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PARTIAL DIFFERENTIAL EQUATIONS

Elliptic Dilation–Contraction Problems on Manifolds with Boundary. C^*-Theory

A. Yu. Savin* and B. Yu. Sternin**
Peoples' Friendship University of Russia, Moscow, Russia
Leibniz Universität Hannover, Hannover, Germany
*e-mail: *antonsavin@mail.ru, **sternin@mail.ru*

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Abstract—We study boundary value problems with dilations and contractions on manifolds with boundary. We construct a C^*- algebra of such problems generated by zero-order operators. We compute the trajectory symbols of elements of this algebra, obtain an analog of the Shapiro–Lopatinskii condition for such problems, and prove the corresponding finiteness theorem.

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1. INTRODUCTION

In the present paper, we study a new class of boundary value problems on a smooth compact manifold M with boundary $X = \partial M$ in which the main and boundary operators are nonlocal and are associated with smooth mappings of the manifold M into itself. As special cases, such problems include some well-known classes of problems: problems for invertible mappings (i.e., for diffeomorphisms) (see [1, p. 3]), problems with homotheties in $\mathbb{R}^n$ (see [2, 3]), and finally the well-known Bitsadze–Samarskii problems [4], in which the values of the unknown function on the boundary are related to its values on a submanifold lying inside the domain. Note that classical boundary value problems for pseudodifferential operators on manifolds with boundary were studied in [5, 6]. (See also the paper [7] on the solvability of pseudodifferential equations on the semiaxis.)

In the present paper, we derive conditions for the Fredholm property of problems for the case in which the problem is associated with a *contraction* that is a mapping of the manifold with boundary strictly into its interior. It is most interesting to derive an analog of the Shapiro–Lopatinskii condition in this situation. Recall that, in the classical theory, thus, in the case without nonlocality where the main and boundary operators are defined by differential expressions, the Shapiro–Lopatinskii condition (providing the Fredholm property of the problem) is obtained as follows: the coefficients of the problem are frozen at an arbitrary point of the boundary, and the Fourier transform is performed with respect to the variables along the boundary; the problem is then reduced to a family of ordinary differential equations with constant coefficients on the half-line (with coefficients defined by the symbol of the main operator) equipped with initial conditions (defined by the symbol of the boundary operator), and the Shapiro–Lopatinskii condition implies the unique solvability of that family. In the present paper, we show that, in the case of nonlocal problems associated with contractions, the ellipticity condition has an essentially new form, because the presence of a contraction operator necessitates freezing the coefficients on the whole orbit of the boundary point generated by the action of iterations of the contraction operator. As a result, the generalized Shapiro–Lopatinskii condition obtained below requires the unique solvability of an *infinite* matrix system corresponding to a trajectory of an arbitrary point on the boundary.

Let us describe methods for the derivation of the above-mentioned results. First, we realize a manifold M with boundary X as a submanifold of some closed manifold W and obtain all results on boundary value problems as corollaries of similar results on the closed manifold. Second, on the closed manifold W, we use methods of the theory of C^*-algebras and their crossed products for the construction of symbols and the proof of the finiteness theorem. To use these methods for the derivation of desired results, on the manifold W, we construct and study an algebra generated by pseudodifferential problems with transmission conditions on the submanifold X and its images under the action of iterations of the contraction. Here we use the results in [8], where the C^*-algebra of transmission problems was considered for the case of a single submanifold. After the construction of the C^*-algebra of transmission problems, we use methods of elliptic theory associated with group actions (see [1, p. 297; 9] and also [10, p. 27]), derive expressions for the symbols of the problem, and prove the finiteness theorem.

Note that only elliptic problems of zero order in the space L^2 are considered in the present paper. However, the results can be used to study problems of arbitrary order with the use of order reduction operators.

2. GEOMETRIC SITUATION

Let $M \subset W$ be a smooth manifold with boundary $X = \partial M$, which is a closed submanifold of codimension zero in a smooth closed manifold W.

Definition 1. A diffeomorphism $g: W \longrightarrow W$ is called a *contraction* of the manifold M if $g(M) \subset M \backslash \partial M$, i.e., if $g(M)$ lies in the interior of M.

Example 1. In the space $\mathbb{R}^n$, consider the mapping $gx = kx$, where $0 < k < 1$. This mapping is a contraction of any ball with center at the origin.

It turns out that the points of the manifold W fall into three types depending on the behavior of their orbits under the action of iterations of the diffeomorphism g.

1. Points whose orbits meet X. The set of such points is the countable union

$$\bigcup_{n\in\mathbb{Z}} g^n(X)$$

of submanifolds of codimension 1; since g is a contraction, it follows that the submanifolds in the union are pairwise disjoint.

2. Points whose orbits lie entirely either in M or outside M. Such points form the closed set

$$W_\infty = \Big(\bigcap_{n\ge 0} g^n(M)\Big) \bigcup \Big(\bigcap_{n<0} g^n(W\backslash M)\Big).$$

(In the general case, the space W_∞ is not necessarily a manifold.)

3. The remaining points (that is, points whose orbits do not meet X and lie neither in M nor in $W\backslash M$) form the open set

$$\bigcup_{n\in\mathbb{Z}} g^n(U), \qquad U = M\backslash(X \cup g(M)). \tag{1}$$

3. PROBLEMS ON CLOSED MANIFOLDS

In this section, we construct a C^*-algebra of pseudodifferential problems on the manifold W with boundary conditions and coconditions on the countable family of submanifolds

$$g^n X \subset W, \qquad n \in \mathbb{Z}. \tag{2}$$

The symbols of these operators may have jump discontinuities along the above-mentioned submanifolds. We introduce the notion of symbol and prove the finiteness theorem.

Definition of the Algebra of Problems

Given $N \ge 0$, consider the finite set of submanifolds

$$g^n X \subset W, \qquad -N \le n \le N+1. \tag{3}$$

These submanifolds are pairwise disjoint (see above). Consider the direct sum

$$\mathcal{H}_N = L^2(W,\mathrm{vol}) \oplus \bigoplus_{-N\le n\le N+1} L^2(g^n X, \mathrm{vol}^n) \tag{4}$$

of L^2-subspaces on the main manifold and on the submanifolds (3), where vol stands for the volume form on manifold W and $\mathrm{vol}^n = (\partial g^{-n})^*\mathrm{vol}_X$ stands for the volume form on the submanifold g^nX. In the representation (4) and below, in the notation of the spaces L^2, we indicate the volume form occurring in their definition. Then we have a well-defined C^*- algebra of operators continuously acting in the space (4), which consists of transmission problems in the sense of [8] and is the closure, in the operator norm, of boundary value problems for pseudodifferential operators on the complement of the union of submanifolds (3). The above-mentioned algebra is denoted by $\Psi_{\mathrm{tr},N}(W)$.

In addition, consider the closure

$$\Psi_{\mathrm{tr}}(W) = \overline{\bigcup_{N\ge 1}\Psi_{\mathrm{tr},N}(W)} \subset \mathcal{BH}, \quad \text{where} \quad \mathcal{H} = L^2(W,\mathrm{vol}) \oplus \bigoplus_{n\in\mathbb{Z}} L^2(g^nX,\mathrm{vol}^n) \tag{5}$$

(here and throughout the following, $\mathcal{BH}$ is the algebra of linear bounded operators in the Hilbert space $\mathcal{H}$) of the increasing sequence of C^*-algebras

$$\Psi_{\mathrm{tr},0}(W) \subset \Psi_{\mathrm{tr},1}(W) \subset \Psi_{\mathrm{tr},2}(W) \subset \cdots \subset \Psi_{\mathrm{tr},N}(W) \subset \cdots$$

Obviously, this closure is a C^*-algebra.

Let us construct the symbol mapping for the operator algebra $\Psi_{\mathrm{tr}}(W)$. First, we define the interior symbol.

Interior Symbol

Let us describe the space on which the interior symbols of the operators in the algebra $\Psi_{\mathrm{tr}}(W)$ are defined. As a set, this space is defined as the disjoint union

$$\widetilde{S}^*W = \left(\bigsqcup_{n=-\infty}^{\infty} S^*(g^n\overline{U})\right) \sqcup S^*W|_{W_\infty} \tag{6}$$

of cosphere bundles of the closed submanifolds $g^n\overline{U} \subset W$ and the restriction of the cosphere bundle of the manifold W to the closed subset $W_\infty \subset W$. The topology on the space (6) is defined as follows: the spaces $S^*(g^n\overline{U})$ are not glued to each other [since the interior symbols of pseudodifferential operators in $\Psi_{\mathrm{tr}}(W)$ can have discontinuities along the submanifolds g^nX], but their disjoint union is glued to the space $S^*W|_{W_\infty}$ by the following rule: a sequence $(x_n,\xi_n) \in S^*(g^n\overline{U})$ converges to $(x,\xi) \in S^*W|_{W_\infty}$ as $n \to \infty$ if $(x_n,\xi_n) \to (x,\xi)$ in the topology of the space S^*W. This definition of convergence corresponds to the fact that the algebra $\Psi_{\mathrm{tr}}(W)$ is obtained as a completion from elements whose interior symbols are continuous in a neighborhood of the set W_∞. One can readily see that the resulting space (6) is compact.

Since the interior symbols of operators in $\Psi_{\mathrm{tr},N}(W)$ with any $N \ge 0$ are continuous functions on S^*W with jumps on the submanifolds g^nX, $-N \le n \le N+1$, it follows that the interior symbol defines a homomorphism

$$\sigma^N_{\mathrm{int}}:\ \Psi_{\mathrm{tr},N}(W)/\mathcal{K} \to C(\widetilde{S}^*W) \tag{7}$$

of C^*-algebras, where $\mathcal{K}$ stands for the ideal of compact operators. This notion is well defined by virtue of the results in [8].

Now we define the boundary symbol.

Boundary Symbol

Given $x_0 \in X \subset W$, the boundary symbol of an operator $\mathcal{D} \in \Psi_{\mathrm{tr},N}(W)$ at the point $g^nx_0 \in g^nX$ is defined as follows (see [8]).

1. First, we freeze the coefficients of the operator $\mathcal{D}$ and pass to an operator on the tangent space $T_{g^n x_0}W$ at that point, which acts in the space

$$L^2(T_{g^n x_0}W, \mathrm{vol}_{g^n x_0}) \oplus L^2(T_{g^n x_0}(g^n X), \mathrm{vol}^n_{g^n x_0}), \tag{8}$$

which is the direct sum of L^2-spaces with respect to the volume forms vol and vol^n of the manifolds W and $g^n X$ at that point. In what follows, for brevity, we omit the notation of points at which the volume forms are computed.

2. Next, we define isomorphisms of the spaces

$$\begin{aligned} L^2(T_{g^n x_0}W, \mathrm{vol}) \oplus L^2(T_{g^n x_0}(g^n X), \mathrm{vol}^n) &\simeq L^2(T^*_{g^n x_0}W, \mathrm{vol}) \oplus L^2(T^*_{g^n x_0}(g^n X), \mathrm{vol}^n) \\ \simeq L^2(T^*_{x_0}W, (\partial g^n)^*\mathrm{vol}) \oplus L^2(T^*_{x_0}X, \mathrm{vol}_X) &\simeq L^2\left(T^*_{x_0}X, \mathrm{vol}_X; L^2\left(\mathbb{R}, \frac{(\partial g^n)^*\mathrm{vol}}{\mathrm{vol}_X}\right) \oplus \mathbb{C}\right) \\ \simeq L^2(T^*_{x_0}X, \mathrm{vol}_X; L^2(\mathbb{R}) \oplus \mathbb{C}). \end{aligned} \tag{9}$$

Here the first isomorphism is defined by the Fourier transform $\mathcal{F}_{x\to p}$ mapping functions on the tangent space with coordinates x into functions on the cotangent space with coordinates p; the second isomorphism is defined by the mapping $(\partial g^n)^*$ induced by the codifferential $\partial g^n : T^*_{x_0}W \to T^*_{g^n x_0}W$ [recall that the codifferential is expressed via the differential by the formula $\partial g^n = ((dg^n)^t)^{-1}$]; to define the third isomorphism in (9), we choose a local coordinate system (x', t) in a neighborhood of a point x_0 of the submanifold $X \subset W$ such that X is defined by the equation $t = 0$. The dual coordinates in the cotangent space are denoted by (η, τ). Then the expansion $T^*_{x_0}W = T^*_{x_0}X \oplus T^*_0\mathbb{R}$ is valid for the cotangent space; it defines the third isomorphism under which the functions on $T^*_{x_0}W$ are considered as L^2-functions on $T^*_{x_0}X$ with respect to the volume form vol_X. Moreover, these functions take values in the space L^2 on the line with the coordinate τ with respect to the volume form

$$\mu_{x_0,n} = \frac{(\partial g^n)^*\mathrm{vol}_{g^n x_0}}{\mathrm{vol}_{X,x_0}};$$

the fourth isomorphism in (9) corresponds to the isomorphism

$$L^2(\mathbb{R}, \mu_{x_0,n}) \longrightarrow L^2(\mathbb{R}), \qquad u \longmapsto u\left(\frac{\mu_{x_0,n}}{d\tau}\right)^{1/2}.$$

We have thereby constructed isomorphisms of the spaces occurring in formula (9). As follows from their definition, these isomorphisms are isometric.

Now the boundary symbol of the operator $\mathcal{D}$ at the point $g^n x_0$ (see [8]) is a continuous operator family

$$\sigma_b(\mathcal{D})(g^n x_0, \eta) \in \mathcal{B}(L^2(\mathbb{R}) \oplus \mathbb{C}), \tag{10}$$

which is twisted-homogeneous with respect to the covariables η; i.e., the relation

$$\sigma_b(\mathcal{D})(g^n x_0, \lambda\eta) = \varkappa_\lambda^{-1}\sigma_b(\mathcal{D})(g^n x_0, \eta)\varkappa_\lambda$$

holds for all $\lambda > 0$ and $|\eta| \neq 0$, where

$$\varkappa_\lambda(f(t), z) = (\lambda^{-1/2} f(\lambda^{-1}t), z)$$

is the unitary dilation group in the space $L^2(\mathbb{R}) \oplus \mathbb{C}$.

Now one can define the boundary symbol mapping on the algebra $\Psi_{\mathrm{tr},N}(W)$ as a homomorphism of C^*-algebras,

$$\sigma_b^N : \Psi_{\mathrm{tr},N}(W)/\mathcal{K} \longrightarrow l^\infty(\mathbb{Z}, \widetilde{C}(T^*_0 X, \mathcal{B}(L^2(\mathbb{R}) \oplus \mathbb{C}))), \tag{11}$$

where on the right-hand side we have the C^*-algebra of norm-bounded sequences of elements of the C^*-algebra $\widetilde{C}(T^*_0 X, \mathcal{B}(L^2(\mathbb{R}) \oplus \mathbb{C}))$ of continuous twisted-homogeneous functions defined on $T^*_0 X$ and taking values in operators acting in the space $L^2(\mathbb{R}) \oplus \mathbb{C}$.

Properties of Operators

Consider the symbol mapping

$$\sigma^N = (\sigma^N_{\rm int}, \sigma^N_b):\ \Psi_{{\rm tr},N}(W)/\mathcal{K} \longrightarrow \Sigma, \tag{12}$$

consisting of the interior and boundary symbols (7) and (11), where Σ stands for the symbol C^*-algebra

$$\Sigma = C(\widetilde{S}^*W) \oplus l^\infty(\mathbb{Z}, \widetilde{C}(T_0^*X, \mathcal{B}(L^2(\mathbb{R}) \oplus \mathbb{C}))).$$

Proposition 1. *The symbol mappings σ^N with distinct N are compatible with the embeddings $\Psi_{{\rm tr},N}(W) \subset \Psi_{{\rm tr},N+j}(W)$ [i.e., the relation $\sigma^N(\mathcal{D}) = \sigma^{N+j}(\mathcal{D})$ holds for any operator $\mathcal{D} \in \Psi_{{\rm tr},N}(W)$ and for arbitrary $j \geq 0$] and extend by continuity up to the monomorphism of C^*-algebras*

$$\sigma = (\sigma_{\rm int}, \sigma_b):\ \Psi_{\rm tr}(W)/\mathcal{K} \longrightarrow \Sigma. \tag{13}$$

Proof. 1. The compatibility of symbol mappings follows from the construction. Therefore, for the operator

$$\mathcal{D} \in \bigcup_{N=1}^{\infty} \Psi_{{\rm tr},N}(W)$$

we denote the symbol by $\sigma(\mathcal{D})$. The equality

$$\|\sigma(\mathcal{D})\| = \|\mathcal{D}\|_{\mathcal{B}/\mathcal{K}} \tag{14}$$

of the norm of a symbol and the norm of the corresponding operator in the Calkin algebra $\mathcal{B}/\mathcal{K}$ was obtained in [8]. It follows from (14) that the symbol can be extended by continuity to the closure of the above-mentioned union of algebras, i.e., to the algebra $\Psi_{\rm tr}(W)$.

2. The homomorphy of the symbol mapping $\mathcal{D} \mapsto \sigma(\mathcal{D})$ follows from the similar property of the symbols σ^N.

3. Let us show that the symbol mapping σ defines a monomorphism of the Calkin algebra (13). Suppose the contrary. Let $\sigma(\mathcal{D}) = 0$ and $\mathcal{D} \notin \mathcal{K}$. Then there exists a sequence of operators $\mathcal{D}_N \in \Psi_{{\rm tr},N}(W)$ such that $\mathcal{D}_N \to \mathcal{D}$ in norm, and, in addition, $\sigma(\mathcal{D}_N) \to 0$ as $N \to \infty$. By (14), hence we obtain

$$\|\mathcal{D}_N\|_{\mathcal{B}/\mathcal{K}} \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

Consequently, the operator $\mathcal{D}$, being the limit of the sequence $\mathcal{D}_N$, defines the zero element in the Calkin algebra, thus, is compact. We obtain a contradiction with the assumption that $\mathcal{D} \notin \mathcal{K}$. The proof of the assertion is complete.

4. DILATION–CONTRACTION PROBLEMS ON THE CLOSED MANIFOLD

We define the action of the group $\mathbb{Z}$ on the space $\mathcal{H}$ [see formula (5)] so as to ensure that the unity of the group is mapped into the operator

$$T_0:\ \mathcal{H} \longrightarrow \mathcal{H},$$
$$T_0(u, \{u_n\}) = (J^{1/2}(g^{-1})^* u, (g^{-1})^* u_{n-1}),$$

where the positive coefficient

$$J = \frac{(g^{-1})^*{\rm vol}}{\rm vol}, \qquad J \in C^\infty(W),$$

is chosen so as to ensure that T_0 is a unitary operator. Then the conjugation with the operator T_0 defines an action of the group $\mathbb{Z}$ on the algebra $\Psi_{\rm tr}(W)$ by automorphisms, and the C^*-crossed product $\Psi_{\rm tr}(W) \rtimes \mathbb{Z}$ is well defined (e.g., see [11]). The elements

$$\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{Z}} \in \Psi_{\rm tr}(W) \rtimes \mathbb{Z}$$

of the crossed product are defined as $\mathbb{Z}$-*operators*

$$\mathcal{D}=\sum_{n\in\mathbb{Z}}\mathcal{D}_nT_0^n:\ \mathcal{H}\longrightarrow\mathcal{H} \tag{15}$$

in the space $\mathcal{H}$. Our aim is to define the symbol of such operators.

Definition 2. The *symbol* of the $\mathbb{Z}$-operator (15) is defined as the element

$$\sigma(\mathcal{D})\in\Sigma\rtimes\mathbb{Z},$$

consisting of the interior symbol

$$\sigma_{\mathrm{int}}(\mathcal{D})\in C(\widetilde{S}^*W)\rtimes\mathbb{Z}$$

and the boundary symbol

$$\sigma_b(\mathcal{D})\in l^\infty(\mathbb{Z},\widetilde{C}(T_0^*X,\mathcal{B}(L^2(\mathbb{R})\oplus\mathbb{C})))\rtimes\mathbb{Z}.$$

Proposition 2. *The symbol mapping $\mathcal{D}\mapsto\sigma(\mathcal{D})$ induces the monomorphism*

$$(\Psi_{\mathrm{tr}}(W)/\mathcal{K})\rtimes\mathbb{Z}\longrightarrow\Sigma\rtimes\mathbb{Z}.$$

Proof. The functor $A\mapsto A\rtimes\mathbb{Z}$ of the crossed product of C^*-algebras A by the group $\mathbb{Z}$ is exact; i.e., the fact that the symbol mapping (13) is monomorphic (see Proposition 1) implies that the symbol mapping

$$\sigma:\ (\Psi_{\mathrm{tr}}(W)/\mathcal{K})\rtimes\mathbb{Z}\longrightarrow\Sigma\rtimes\mathbb{Z} \tag{16}$$

is monomorphic. The proof of the proposition is complete.

Since the algebra $\Psi_{\mathrm{tr}}(W)$ is nonunital, we find that its elements [as well as elements of the crossed product $\Psi_{\mathrm{tr}}(W)\rtimes\mathbb{Z}$] do not define Fredholm operators. However, the Fredholm property can be obtain by restricting operators to appropriate subspaces or by unitization of the algebra.

5. DILATION–CONTRACTION PROBLEMS ON THE MANIFOLD WITH BOUNDARY

Consider the orthogonal projection

$$P:\ \mathcal{H}\longrightarrow\mathcal{H}$$

onto the subspace

$$L^2(M)\oplus L^2(X)=\{(u,\{u_n\})|u|_{W\setminus M}=0,\ u_n=0\text{ if }n\neq0\}.$$

Obviously, this projection, defined via the characteristic function of the manifold $M\subset W$, lies in the algebra $\Psi_{\mathrm{tr}}(W)$ (as follows from the definition of the above-mentioned algebra). We introduce the following notion.

Definition 3. A *dilation–contraction problem* on the manifold M with boundary is defined as the restriction

$$P\mathcal{D}P:\ L^2(M)\oplus L^2(X)\longrightarrow L^2(M)\oplus L^2(X),\qquad \mathcal{D}\in\Psi_{\mathrm{tr}}(W)\rtimes\mathbb{Z},$$

of the $\mathbb{Z}$-operator $\mathcal{D}$ on the ambient manifold W to the subspace

$$\operatorname{Im}P\simeq L^2(M)\oplus L^2(X).$$

Definition 4. The dilation–contraction problem $P\mathcal{D}P$ is said to be *elliptic* if there exists a symbol $\sigma(\mathcal{D})^{-1} \in \Sigma \rtimes \mathbb{Z}$ such that the relations

$$\sigma(P)\sigma(\mathcal{D})\sigma(P)\sigma(\mathcal{D})^{-1}\sigma(P) = \sigma(P), \qquad \sigma(P)\sigma(\mathcal{D})^{-1}\sigma(P)\sigma(\mathcal{D})\sigma(P) = \sigma(P) \tag{17}$$

hold in the algebra $\Sigma \rtimes \mathbb{Z}$, i.e., if the symbol of the original problem is invertible on the range of the projection $\sigma(P)$.

Theorem 1. *An elliptic dilation–contraction problem is Fredholm.*

Proof. Let the problem $P\mathcal{D}P$ be elliptic. It follows that there exists an inverse symbol $\sigma(\mathcal{D})^{-1}$. By $\mathcal{D}' \in \Psi_{\mathrm{tr}} \rtimes \mathbb{Z}$ we denote an arbitrary dilation–contraction operator corresponding to that symbol. Now let us show that the operator $P\mathcal{D}'P: \ \operatorname{Im} P \to \operatorname{Im} P$ is an almost inverse of the operator $P\mathcal{D}P: \ \operatorname{Im} P \to \operatorname{Im} P$. Indeed, it follows from (17) that the symbols of the operators

$$P\mathcal{D}'P\mathcal{D}P - P \quad \text{and} \quad P\mathcal{D}P\mathcal{D}'P - P$$

are zero, and (by virtue of Proposition 1) these operators are compact. Hence we obtain the desired Fredholm property. The proof of the theorem is complete.

6. TRAJECTORY SYMBOLS

To verify the ellipticity condition in Theorem 1 for particular operators, it is convenient to pass to the trajectory representation of the crossed product (see [1, 9]). This representation and the corresponding trajectory symbol will be described in this section.

Trajectory Symbols for Dilation–Contraction Problems on the Closed Manifold

For the dilation–contraction problem, one has

$$\mathcal{D} = \sum_k \mathcal{D}_k T_0^k : \ \mathcal{H} \longrightarrow \mathcal{H},$$

where

$$\mathcal{H} = L^2(W, \mathrm{vol}) \oplus \bigoplus_{n\in\mathbb{Z}} L^2(g^n X, \mathrm{vol}^n),$$

and, with respect to this expansion, the operators $\mathcal{D}_k \in \Psi_{\mathrm{tr}}(W)$ have the form of matrices

$$\mathcal{D}_k = \begin{pmatrix} D_k & C_k \\ B_k & Q_k \end{pmatrix}$$

with pseudodifferential operators D_k, boundary operators B_k, coboundary operators C_k, and pseudodifferential operators Q_k.

Then the interior trajectory symbol

$$\sigma_{\mathrm{int}}(\mathcal{D})(x,\xi) = \sum_{k\in\mathbb{Z}} \sigma_{\mathrm{int}}(D_k)(\partial g^n(x,\xi))\mathcal{T}^k : \ l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z}), \qquad \mathcal{T}u(n) = u(n-1), \tag{18}$$

is defined for each point $(x,\xi) \in \tilde{S}^*W$ of the cosphere bundle; it is a finite-difference operator acting in the space l^2 of functions on the trajectory of the point (x,ξ). [The trajectory $\{\partial g^n(x,\xi)\}$ is isomorphic to the set of integers.]

For each point $(x',\eta) \in T_0^*X$ of the cotangent bundle of the submanifold X, we define the boundary trajectory symbol

$$\sigma_b(\mathcal{D})(x',\eta) = \sum_{k\in\mathbb{Z}} \sigma_b(\mathcal{D}_k)(x',\eta)\mathcal{T}^k : \ l^2(\mathbb{Z}, L^2(\mathbb{R}) \oplus \mathbb{C}) \longrightarrow l^2(\mathbb{Z}, L^2(\mathbb{R}) \oplus \mathbb{C}), \tag{19}$$

where

$$\mathcal{T}(u(n,\tau), v(n)) = (u(n-1,\tau), v(n-1)).$$

Proposition 3. *A symbol $a \in \Sigma^+ \rtimes \mathbb{Z}$, where Σ^+ is the algebra of symbols with adjoint unit, is invertible if and only if the following symbols are invertible.*

1. *The interior trajectory symbol $a_{\mathrm{int}}(x,\xi)$ for all $(x,\xi) \in \tilde{S}^*W$ [see relation (18)].*
2. *The boundary trajectory symbol $a_b(x',\eta)$ for all $(x',\eta) \in S^*X$ [see (19)].*

Proof. It follows from the properties of crossed products that a symbol a is invertible in the crossed product $\Sigma^+ \rtimes \mathbb{Z}$ if and only if its interior and boundary components

$$a_{\mathrm{int}} \in C(\tilde{S}^*W) \rtimes \mathbb{Z}, \qquad a_b \in l^\infty(\mathbb{Z}, \widetilde{C}(T_0^*X, \mathcal{B}(L^2(\mathbb{R}) \oplus \mathbb{C}))) \rtimes \mathbb{Z}$$

are invertible. Next, the invertibility of the interior symbol is equivalent to the invertibility of all of its restrictions $a_{\mathrm{int}}(x,\xi)$ to trajectories (see [1, p. 297]). Finally, the boundary symbol a_b is an operator function $a_b(x',\eta)$ on T^*X that ranges in bounded operators acting in the space $l^2(\mathbb{Z}, L^2(\mathbb{R}) \oplus \mathbb{C})$. The invertibility of the latter function is equivalent to its invertibility at each point. The proof of the assertion is complete.

Trajectory Symbols for Dilation–Contraction Problems on the Manifold with Boundary

Now let

$$\mathcal{D}_P = P\mathcal{D}P : \ L^2(M) \oplus L^2(X) \longrightarrow L^2(M) \oplus L^2(X), \quad \text{where} \quad \mathcal{D} \in \Psi_{\mathrm{tr}}(W) \rtimes \mathbb{Z},$$

be a dilation–contraction problem on the manifold M with boundary X. We have the following expression in closed form for such a problem:

$$\mathcal{D}_P = P \begin{pmatrix} \sum_k D_k T_0^k & \sum_{k\ge 0} C_k T_0^k \\ \sum_{k\le 0} B_k T_0^k & D_X \end{pmatrix}, \tag{20}$$

where D_X is a pseudodifferential operator on X, D_k is a pseudodifferential operator on M, and C_k and B_k are (co)boundary operators on X.

To derive expressions for the trajectory symbols of problem (20), it suffices to restrict the trajectory symbols of the problem $\mathcal{D}$ to the range of the trajectory symbol of the projection P. The following expressions for symbols can be obtained by straightforward computations.

Just as above, the interior symbol of problem (20) at points $(x,\xi) \in \tilde{S}^*M$, where $x \in M_\infty$ (i.e., the orbit of the point x lies entirely in M), is expressed in the form

$$\sigma_{\mathrm{int}}(\mathcal{D}_P)(x,\xi) = \sum_{k\in\mathbb{Z}} \sigma_{\mathrm{int}}(D_k)(\partial g^n(x,\xi))\mathcal{T}^k : \ l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z}), \qquad \mathcal{T}u(n) = u(n-1). \tag{21}$$

But if the orbit of the point x does not lie entirely in M, then the trajectory symbol of the projection P is simply the projection onto the subspace of half-infinite sequences. As a result, we obtain the expression

$$\sigma_{\mathrm{int}}(\mathcal{D}_P)(x,\xi) = \pi_+ \sum_{k\in\mathbb{Z}} \sigma_{\mathrm{int}}(D_k)(\partial g^n(x,\xi))\mathcal{T}^k : \ l^2(\mathbb{Z}_+) \longrightarrow l^2(\mathbb{Z}_+), \qquad \mathcal{T}u(n) = u(n-1), \tag{22}$$

for the interior symbol at the points $(x,\xi) \in \tilde{S}^*M$, where $x \in M \backslash g(M)$, $\mathbb{Z}_+ = \{n \ge 0\}$, and $\pi_+ : \ l^2(\mathbb{Z}) \to l^2(\mathbb{Z}_+)$ stands for the operator of "forgetting" the negative part of sequences.

Now let us compute the boundary trajectory symbol of the problem $\mathcal{D}_P$. Note that the boundary symbol $\sigma_b(P)(x',\eta)$ of the projection P is the projection onto the subspace

$$\operatorname{Im}\Pi_+ \oplus l^2(\mathbb{Z}_{>0}, L^2(\mathbb{R})) \oplus \mathbb{C} \subset l^2(\mathbb{Z}_+, L^2(\mathbb{R})) \oplus \mathbb{C}, \qquad \mathbb{Z}_{>0} = \{n > 0\} \cap \mathbb{Z}.$$

Here

$$\Pi_+ : \ L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

is the projection onto the subspace of functions that admit an analytic continuation to the lower half-plane (or, equivalently, the Fourier transform of the subspace of functions vanishing for $t < 0$).

Hence it follows that, for the problem $\mathcal{D}_P$, its *boundary trajectory symbol* at a point $(x', \eta) \in T_0^*X$ of the cotangent bundle of the submanifold X is an operator acting in the space

$$\sigma_b(\mathcal{D}_P)(x',\eta): \ \operatorname{Im}\Pi_+ \oplus l^2(\mathbb{Z}_{>0}, L^2(\mathbb{R})) \oplus \mathbb{C} \longrightarrow \operatorname{Im}\Pi_+ \oplus l^2(\mathbb{Z}_{>0}, L^2(\mathbb{R})) \oplus \mathbb{C}. \tag{23}$$

By a straightforward computation, one can obtain the following expression for the boundary trajectory symbol:

$$\sigma_b(\mathcal{D}_P)(x',\eta) = \sigma(P) \begin{pmatrix} \sum_k \sigma_b(D_k)(x',\eta)\mathcal{T}^k & \sum_{k\geq 0} \sigma_b(C_k)(x',\eta)\mathcal{T}^k \\ \sum_{k\leq 0} \sigma_b(B_k)(x',\eta)\mathcal{T}^k & \sigma(D_X)(x',\eta) \end{pmatrix}.$$

Via components, the action of the symbol can be written out as

$$\begin{aligned} &(u_0, u_1, u_2, \ldots, z) \overset{\sigma_b(\mathcal{D}_P)(x',\eta)}{\longmapsto} (\Pi_+ v_0, v_1, v_2, \ldots, w), \\ &v_n = \sum_{k\leq n} [\sigma_b(D_k)](x',\eta) u_{n-k} + \sigma_b(C_n)(x',\eta) z, \qquad n \geq 0, \\ &w = \sum_{n\geq 0} \sigma_b(B_{-n})(x',\eta) u_n + \sigma(D_X)(x',\eta) z. \end{aligned}$$

The last operator can also be written out in the form of the infinite operator matrix

$$\sigma_b(\mathcal{D}_P) = \begin{pmatrix} \Pi_+\sigma_b(D_0) & \Pi_+\sigma_b(D_{-1}) & \Pi_+\sigma_b(D_{-2}) & \ldots & \Pi_+\sigma_b(C_0) \\ \sigma_b(D_1) & \sigma_b(D_0) & \sigma_b(D_{-1}) & \ldots & \sigma_b(C_1) \\ \sigma_b(D_2) & \sigma_b(D_1) & \sigma_b(D_0) & \ldots & \sigma_b(C_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sigma_b(B_0) & \sigma_b(B_{-1}) & \sigma_b(B_{-2}) & \ldots & \sigma(D_X) \end{pmatrix}. \tag{24}$$

Here, for brevity, we omit the arguments (x', η) of symbols. The matrix (24) is an analog of the Lopatinskii operator in the theory of classical boundary value problems.

As a consequence of Theorem 1 and Proposition 3, we obtain the following theorem, which is the main result of the present paper.

Theorem 2. *The dilation–contraction problem $\mathcal{D}_P$ on a manifold with boundary* [*see* (20)] *is Fredholm if it is trajectory elliptic, i.e., if the following conditions are satisfied.*

1. *The interior trajectory symbol $\sigma_{\text{int}}(\mathcal{D}_P)(x,\xi)$ is invertible for all $(x,\xi) \in \tilde{S}^*M$* [*see relations* (21) *and* (22)].
2. *The boundary trajectory symbol $\sigma_b(\mathcal{D}_P)(x',\eta)$ is invertible for all $(x',\eta) \in S^*X$* [*see* (23)].

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