Fe\,ds\,dt. \tag{14}$$

By the Gauss divergence theorem, we have the relation

$$\int_{Q_T} \operatorname{div} ae^2\,dx\,dt = -\int_{Q_T} a\cdot\nabla(e^2)\,dx\,dt + \int_{S_N} (a\cdot n)e^2\,ds\,dt = -2\int_{Q_T} (a\cdot\nabla e)e\,dx\,dt + \int_{S_N} \varkappa e^2\,ds\,dt,$$

whence it follows that

$$\int_{Q_T} (a\cdot\nabla e)e\,dx\,dt = \frac{1}{2}\left(\int_{S_N} \varkappa e^2\,ds\,dt - \int_{Q_T} \operatorname{div} ae^2\,dx\,dt\right). \tag{15}$$

Therefore, by substituting the expression (15) into relation (14), we obtain the energy identity

$$[e]^2_{(1,\varrho,\sqrt{\varkappa/2},1/2)} := \int_0^T \left(\|\nabla e\|^2_{\Omega,A} + \|\varrho e\|^2_\Omega + \left\| \sqrt{\frac{\varkappa}{2}} e \right\|^2_{\Gamma_N} \right) dt + \frac{1}{2}\|e(\cdot,T)\|^2_\Omega$$
$$= \frac{1}{2}\|e(\cdot,0)\|^2_\Omega + \int_{Q_T} ((f - v_t - \lambda^2 v - a\cdot\nabla v)e - A\nabla v\cdot\nabla e)\,dx\,dt + \int_{S_N} Fe\,ds\,dt. \quad (16)$$

Next, we transform the right-hand side of relation (16) with the use of an auxiliary vector function

$$y \in Y^{S_N}_{\operatorname{div}}(Q_T) := \{y \in L^2(0,T;L^2(\Omega,\mathbb{R}^d))|\ \operatorname{div} y \in L^2(Q_T),\ y\cdot n \in L^2(S_N)\},$$

satisfying the identity

$$\int_{Q_T} (\operatorname{div} yw + y\cdot\nabla w)\,dx\,dt = \int_{S_N} y\cdot nw\,ds\,dt, \qquad w \in H^{1,1}_0(Q_T). \quad (17)$$

By using property (17), one can represent relation (16) in the form

$$[e]^2_{(1,\varrho,\sqrt{\varkappa/2},1/2)} = \frac{1}{2}\|e(\cdot,0)\|^2_\Omega + \int_{Q_T} (\mathfrak{R}_f(v,y)e + \mathfrak{R}_A(v,y)\cdot\nabla e)\,dx\,dt + \int_{S_N} \mathfrak{R}_F(v,y)e\,ds\,dt, \quad (18)$$

where

$$\mathfrak{R}_f(v,y) := f - v_t - \lambda^2 v - a\cdot\nabla v + \operatorname{div} y, \quad (19)$$
$$\mathfrak{R}_A(v,y) := y - A\nabla v, \quad (20)$$
$$\mathfrak{R}_F(v,y) := F - y\cdot n. \quad (21)$$

The functionals $\mathfrak{R}_f(v,y)$, $\mathfrak{R}_A(v,y)$, and $\mathfrak{R}_F(v,y)$ are the residuals in the differential relations forming system (1)–(5). In what follows, we also use the "weighted" residuals

$$\mathfrak{R}^\mu_f(v,y) := \mu\mathfrak{R}_f, \qquad \mathfrak{R}^{1-\mu}_f(v,y) := (1-\mu)\mathfrak{R}_f, \quad (22)$$
$$\mathfrak{R}^\eta_F(v,y) := \eta\mathfrak{R}_F, \qquad \mathfrak{R}^{1-\eta}_F(v,y) := (1-\eta)\mathfrak{R}_F. \quad (23)$$

Here μ and η are real-valued functions ranging on the interval $[0,1]$; more precisely, the inclusions

$$\mu \in L^\infty_{[0,1]}(\Omega) := \{\xi \in L^\infty(\Omega)|\ 0 \le \xi \le 1 \ \text{ almost everywhere on } \ \Omega\}, \quad (24)$$
$$\eta \in L^\infty_{[0,1]}(\Gamma_N) := \{\xi \in L^\infty(\Gamma_N)|\ 0 \le \xi \le 1 \ \text{ almost everywhere on } \ \Gamma_N\} \quad (25)$$

hold for almost all $t \in (0,T)$. The functions ν and η are introduced to counterbalance the contribution of ϱ and $\varkappa$ in the corresponding residuals by dividing them into two integrals with different weights. The resulting estimate is more stable under strong changes in ϱ and $\varkappa$ on some subdomains of Ω or the boundary parts Γ_N. These properties of the majorant were noted in the paper [14] for an example of a parabolic reaction–diffusion equation.

In what follows, we need two sets $\Omega^{\varrho,+}$ and $\Gamma^{\varkappa,+}_N$, where ϱ and $\varkappa$, respectively, are nonzero for almost all $t \in (0,T)$,

$$\Omega^{\varrho,+} := \{x \in \Omega|\ \varrho(x) \neq 0\} \quad \text{and} \quad \Gamma^{\varkappa,+}_N := \{x \in \Gamma_N|\ \varkappa(x) \neq 0\}.$$

The following theorem shows that a weighted combination of norms of the residuals (19)–(23) controls the distance between u and v measured in the norm (12).

Theorem 1. *For arbitrary $v \in H^{1,1}_g(Q_T)$ and $y \in Y^{S_N}_{\rm div}(Q_T)$, one has the estimate*

$$
\begin{aligned}
[e]^2_{(\nu,\theta,\chi,1)} &\le \overline{\mathrm{M}}^2_{\mathrm{I}}(v,y;\delta,\gamma_j,\mu_+,\eta_+,\alpha_i) \\
&:= \|e(\cdot,0)\|^2_\Omega + \int\limits_0^T \Bigg(\gamma_1 \left\| \frac{1}{\varrho}\,\mathfrak{R}^{\mu_+}_f \right\|^2_{\Omega^{\varrho,+}} + \gamma_2 \left\| \frac{1}{\sqrt{\varkappa}}\,\mathfrak{R}^{\eta_+}_F \right\|^2_{\Gamma^{\varkappa,+}_N} + \alpha_1 \|\mathfrak{R}_A\|^2_{\Omega,A^{-1}} \\
&\quad + \alpha_2 \frac{C^2_{\mathrm{F}\Omega}}{\underline{\nu}_A} \|\mathfrak{R}^{1-\mu_+}_f\|^2_{\Omega^{\varrho,+}} + \alpha_3 \frac{\widetilde{C}^2_{\mathrm{Tr}\,\Gamma_N}}{\underline{\nu}_A} \|\mathfrak{R}^{1-\eta_+}_F\|^2_{\Gamma^{\varkappa,+}_N} \Bigg)\,dt, \qquad (26)
\end{aligned}
$$

where $\delta \in (0,2]$, $\gamma_1 \in [1/2,+\infty)$, and $\gamma_2 \in [1,+\infty)$ are parameters of the majorant. In addition, for almost all $t \in (0,T)$, the weight functions μ_+ and η_+ satisfy the inclusions

$$\mu_+ \in L^{\varrho,+}_{[0,1]}(\Omega) := \{\mu \in L^\infty_{[0,1]}(\Omega) |\ \mu(x) \equiv 0 \quad \text{on} \quad \Omega \backslash \Omega^{\varrho,+}\}, \qquad (27)$$

$$\eta_+ \in L^{\varkappa,+}_{[0,1]}(\Gamma_N) := \{\eta \in L^\infty_{[0,1]}(\Gamma_N) |\ \eta(s) \equiv 0 \quad \text{on} \quad \Gamma_N \backslash \Gamma^{\varkappa,+}_N\} \qquad (28)$$

and the $\alpha_i(t)$, $i = 1,2,3$, are positive definite real-valued functions satisfying the identity

$$\sum_{i=1}^{3} \frac{1}{\alpha_i(t)} = \delta. \qquad (29)$$

The weights in $[e]^2_{(\nu,\theta,\chi,1)}$ are defined as follows:

$$\nu = 2-\delta, \quad \theta(x,t) = \varrho(x)\left(2 - \frac{1}{\gamma_1(t)}\right)^{1/2}, \quad \text{and} \quad \chi(s,t) := \left(\varkappa(s)\left(1 - \frac{1}{\gamma_2(t)}\right)\right)^{1/2}.$$

The constant $\widetilde{C}_{\mathrm{Tr}\,\Gamma_N} = C_{\mathrm{Tr}\,\Gamma_N}(1 + C_{\mathrm{F}\,\Omega})$ includes a constant from the Friedrichs inequality [see inequality (40) below] and a constant from the inequality for the trace of a function on the boundary [see inequality (41) below].

Proof. Consider identity (18) and state it as follows:

$$[e]^2_{(1,\varrho,\sqrt{\varkappa/2},1/2)} = \mathfrak{I}_f + \mathfrak{I}_A + \mathfrak{I}_F + \frac{1}{2}\|e(\cdot,0)\|^2_\Omega,$$

where

$$\mathfrak{I}_f := \int\limits_{Q_T} \mathfrak{R}_f e\,dx\,dt, \quad \mathfrak{I}_A := \int\limits_{Q_T} \mathfrak{R}_A \cdot \nabla e\,dx\,dt, \quad \text{and} \quad \mathfrak{I}_F := \int\limits_{S_N} \mathfrak{R}_F e\,ds\,dt. \qquad (30)$$

The term $\mathfrak{I}_A$ can be estimated with the use of the Hölder inequality

$$\mathfrak{I}_A \le \int\limits_0^T \|\mathfrak{R}_A\|_{\Omega,A^{-1}} \|\nabla e\|_{\Omega,A}\,dt. \qquad (31)$$

Next, to estimate the term $\mathfrak{I}_f$, we introduce the function $\mu_+ \in L^{\varrho,+}_{[0,1]}$ [see (27)], which counterbalances the contribution of the term with factor $1/\varrho$ sharply increasing for sufficiently small ϱ. (A similar method was used in [7, 23].) We thereby obtain the estimate

$$\mathfrak{I}_f = \int\limits_{Q_T} (\mathfrak{R}^{\mu_+}_f + \mathfrak{R}^{1-\mu_+}_f) e\,dx\,dt \le \int\limits_0^T \left(\left\| \frac{1}{\varrho}\,\mathfrak{R}^{\mu_+}_f \right\|_{\Omega^{\varrho,+}} \|\varrho e\|_{\Omega^{\varrho,+}} + \frac{C_{\mathrm{F}\,\Omega}}{\sqrt{\underline{\nu}_A}} \|\mathfrak{R}^{1-\mu_+}_f\|_\Omega \|\nabla e\|_{\Omega,A} \right) dt. \qquad (32)$$

A similar argument is used to estimate the term $\mathfrak{I}_F$. By using the auxiliary function η_+ [see relation (28)] and the trace inequality [see relation (41) below], one can counterbalance the contribution of the function $\varkappa$ into the majorant,

$$\mathfrak{I}_F \leq \int\limits_0^T \left(\left\| \frac{1}{\sqrt{\varkappa}} \mathfrak{R}_F^{\eta_+} \right\|_{\Gamma_N^{\varkappa,+}} \|\sqrt{\varkappa}\, e\|_{\Gamma_N^{\varkappa,+}} + \frac{\widetilde{C}_{\mathrm{Tr}\,\Gamma_N}}{\sqrt{\underline{\nu}_A}} \|\mathfrak{R}_F^{1-\eta_+}\|_{\Gamma_N} \|\nabla e\|_{\Omega,A} \right) dt. \tag{33}$$

By adding the resulting estimates (31)–(33), we obtain

$$\begin{aligned}[e]^2_{(1,1,\sqrt{\varkappa/2},1/2)} \leq \frac{1}{2}\|e(x,0)\|^2_\Omega &+ \int\limits_0^T \|\mathfrak{R}_A\|_{A^{-1}} \|\nabla e\|_A \, dt \\ &+ \int\limits_0^T \left(\left\| \frac{1}{\varrho} \mathfrak{R}_f^{\mu_+} \right\|_{\Omega^{\varrho,+}} \|\varrho e\|_{\Omega^{\varrho,+}} + \frac{C_{\mathrm{F}\,\Omega}}{\sqrt{\underline{\nu}_A}} \|\mathfrak{R}_f^{1-\mu_+}\|_\Omega \|\nabla e\|_{\Omega,A} \right) dt \\ &+ \int\limits_0^T \left(\left\| \frac{1}{\sqrt{\varkappa}} \mathfrak{R}_F^{\eta_+} \right\|_{\Gamma_N^{\varkappa,+}} \|\sqrt{\varkappa} e\|_{\Gamma_N^{\varkappa,+}} + \frac{\widetilde{C}_{\mathrm{Tr}\,\Gamma_N}}{\sqrt{\underline{\nu}_A}} \|\mathfrak{R}_F^{1-\eta_+}\|_{\Gamma_N} \|\nabla e\|_{\Omega,A} \right) dt. \end{aligned} \tag{34}$$

The terms on the right-hand side in inequality (34) containing $\|\varrho e\|_{\Omega^{\varrho,+}}$ and $\|\sqrt{\varkappa}\, e\|_{\Gamma_N^{\varkappa,+}}$ can be estimated with the use of the Young inequality,

$$\int\limits_0^T \left\| \frac{1}{\varrho} \mathfrak{R}_f^{\mu_+} \right\|_{\Omega^{\varrho,+}} \|\varrho e\|_\Omega \, dt \leq \int\limits_0^T \left(\frac{\gamma_1(t)}{2} \left\| \frac{1}{\varrho} \mathfrak{R}_f^{\mu_+} \right\|^2_{\Omega^{\varrho,+}} + \frac{1}{2\gamma_1(t)} \|\varrho e\|^2_{\Omega^{\varrho,+}} \right) dt, \tag{35}$$

$$\int\limits_0^T \left\| \frac{1}{\sqrt{\varkappa}} \mathfrak{R}_F^{\eta_+} \right\|_{\Gamma_N^{\varkappa,+}} \|\sqrt{\varkappa} e\|_{\Gamma_N^{\varkappa,+}} \, dt \leq \int\limits_0^T \left(\frac{\gamma_2(t)}{2} \left\| \frac{1}{\sqrt{\varkappa}} \mathfrak{R}_F^{\eta_+} \right\|^2_{\Gamma_N^{\varkappa,+}} + \frac{1}{2\gamma_2(t)} \|\sqrt{\varkappa} e\|^2_{\Gamma_N^{\varkappa,+}} \right) dt, \tag{36}$$

where the $\gamma_j(t)$, $j = 1, 2$, are arbitrary real-valued functions ranging in the intervals $[1/2, +\infty)$ and $[1, +\infty)$, respectively. In a similar way, we have

$$\int\limits_0^T \frac{C_{\mathrm{F}\,\Omega}}{\sqrt{\underline{\nu}_A}} \|\mathfrak{R}_f^{1-\mu_+}\|_\Omega \|\nabla e\|_{\Omega,A} \, dt \leq \frac{1}{2} \int\limits_0^T \left(\alpha_1(t) \frac{C^2_{\mathrm{F}\,\Omega}}{\underline{\nu}_A} \|\mathfrak{R}_f^{1-\mu_+}\|^2_\Omega + \frac{1}{\alpha_1(t)} \|\nabla e\|^2_{\Omega,A} \right) dt, \tag{37}$$

$$\int\limits_0^T \|\mathfrak{R}_A\|_{\Omega,A^{-1}} \|\nabla e\|_{\Omega,A} \, dt \leq \frac{1}{2} \int\limits_0^T \left(\alpha_2(t) \|\mathfrak{R}_A\|^2_{\Omega,A^{-1}} + \frac{1}{\alpha_2(t)} \|\nabla e\|^2_{\Omega,A} \right) dt, \tag{38}$$

and

$$\int\limits_0^T \frac{\widetilde{C}_{\mathrm{Tr}\,\Gamma_N}}{\sqrt{\underline{\nu}_A}} \|\mathfrak{R}_F^{1-\eta_+}\|_{\Gamma_N^{\varkappa,+}} \|\nabla e\|_A \, dt \leq \frac{1}{2} \int\limits_0^T \left(\alpha_3(t) \frac{\widetilde{C}^2_{\mathrm{Tr}\,\Gamma_N}}{\underline{\nu}_A} \|\mathfrak{R}_F^{1-\eta_+}\|^2_{\Gamma_N^{\varkappa,+}} + \frac{1}{\alpha_3(t)} \|\nabla e\|^2_{\Omega,A} \right) dt, \tag{39}$$

where the $\alpha_i(t)$, $i = 1, 2, 3$, are real-valued positive definite functions satisfying identity (29). Then, by combining the estimates (35)–(39), we obtain the majorant (26). The proof of the theorem is complete.

3. MAJORANT USING A DISCONTINUOUS FLUX FIELD

The error majorant obtained in Section 2 contains the Friedrichs constant from the inequality

$$\|w\|_\Omega \le C_{\mathrm{F}\,\Omega}\|\nabla w\|_\Omega, \qquad w \in H_0^1(\Omega), \tag{40}$$

and the constant from the trace inequality

$$\|w\|_{\Gamma_N} \le C_{\mathrm{Tr}\,\Gamma_N}\|w\|_{H^1(\Omega)}, \qquad w \in H_0^1(\Omega), \tag{41}$$

on the boundary Γ_N. If the domain has a complicated geometric structure, then the computation of these constants (or their guaranteed upper bounds) is a quite complicated technical problem. To avoid these difficulties, the method of decomposition of a polygonal domain Ω into a set of nonoverlapping convex subdomains was used in [21, 24], and similar functional inequalities on the local level were used for these subdomains. In what follows, we show that the use of special classical and Poincaré-type inequalities permits maximally weaken the constraints in the space of admissible fluxes. The exact values, as well as upper bounds, for the corresponding constants were obtained in [22, 25–27].

Assume that

$$\overline{\Omega} := \bigcup_{\Omega_i \in \mathcal{O}_\Omega} \overline{\Omega}_i, \qquad \Omega_i \cap \Omega_j = \varnothing, \qquad i \ne j, \qquad i, j = 1, \dots, N, \tag{42}$$

where Ω_i is a convex domain with Lipschitz boundary. The subdomain Ω_i is an element of the set $\mathcal{O}_\Omega$. In practice, $\{\Omega_i\}_{i=1}^N$ are usually represented by disjoint simplices or convex polygons. The set of all faces of the resulting subpartition is denoted by $\mathcal{G}$, and the elements of this set form the subsets

$$\mathcal{G}_{\mathrm{int}} = \{\Gamma_{ij} \in \mathcal{G} |\ \Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j\}, \tag{43}$$

$$\mathcal{G}_D = \{\Gamma_{Di} \in \mathcal{G} |\ \Gamma_{Di} = \overline{\Omega}_i \cap \Gamma_D\}, \tag{44}$$

$$\mathcal{G}_N = \{\Gamma_{Ni} \in \mathcal{G} |\ \Gamma_{Ni} = \overline{\Omega}_i \cap \Gamma_N\}. \tag{45}$$

For each Ω_i, the Poincaré inequality has the form

$$\|w\|_{\Omega_i} \le C^{\mathrm{P}}_{\Omega_i}\|\nabla w\|_{\Omega_i} \le \frac{\operatorname{diam}\Omega_i}{\pi}\|\nabla w\|_{\Omega_i}, \qquad w \in \widetilde{H}^1(\Omega_i), \tag{46}$$

where $\widetilde{H}^1(\Omega_i) := \{u \in H^1(\Omega_i) \,|\, \{|u|\}_{\Omega_i} = 0\}$ and $\{|u|\}_{\Omega_i} := |\Omega_i|^{-1}\int_{\Omega_i} w\,dx$. Here we use the estimate obtained for the constant $C^{\mathrm{P}}_{\Omega_i}$ in [25]. In addition, we use the following so-called Poincaré-type inequalities for the functions $\widetilde{H}^1(\Omega_i, \Gamma_i) := \{u \in H^1(\Omega_i) \,|\, \{|u|\}_{\Gamma_i} = 0\}$, where Γ_i is a part of the boundary $\partial\Omega_i$ (coinciding with Γ_{ij} or Γ_{Ni}) :

$$\|w\|_{\Omega_i} \le C^{\mathrm{P}}_{\Gamma_i}\|\nabla w\|_{\Omega_i}, \tag{47}$$

$$\|w\|_{\Gamma_i} \le C^{\mathrm{Tr}}_{\Gamma_i}\|\nabla w\|_{\Omega_i}. \tag{48}$$

In the new form of the majorant obtained in this section, we use the following space of admissible fluxes: $y \in \hat{Y}^{S_N}_{\mathrm{div}}(Q_T, \mathcal{O}_\Omega)$, where

$$\begin{aligned}\hat{Y}^{S_N}_{\mathrm{div}}(Q_T, \mathcal{O}_\Omega) := \Big\{ & y(x) = y_i(x), \ \text{ if } \ x \in \Omega_i \,|\, y_i \in L^2(\Omega_i, \mathbb{R}^d), \ \operatorname{div} y_i \in L^2(\Omega_i), \\ & \{|\operatorname{div} y_i + f - a\cdot\nabla v - \lambda^2 v|\}_{\Omega_i} = 0, \ \Omega_i \in \mathcal{O}_\Omega, \\ & \{|(y_i - y_j)\cdot n_{ij}|\}_{\Gamma_{ij}} = 0, \ \Gamma_{ij} \in \mathcal{G}_{\mathrm{int}}, \\ & \{|y_i\cdot n_i - F|\}_{\Gamma_{Ni}} = 0, \ \Gamma_{Ni} \in \mathcal{G}_{\mathrm{N}} \Big\}\end{aligned} \tag{49}$$

for almost all $t \in (0, T)$.

The set $\hat{Y}^{S_N}_{\rm div}(Q_T, \mathcal{O}_\Omega)$ substantially differs from $Y^{S_N}_{\rm div}(Q_T)$. Here the requirement of continuity of the normal components of fluxes are maximally weakened. For a simpler elliptic boundary value problem, a similar space was considered in [28]. The extension of the space $Y^{S_N}_{\rm div}(Q_T)$ is useful from the practical viewpoint. Indeed, the functions in $Y^{S_N}_{\rm div}(Q_T)$ should have a continuous normal component on all $\Gamma_{ij} \in \mathcal{G}_{\rm int}$ and satisfy the pointwise boundary condition on $\Gamma_{Ni} \in \mathcal{G}_N$. This imposes very restrictive conditions on the corresponding approximation. A function in $\hat{Y}^{S_N}_{\rm div}(Q_T, \mathcal{O}_\Omega)$ satisfies a weaker condition; more precisely, the normal flux components should be continuous on the boundaries of subdomains only in the integral sense. In a similar way, the Neumann condition is required to be valid only in the sense of average values on the boundary parts Γ_N. These properties simplify the practical construction of the vector function y.

Integration by parts for $y \in \hat{Y}^{S_N}_{\rm div}(Q_T, \mathcal{O}_\Omega)$ has the following form: the relation

$$\sum_{\Omega_i \in \mathcal{O}_\Omega} \int\limits_{\Omega_i} (y \cdot \nabla w + \operatorname{div} y w)\, dx = \sum_{\Gamma_{ij} \in \mathcal{G}_{\rm int}} \int\limits_{\Gamma_{ij}} (y_i - y_j) \cdot n_{ij} w\, ds + \sum_{\Gamma_{Ni} \in \mathcal{G}_N} \int\limits_{\Gamma_{Ni}} (y_i \cdot n_i - F) w\, ds$$

holds for any function $w \in H^{1,1}_0(Q_T)$ and for almost all $t \in (0, T)$. Now identity (11) can be represented in the form

$$\|\nabla e\|^2_{Q_T, A} + \|\varrho e\|^2_{Q_T} + \frac{1}{2}\|e(\cdot, T)\|^2_\Omega = \mathfrak{I}_f + \mathfrak{I}_A + \mathfrak{I}^{\rm jmp}_{\Gamma_{ij}} + \mathfrak{I}^{\rm jmp}_{\Gamma_{Ni}} + \frac{1}{2}\|e(x, 0)\|^2_\Omega, \tag{50}$$

where $\mathfrak{I}_f$ and $\mathfrak{I}_A$ are defined in (30) and

$$\mathfrak{I}^{\rm jmp}_{\Gamma_{ij}} := \int\limits_0^T \sum_{\Gamma_{ij} \in \mathcal{G}_{\rm int}} \int\limits_{\Gamma_{ij}} (y_i - y_j) \cdot n_{ij} e\, ds\, dt, \qquad \mathfrak{I}^{\rm jmp}_{\Gamma_{Ni}} := \int\limits_0^T \sum_{\Gamma_{Ni} \in \mathcal{G}_N} \int\limits_{\Gamma_{Ni}} (y_i \cdot n_i - F) e\, ds\, dt.$$

Next, we use the following complexes based on local residuals:

$$R^{1-\mu_+}_{\mathcal{O}}(t) := \sum_{\Omega_l \in \mathcal{O}_{\rm P}} \frac{(C^{\rm P}_{\Omega_l})^2}{\underline{\nu}_A} \|\mathfrak{R}^{1-\mu_+}_f(v, y)\|^2_{\Omega_l}, \tag{51}$$

$$R_{\mathcal{G}_{\rm jmp}}(t) := \sum_{\Omega_i \in \mathcal{O}_\Omega} \frac{(C^{\rm Tr}_{i,\max})^2}{\underline{\nu}_A} \eta^2_i(y), \tag{52}$$

and

$$\eta^2_i(y) = \sum_{\substack{\Gamma_{ij} \in \mathcal{G}_{\rm int} \\ \Gamma_{ij} \cap \partial\Omega_i \neq \varnothing}} \frac{1}{4} r^2_{ij}(y) + \sum_{\substack{\Gamma_{Ni} \in \mathcal{G}_N \\ \Gamma_{Ni} \cap \partial\Omega_i \neq \varnothing}} \varrho^2_i(y), \tag{53}$$

where

$$r_{ij}(y) := \|(y_i - y_j) \cdot n_{ij}\|_{\Gamma_{ij}}, \qquad \varrho_i(y) := \|y_i \cdot n_i - F\|_{\Gamma_{Ni}}.$$

Theorem 2. (i) *The inequality*

$$[e]^2_{(\nu,\theta,2,1)} \le \overline{\rm M}^2_{{\rm I},N,\hat{Y}}(v, y; \delta, \varrho, \mu_+, \alpha_i) := \|e(x, 0)\|^2_\Omega$$
$$+ \int\limits_0^T \left(\varrho \left\| \frac{1}{\varrho} \mathfrak{R}^{\mu_+}_f(v, y) \right\|^2_\Omega + \alpha_1(t) \|\mathbf{r}_A(v, y)\|^2_{\Omega, A^{-1}} + \alpha_2(t) R^{1-\mu_+}_{\mathcal{O}}(t) + \alpha_3(t) R_{\mathcal{G}_{\rm jmp}}(t) \right) dt \tag{54}$$

holds for arbitrary $v \in H^{1,1}_g(Q_T)$ *and* $y \in \hat{Y}^{S_N}_{\rm div}(Q_T, \mathcal{O}_\Omega)$, *where* $\delta \in (0, 2]$, $\varrho \in [1/2, +\infty)$, $\mu_+ \in L^{\varrho,+}_{[0,1]}(Q_T)$, *and the* $\alpha_i(t)$, $i = 1, 2, 3$, *are positive definite real-valued functions satisfying identity* (29). *The weights in relation* (16) *are defined as*

$$\nu = 2 - \delta, \qquad \theta(x, t) = \varrho(x) \left(2 - \frac{1}{\varrho(t)}\right)^{1/2}.$$

(ii) *For the parameters defined in the assertion* (i), *the variational problem*

$$\inf_{\substack{v\in H_g^{1,1}(Q_T)\\ y\in \hat{Y}_{\mathrm{div}}^{S_N}(Q_T,\mathcal{O}_\Omega)}} \overline{\mathrm{M}}^2_{\mathrm{I},N,\hat{Y}}(v,y;\delta,\varrho,\alpha_i,\mu_+) \tag{55}$$

has a solution, and the functional $\overline{\mathrm{M}}^2_{\mathrm{I},N,\hat{Y}}$ *attains its minimum* (*which is zero*) *if and only if* $v=u$ *and* $y=A\nabla u$.

Proof. (i) Consider identity (50) by splitting the integral $\mathfrak{I}_f$ into the sum of two integrals

$$\mathfrak{I}_f=\mathfrak{I}_f^{\mu_+}+\mathfrak{I}_f^{1-\mu_+}. \tag{56}$$

We estimate the terms $\mathfrak{I}_A$ and $\mathfrak{I}_f^{\mu_+}$ with the use of the Hölder inequality. By taking into account the relation

$$y\in \hat{Y}_{\mathrm{div}}^{S_N}(Q_T,\mathcal{O}_\Omega),$$

we estimate the integral $\mathfrak{I}_f^{1-\mu_+}$ with regard of inequality (46),

$$\mathfrak{I}_f^{1-\mu_+}\le\int_0^T (R_{\mathcal{O}_\Omega}^{1-\mu_+})^{1/2}\|\nabla e\|_{\Omega,A}\,dt. \tag{57}$$

Any subdomain $\Omega_i\in\mathcal{O}_\Omega$ can be represented as the sum of simplices T_{ij} with edge ∂T_{ij} that is a part of the boundary $\partial\Omega_i$; i.e., the domain Ω_i forms a set of finite elements with one vertex. Suppose that $C_{i,\max}^{\mathrm{Tr}}$ stands for the maximum among constants in the corresponding Poincaré-type inequalities (48) specifying the faces $\partial T_{ij}\in\partial\Omega_i$; then the terms $\mathfrak{I}_{\Gamma_{ij}}^{\mathrm{jmp}}$ and $\mathfrak{I}_{\Gamma_{Ni}}^{\mathrm{jmp}}$ occurring in the estimate (50) can be estimated as follows:

$$\begin{aligned}\mathfrak{I}_{\Gamma_{ij}}^{\mathrm{jmp}}+\mathfrak{I}_{\Gamma_{Ni}}^{\mathrm{jmp}}&=\int_0^T\Bigg(\sum_{\Gamma_{ij}\in\mathcal{G}_{\mathrm{int}}} r_{ij}(y)\|e-\{|e|\}_{\Gamma_{ij}}\|_{\Gamma_{ij}}+\sum_{\Gamma_{Ni}\in\mathcal{G}}\varrho_k(y,v)\|e-\{|e|\}_{\Gamma_{Ni}}\|_{\Gamma_{Ni}}\Bigg)\,dt\\ &\le\int_0^T R_{\mathcal{G}_{\mathrm{jmp}}}^{1/2}(t)\|\nabla e\|_{\Omega,A}\,dt,\end{aligned}$$

where the complex $R_{\mathcal{G}_{\mathrm{jmp}}}^{1/2}$ is defined by relation (52). By using the Hölder inequality once more and the Young inequality, we obtain

$$\int_0^T\left\|\frac{1}{\varrho}\mathbf{r}_{f,\mu}\right\|_\Omega\|\varrho e\|_\Omega\,dt\le\frac12\int_0^T\left(\gamma_1(t)\left\|\frac{1}{\varrho}\mathbf{r}_{f,\mu}\right\|_\Omega^2+\frac{1}{\gamma_1(t)}\|\varrho e\|_\Omega^2\right)dt, \tag{58}$$

$$\int_0^T\|\mathbf{r}_A\|_{\Omega,A^{-1}}\|\nabla e\|_{\Omega,A}\,dt\le\frac12\int_0^T\left(\alpha_1(t)\|\mathbf{r}_A\|_{\Omega,A^{-1}}^2+\frac{1}{\alpha_1(t)}\|\nabla e\|_{\Omega,A}^2\right)dt, \tag{59}$$

$$\int_0^T(R_{\mathcal{O}}^{1-\mu_+})^{1/2}\|\nabla e\|_{\Omega,A}\,dt\le\frac12\int_0^T\left(\alpha_2(t)R_{\mathcal{O}}^{1-\mu_+}+\frac{1}{\alpha_2(t)}\|\nabla e\|_{\Omega,A}^2\right)dt, \tag{60}$$

$$\int_0^T R_{\mathcal{G}_{ij}}^{1/2}(t)\|\nabla e\|_{\Omega,A}\,dt\le\frac12\int_0^T\left(\alpha_3(t)R_{\mathcal{G}_{ij}}(t)+\frac{1}{\alpha_3(t)}\|\nabla e\|_{\Omega,A}^2\right)dt. \tag{61}$$

By combining relations (58)–(61), we obtain inequality (54).

(ii) To prove the existence of a pair $(v, y) \in H_g^{1,1}(Q_T) \times \hat{Y}_{\mathrm{div}}^{S_N}(Q_T, \mathcal{O}_\Omega)$ minimizing the functional $\overline{\mathrm{M}}^2_{\mathrm{I},N,\hat{Y}}$, it suffices to set $v = u$ and $y = A\nabla u$. In this case, we have $e(x, 0) = 0$, $\mathfrak{R}_f^{\mu_+}(v, y) = 0$, $\mathbf{r}_A(v, y) = 0$, $R_{\mathcal{O}}^{1-\mu_+}(v, y) = 0$, and $R_{\mathcal{G}_{\mathrm{jmp}}}(v, y) = 0$. Since the functional $\overline{\mathrm{M}}^2_{\mathrm{I},N,\hat{Y}}$ is nonnegative, we find that this choice of the functions v and y corresponds to the minimizer. On the other hand, if $\overline{\mathrm{M}}^2_{\mathrm{I},N,\hat{Y}} = 0$, then $\mathfrak{R}_f^{\mu_+} = 0$, $\mathbf{r}_A = 0$, $R_{\mathcal{O}}^{1-\mu_+} = 0$, and $R_{\mathcal{G}_{\mathrm{jmp}}} = 0$. It follows that the function v is a solution of problem (1)–(5). By virtue of the uniqueness of the solution of the initial–boundary value problem, we find that the other minimizing elements are absent. The proof of the theorem is complete.

In conclusion, note that the use of local inequalities (46)–(48) permits one to obtain sharper majorants (for example, in comparison with [12, 14]), because the constants $C_{\Omega_i}^{\mathrm{P}}$, $C_{\Gamma_i}^{\mathrm{P}}$, and $C_{\Gamma_i}^{\mathrm{Tr}}$ depend on the domain diameter. In addition, as usual, the subdomains Ω_i have a simpler form than Ω, which simplifies the computation of the exact values $C_{\Omega_i}^{\mathrm{P}}$, $C_{\Gamma_i}^{\mathrm{P}}$, and $C_{\Gamma_i}^{\mathrm{Tr}}$ (or the corresponding majorants) (see [22, 27]).

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