ISSN 0012-2661, Differential Equations, 2016, Vol. 52, No. 7, pp. 877–886. © Pleiades Publishing, Ltd., 2016.
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NUMERICAL METHODS

Numerical Solution Method for the Electric Impedance Tomography Problem in the Case of Piecewise Constant Conductivity and Several Unknown Boundaries

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Received February 2, 2016

Abstract—We study the electrical impedance tomography problem with piecewise constant electric conductivity coefficient, whose values are assumed to be known. The problem is to find the unknown boundaries of domains with distinct conductivities. The input information for the solution of this problem includes several pairs of Dirichlet and Neumann data on the known external boundary of the domain, i.e., several cases of specification of the potential and its normal derivative. We suggest a numerical solution method for this problem on the basis of the derivation of a nonlinear operator equation for the functions that define the unknown boundaries and an iterative solution method for this equation with the use of the Tikhonov regularization method. The results of numerical experiments are presented.

DOI: 10.1134/S0012266116070077

1. INTRODUCTION

The electric impedance tomography problem in the case of a piecewise constant conductivity is the problem of finding the boundaries of inhomogeneities inside the domain Ω on the basis of measurements of the potential (Dirichlet data) and its normal derivative (Neumann data) on the external boundary of the domain Ω. The uniqueness of the solution of this problem in some special cases was studied in [1–4]. Numerical solution methods for this problem were suggested in [5–14]. Either problems of finding one boundary were solved, or methods for finding several boundaries of a very simple form were constructed in that connection. Note that even in the case of one unknown boundary, the electric impedance tomography problem is strongly unstable. Therefore, for a sufficiently accurate determination of the unknown boundary, one should use several pairs of Dirichlet–Neumann data rather than one pair and apply regularizing algorithms [15, p. 53]. If the problem of finding several unknown boundaries is to be solved, then it is so much the more necessary to use several pairs of Dirichlet–Neumann data. In the present paper, we suggest a numerical method for finding the unknown boundaries on the basis of several pairs of Dirichlet–Neumann data. To simplify the formulas, we consider the case of two unknown boundaries, although the suggested scheme of the numerical method can be used in the case of more boundaries as well.

Let Ω be a bounded connected domain on the plane, let the curve Γ_0 be its boundary, let Ω_1 and Ω_2 be connected domains bounded by curves Γ_1 and Γ_2, respectively, such that $\overline{\Omega}_1, \overline{\Omega}_2 \subset \Omega$ and $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \varnothing$. The curves Γ_0, Γ_1, and Γ_2 are sufficiently smooth. Let $\Omega_0 = \Omega \backslash (\overline{\Omega}_1 \cup \overline{\Omega}_2)$.

Consider the functions $f^j(M)$, $j = 1, \dots, k$, that are continuous and not constant on Γ_0. Let the functions $u^j(M)$ satisfy the conditions $u^j \in C(\overline{\Omega})$ and $u^j(M) = u_i^j(M)$, $M \in \Omega_i$ $(i = 0, 1, 2)$, where $u_i^j \in C^2(\Omega_i) \cap C^1(\overline{\Omega}_i)$,

$$\Delta u_i^j(M) = 0, \qquad M \in \Omega_i, \qquad i = 0, 1, 2, \tag{1.1}$$

$$u_0^j(M) = u_i^j(M), \qquad M \in \Gamma_i, \qquad i = 1, 2, \tag{1.2}$$

$$\sigma_0 \frac{\partial u_0^j(M)}{\partial n} = \sigma \frac{\partial u_i^j(M)}{\partial n}, \qquad M \in \Gamma_i, \qquad i = 1, 2, \tag{1.3}$$

$$u_0^j(M) = f^j(M), \qquad M \in \Gamma_0, \tag{1.4}$$

and σ_0 and σ are positive constants.

The electric impedance tomography problem can be stated as the problem inverse to the Dirichlet problems (1.1)–(1.4). Let the curve Γ_0, the constants σ_0 and σ, and the functions $f^j(M)$, $j = 1, \ldots, k$ (the Dirichlet data), be given in the Dirichlet problems (1.1)–(1.4). The problem is to find the curves Γ_1 and Γ_2 on the basis of the following additional information on the solutions $u^j(M)$ of the Dirichlet problems (1.1)–(1.4):

$$\frac{\partial u^j(M)}{\partial n} = g^j(M), \qquad M \in \Gamma_0, \tag{1.5}$$

where n is the inward normal on Γ_0 and the $g^j(M)$ are given functions (Neumann data) continuous on the curve Γ_0.

2. NUMERICAL METHOD

The numerical method for solving the stated problem includes the derivation of a nonlinear operator equation for the unknown boundaries and the construction of an iterative method for the solution of this operator equation.

To construct the solutions $u^j(M)$, $j = 1, \ldots, k$, of the Dirichlet problems (1.1)–(1.4), we use the theory of potential [16, p. 348].

For each $j = 1, \ldots, k$, consider the following system of integral equations for the densities $\mu^j(P)$, $\nu_1^j(P)$, and $\nu_2^j(P)$:

$$\int_{\Gamma_0} \mu^j(P) \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_1} \nu_1^j(P) \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_2} \nu_2^j(P) \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P = f^j(M), \qquad M \in \Gamma_0, \tag{2.1}$$

$$\pi \nu_1^j(M) + \int_{\Gamma_0} \mu^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_1} \nu_1^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_2} \nu_2^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P = 0, \qquad M \in \Gamma_1, \tag{2.2}$$

$$\pi \nu_2^j(M) + \int_{\Gamma_0} \mu^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_1} \nu_1^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_2} \nu_2^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P = 0, \qquad M \in \Gamma_2, \tag{2.3}$$

where $\sigma^* = (\sigma_0 - \sigma)/(\sigma_0 + \sigma)$ and n_m is the inward normal to the curve Γ_1 or Γ_2.

Assertion. *If continuous functions* $\mu^j(P)$, $\nu_1^j(P)$, *and* $\nu_2^j(P)$ *satisfy the system of integral equations* (2.1)–(2.3), *then the function*

$$u^j(M) = \int_{\Gamma_0} \mu^j(P) \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_1} \nu_1^j(P) \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int_{\Gamma_2} \nu_2^j(P) \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P, \tag{2.4}$$

$M \in \Omega$, *is a solution of the Dirichlet problem* (1.1)–(1.4).

Proof. Since the simple layer potentials occurring in the representation (2.4) are continuous in the domain $\overline{\Omega}$, it follows that $u^j(M)$ belongs to $C(\overline{\Omega})$. Since the simple layer potentials are twice continuously differentiable and satisfy the Laplace equation in the domain $\Omega_0 \cup \Omega_1 \cup \Omega_2$, it follows that the function $u^j(M)$ has the same properties. Therefore, the function $u^j(M)$ satisfies Eq. (1.1) and condition (1.2). The Dirichlet condition (1.4) holds, because the potential densities $\mu^j(P)$, $\nu_1^j(P)$, and $\nu_2^j(P)$ satisfy Eq. (2.1).

Let us show that the function $u^j(M)$ determined by relation (2.4) satisfies condition (1.3). Let the point M belong to the curve Γ_1. By computing the exterior limit of the normal derivative, we obtain

$$\sigma_0 \frac{\partial u^j(M)}{\partial n_m} = \sigma_0 \int\limits_{\Gamma_0} \mu^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma_0\sigma^* \int\limits_{\Gamma_1} \nu_1^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P$$
$$+ \sigma_0\sigma^* \int\limits_{\Gamma_2} \nu_2^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \pi\sigma_0\sigma^*\nu_1^j(M), \qquad M \in \Gamma_1, \tag{2.5}$$

and by computing the interior limit of the normal derivative, we obtain the relation

$$\sigma \frac{\partial u^j(M)}{\partial n_m} = \sigma \int\limits_{\Gamma_0} \mu^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma\sigma^* \int\limits_{\Gamma_1} \nu_1^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P$$
$$+ \sigma\sigma^* \int\limits_{\Gamma_2} \nu_2^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P - \pi\sigma\sigma^*\nu_1^j(M), \qquad M \in \Gamma_1. \tag{2.6}$$

It follows from relations (2.5) and (2.6) and Eq. (2.2) that the function $u^j(M)$ satisfies condition (1.3) everywhere on the curve Γ_1.

In a similar way, by using Eq. (2.3), one can prove the validity of relation (1.3) for the function $u^j(M)$ on the curve Γ_2. The proof of the assertion is complete.

By using the representation (2.4) and the additional conditions (1.5), we obtain the equation

$$-\pi\mu^j(M) + \int\limits_{\Gamma_0} \mu^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P + \sigma^* \int\limits_{\Gamma_1} \nu_1^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P$$
$$+ \sigma^* \int\limits_{\Gamma_2} \nu_2^j(P) \frac{\partial}{\partial n_m} \ln\left(\frac{1}{\varrho_{MP}}\right) dl_P = g^j(M), \qquad M \in \Gamma_0. \tag{2.7}$$

We pass in Eqs. (2.1)–(2.3) and (2.7) to the polar coordinates. Let the curve Γ_0 be defined in the polar coordinate system with origin M_0 by the function $R(\psi) \in C^2[0, 2\pi]$. Take a Cartesian coordinate system with origin M_0. The points $M(x, y)$ on the curve Γ_0 have the coordinates

$$x = R(\psi)\cos\psi, \qquad y = R(\psi)\sin\psi, \qquad 0 \le \psi \le 2\pi.$$

Consider the class of unknown curves Γ_1 and Γ_2 such that $M_{01}(x_{01}, y_{01})$ and $M_{02}(x_{02}, y_{02})$ are known points that are centers of star shapes for all curves Γ_1 and Γ_2, respectively. For the parametrization of the curves Γ_1 and Γ_2, we use two polar coordinate systems with origins $M_{01}(x_{01}, y_{01})$ and $M_{02}(x_{02}, y_{02})$. Let the points $M(x, y)$ of the curve Γ_1 be defined by the function $r_1(\psi) \in C^2[0, 2\pi]$,

$$x = r_1(\psi)\cos\psi + x_{01}, \qquad y = r_1(\psi)\sin\psi + y_{01}, \qquad 0 \le \psi \le 2\pi,$$

and let the points $M(x, y)$ of the curve Γ_2 be defined by the function $r_2(\psi) \in C^2[0, 2\pi]$,

$$x = r_2(\psi)\cos\psi + x_{02}, \qquad y = r_2(\psi)\sin\psi + y_{02}, \qquad 0 \le \psi \le 2\pi.$$

We impose an additional condition on the class of the curves Γ_1 and Γ_2. Let $\|r_i\|_{C^2[0,2\pi]} \le c_i$, $i = 1, 2$, where the c_i, $i = 1, 2$, are given numbers.

Let us rewrite Eqs. (2.1)–(2.3), (2.7) in polar coordinates. We introduce the functions

$$
\begin{aligned}
&E(p,q) = \sqrt{p^2+q^2},\\
&N(\varphi,\psi) = -\frac{1}{2}\sqrt{R^2(\varphi)+(R'(\varphi))^2}\ln(R^2(\varphi)+R^2(\psi)-2R(\varphi)R(\psi)\cos(\varphi-\psi)),\\
&D_i(\zeta,\psi,p,q)\\
&\qquad = -\frac{1}{2}E(p,q)\ln([R(\psi)\cos\psi - p\cos\zeta - x_{0i}]^2 + [R(\psi)\sin\psi - p\sin\zeta - y_{0i}]^2), \quad i=1,2.
\end{aligned}
$$

By passing to polar coordinates in Eq. (2.1), we obtain

$$
\int_0^{2\pi} N(\varphi,\psi)\mu^j(\varphi)\,d\varphi + \sigma^* \int_0^{2\pi} D_1(\zeta,\psi,r_1(\zeta),r_1'(\zeta))\nu_1^j(\zeta)\,d\zeta
+ \sigma^* \int_0^{2\pi} D_2(\theta,\psi,r_2(\theta),r_2'(\theta))\nu_2^j(\theta)\,d\theta = f^j(\psi), \qquad 0\le\psi\le 2\pi. \tag{2.8}
$$

We introduce the functions

$$
\begin{aligned}
&Q(\varphi,\psi,p_1,q_1,p_2,q_2) = \frac{E(p_2,q_2)}{E(p_1,q_1)}\,\frac{[p_1 - p_2\cos(\psi-\varphi)]p_1 - p_2q_1\sin(\psi-\varphi)}{p_1^2+p_2^2-2p_1p_2\cos(\psi-\varphi)},\\
&W(\varphi,\psi,p_1,q_1,p_2,q_2,a,b,c,d) = \frac{E(p_2,q_2)}{E(p_1,q_1)}([p_1\cos\psi + a - p_2\cos\varphi - c][q_1\sin\psi + p_1\cos\psi]\\
&\quad - [p_1\sin\psi + b - p_2\sin\varphi - d][q_1\cos\psi - p_1\sin\psi])([p_1\cos\psi + a - p_2\cos\varphi - c]^2\\
&\quad + [p_1\sin\psi + b - p_2\sin\varphi - d]^2)^{-1}.
\end{aligned}
$$

We pass in Eqs. (2.2) and (2.3) to polar coordinates,

$$
\pi\nu_1^j(\psi) + \int_0^{2\pi} W_1(\varphi,\psi,r_1(\psi),r_1'(\psi))\mu^j(\varphi)\,d\varphi + \sigma^*\int_0^{2\pi} Q(\zeta,\psi,r_1(\psi),r_1'(\psi),r_1(\zeta),r_1'(\zeta))\nu_1^j(\zeta)\,d\zeta
+ \sigma^*\int_0^{2\pi} W_2(\theta,\psi,r_1(\psi),r_1'(\psi),r_2(\theta),r_2'(\theta))\nu_2^j(\theta)\,d\theta = 0, \qquad 0\le\psi\le 2\pi, \tag{2.9}
$$

$$
\pi\nu_2^j(\psi) + \int_0^{2\pi} W_3(\varphi,\psi,r_2(\psi),r_2'(\psi))\mu^j(\varphi)\,d\varphi + \sigma^*\int_0^{2\pi} W_4(\zeta,\psi,r_2(\psi),r_2'(\psi),r_1(\zeta),r_1'(\zeta))\nu_1^j(\zeta)\,d\zeta
+ \sigma^*\int_0^{2\pi} Q(\theta,\psi,r_2(\psi),r_2'(\psi),r_2(\theta),r_2'(\theta))\nu_2^j(\theta)\,d\theta = 0, \qquad 0\le\psi\le 2\pi, \tag{2.10}
$$

where

$$
\begin{aligned}
W_1(\varphi,\psi,p,q) &= W(\varphi,\psi,p,q,R(\varphi),R'(\varphi),x_{01},y_{01},0,0),\\
W_2(\theta,\psi,p_1,q_1,p_2,q_2) &= W(\theta,\psi,p_1,q_1,p_2,q_2,x_{01},y_{01},x_{02},y_{02}),\\
W_3(\varphi,\psi,p,q) &= W(\varphi,\psi,p,q,R(\varphi),R'(\varphi),x_{02},y_{02},0,0),\\
W_4(\zeta,\psi,p_1,q_1,p_2,q_2) &= W(\zeta,\psi,p_1,q_1,p_2,q_2,x_{02},y_{02},x_{01},y_{01}).
\end{aligned}
$$

We pass to polar coordinates in Eqs. (2.7),

$$-\pi\mu^j(\psi) + \int_0^{2\pi} Q_R(\varphi,\psi)\mu^j(\varphi)\,d\varphi + \sigma^* \int_0^{2\pi} W_5(\zeta,\psi,r_1(\zeta),r_1'(\zeta))\nu_1^j(\zeta)\,d\zeta$$

$$+\,\sigma^* \int_0^{2\pi} W_6(\theta,\psi,r_2(\theta),r_2'(\theta))\nu_2^j(\theta)\,d\theta = g^j(\psi), \qquad 0 \le \psi \le 2\pi, \tag{2.11}$$

where the functions $Q_R(\varphi,\psi)$, $W_5(\zeta,\psi,p,q)$, and $W_6(\theta,\psi,p,q)$ are defined as follows:

$$Q_R(\varphi,\psi) = Q(\varphi,\psi,R(\psi),R'(\psi),R(\varphi),R'(\varphi)),$$
$$W_5(\zeta,\psi,p,q) = W(\zeta,\psi,R(\psi),R'(\psi),p,q,0,0,x_{01},y_{01}),$$
$$W_6(\theta,\psi,p,q) = W(\theta,\psi,R(\psi),R'(\psi),p,q,0,0,x_{02},y_{02}).$$

For each $j = 1,\dots,k$, consider the nonlinear operator A^j that takes the functions $r_1(\psi)$ and $r_2(\psi)$ to the values of the normal derivative of the solution of the Dirichlet problem (1.1)–(1.4) on the boundary Γ_0. The operator A^j acts as follows. For given functions $r_1(\psi)$ and $r_2(\psi)$, the system of integral equations (2.8)–(2.10) is solved, and the densities of the potentials $\mu^j(\psi)$, $\nu_1^j(\psi)$, and $\nu_2^j(\psi)$ are computed. Then, for these densities, one computes the left-hand side of Eq. (2.11), that is, the value of the operator $A^j(r_1,r_2)$. By $\overline{A}(r_1,r_2)$ we denote the operator defined by all operators $A^j(r_1,r_2)$, that is, the operator taking the functions $r_1(\psi)$ and $r_2(\psi)$ to the values of normal derivatives of solutions of the Dirichlet problems (1.1)–(1.4) on the boundary Γ_0 for all $j = 1,\dots,k$. Then the posed electric impedance tomography problem can be represented in the form of a nonlinear operator equation for the unknown functions $r_1(\psi)$ and $r_2(\psi)$,

$$\overline{A}(r_1,r_2) = \overline{g}(\psi), \tag{2.12}$$

where $\overline{A} = \{A^1, A^2, \dots, A^k\}$, and $\overline{g}(\psi) = \{g^1(\psi), g^2(\psi), \dots, g^k(\psi)\}$.

Consider an iterative method for solving the operator equation (2.12). Since the two functions $r_1(\psi)$ and $r_2(\psi)$ are unknown, we perform iterations successively for each of them; i.e., we make several steps for one function, then several steps for the other, and so on. Let $r_{1n}(\psi)$ and $r_{2m}(\psi)$ be some functions obtained in the iteration process. Consider the linearization of Eq. (2.12) with respect to $r_1(\psi)$ in a neighborhood of $r_{1n}(\psi)$ and $r_{2m}(\psi)$; as a result, we obtain a linear equation for the unknown increment $\varrho_{1n}(\psi)$ of the function $r_{1n}(\psi)$. By $\mu^j(\psi;r_{1n},r_{2m})$, $\nu_1^j(\psi;r_{1n},r_{2m})$, and $\nu_2^j(\psi;r_{1n},r_{2m})$ we denote the solutions of system (2.8)–(2.10) with the functions $r_{1n}(\psi)$ and $r_{2m}(\psi)$.

We introduce the functions

$$L_0^j(\psi;r_{1n},r_{2m},\varrho_{1n}) = \sigma^* \int_0^{2\pi} \frac{\partial Q}{\partial p_1}(\zeta,\psi,r_{1n}(\psi),r_{1n}'(\psi),r_{1n}(\zeta),r_{1n}'(\zeta))\nu_1^j(\zeta;r_{1n},r_{2m})\varrho_{1n}(\psi)\,d\zeta$$

$$+\,\sigma^* \int_0^{2\pi} \frac{\partial Q}{\partial q_1}(\zeta,\psi,r_{1n}(\psi),r_{1n}'(\psi),r_{1n}(\zeta),r_{1n}'(\zeta))\nu_1^j(\zeta;r_{1n},r_{2m})\varrho_{1n}'(\psi)\,d\zeta$$

$$+\,\sigma^* \int_0^{2\pi} \frac{\partial Q}{\partial p_2}(\zeta,\psi,r_{1n}(\psi),r_{1n}'(\psi),r_{1n}(\zeta),r_{1n}'(\zeta))\nu_1^j(\zeta;r_{1n},r_{2m})\varrho_{1n}(\zeta)\,d\zeta$$

$$+\,\sigma^* \int_0^{2\pi} \frac{\partial Q}{\partial q_2}(\zeta,\psi,r_{1n}(\psi),r_{1n}'(\psi),r_{1n}(\zeta),r_{1n}'(\zeta))\nu_1^j(\zeta;r_{1n},r_{2m})\varrho_{1n}'(\zeta)\,d\zeta$$

and

$$L_1^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}) = \sigma^* \int\limits_0^{2\pi} \frac{\partial D_1}{\partial p}(\zeta, \psi, r_{1n}(\zeta), r'_{1n}(\zeta))\nu_1^j(\zeta; r_{1n}, r_{2m})\varrho_{1n}(\zeta)\, d\zeta + \sigma^* \int\limits_0^{2\pi} \frac{\partial D_1}{\partial q}(\zeta, \psi, r_{1n}(\zeta), r'_{1n}(\zeta))\nu_1^j(\zeta; r_{1n}, r_{2m})\varrho'_{1n}(\zeta)\, d\zeta.$$

Consider the following system of linear integral equations for the unknown functions $\hat{\mu}^j(\psi; r_{1n}, r_{2m})$, $\hat{\nu}_1^j(\psi; r_{1n}, r_{2m})$, and $\hat{\nu}_2^j(\psi; r_{1n}, r_{2m})$:

$$\int\limits_0^{2\pi} N(\varphi, \psi)\hat{\mu}^j(\varphi; r_{1n}, r_{2m})\, d\varphi + \sigma^* \int\limits_0^{2\pi} D_1(\zeta, \psi, r_{1n}(\zeta), r'_{1n}(\zeta))\hat{\nu}_1^j(\zeta; r_{1n}, r_{2m})\, d\zeta + \sigma^* \int\limits_0^{2\pi} D_2(\theta, \psi, r_{2m}(\theta), r'_{2m}(\theta))\hat{\nu}_2^j(\theta; r_{1n}, r_{2m})\, d\theta = -L_1^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}), \quad 0 \le \psi \le 2\pi, \tag{2.13}$$

$$\pi\hat{\nu}_1^j(\psi; r_{1n}, r_{2m}) + \int\limits_0^{2\pi} W_1(\varphi, \psi, r_{1n}(\psi), r'_{1n}(\psi))\hat{\mu}^j(\varphi; r_{1n}, r_{2m})\, d\varphi + \sigma^* \int\limits_0^{2\pi} Q(\zeta, \psi, r_{1n}(\psi), r'_{1n}(\psi), r_{1n}(\zeta), r'_{1n}(\zeta))\hat{\nu}_1^j(\zeta; r_{1n}, r_{2m})\, d\zeta + \sigma^* \int\limits_0^{2\pi} W_2(\theta, \psi, r_{1n}(\psi), r'_{1n}(\psi), r_{2m}(\theta), r'_{2m}(\theta))\hat{\nu}_2^j(\theta; r_{1n}, r_{2m})\, d\theta = -L_2^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}), \qquad 0 \le \psi \le 2\pi, \tag{2.14}$$

$$\pi\hat{\nu}_2^j(\psi; r_{1n}, r_{2m}) + \int\limits_0^{2\pi} W_3(\varphi, \psi, r_{2m}(\psi), r'_{2m}(\psi))\hat{\mu}^j(\varphi; r_{1n}, r_{2m})\, d\varphi + \sigma^* \int\limits_0^{2\pi} W_4(\zeta, \psi, r_{2m}(\psi), r'_{2m}(\psi), r_{1n}(\zeta), r'_{1n}(\zeta))\hat{\nu}_1^j(\zeta; r_{1n}, r_{2m})\, d\zeta + \sigma^* \int\limits_0^{2\pi} Q(\theta, \psi, r_{2m}(\psi), r'_{2m}(\psi), r_{2m}(\theta), r'_{2m}(\theta))\hat{\nu}_2^j(\theta; r_{1n}, r_{2m})\, d\theta = -L_3^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}), \qquad 0 \le \psi \le 2\pi, \tag{2.15}$$

where

$$L_2^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}) = \int\limits_0^{2\pi} \frac{\partial W_1}{\partial p}(\varphi, \psi, r_{1n}(\psi), r'_{1n}(\psi))\mu^j(\varphi; r_{1n}, r_{2m})\varrho_{1n}(\psi)\, d\varphi + \int\limits_0^{2\pi} \frac{\partial W_1}{\partial q}(\varphi, \psi, r_{1n}(\psi), r'_{1n}(\psi))\mu^j(\varphi; r_{1n}, r_{2m})\varrho'_{1n}(\psi)\, d\varphi + L_0^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}) + \sigma^* \int\limits_0^{2\pi} \frac{\partial W_2}{\partial p_1}(\theta, \psi, r_{1n}(\psi), r'_{1n}(\psi), r_{2m}(\theta), r'_{2m}(\theta))\nu_2^j(\theta; r_{1n}, r_{2m})\varrho_{1n}(\psi)\, d\theta$$

$$+\sigma^* \int_0^{2\pi} \frac{\partial W_2}{\partial q_1}(\theta, \psi, r_{1n}(\psi), r'_{1n}(\psi), r_{2m}(\theta), r'_{2m}(\theta))\nu_2^j(\theta; r_{1n}, r_{2m})\varrho'_{1n}(\psi)\, d\theta,$$

$$L_3^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}) = \sigma^* \int_0^{2\pi} \frac{\partial W_4}{\partial p_2}(\zeta, \psi, r_{2m}(\psi), r'_{2m}(\psi), r_{1n}(\zeta), r'_{1n}(\zeta))\nu_1^j(\zeta; r_{1n}, r_{2m})\varrho_{1n}(\zeta)\, d\zeta$$
$$+\sigma^* \int_0^{2\pi} \frac{\partial W_4}{\partial q_2}(\zeta, \psi, r_{2m}(\psi), r'_{2m}(\psi), r_{1n}(\zeta), r'_{1n}(\zeta))\nu_1^j(\zeta; r_{1n}, r_{2m})\varrho'_{1n}(\zeta)\, d\zeta.$$

The functions $L_1^j(\psi; r_{1n}, r_{2m}, \varrho_{1n})$, $L_2^j(\psi; r_{1n}, r_{2m}, \varrho_{1n})$, and $L_3^j(\psi; r_{1n}, r_{2m}, \varrho_{1n})$, which are the right-hand sides of Eqs. (2.13)–(2.15), linearly depend on $\varrho_{1n}(\psi)$.

By linearizing Eqs. (2.11) for all $j = 1, \ldots, k$, we obtain

$$-\pi\hat{\mu}^j(\psi; r_{1n}, r_{2m}) + L_4^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}) + \int_0^{2\pi} Q_R(\varphi, \psi)\hat{\mu}^j(\varphi; r_{1n}, r_{2m})\, d\varphi$$
$$+\sigma^* \int_0^{2\pi} W_5(\zeta, \psi, r_{1n}(\zeta), r'_{1n}(\zeta))\hat{\nu}_1^j(\zeta; r_{1n}, r_{2m})\, d\zeta$$
$$+\sigma^* \int_0^{2\pi} W_6(\theta, \psi, r_{2m}(\theta), r'_{2m}(\theta))\hat{\nu}_2^j(\theta; r_{1n}, r_{2m})\, d\theta$$
$$= g^j(\psi) - A^j(r_{1n}, r_{2m})(\psi), \qquad 0 \le \psi \le 2\pi, \tag{2.16}$$

where

$$L_4^j(\psi; r_{1n}, r_{2m}, \varrho_{1n}) = \sigma^* \int_0^{2\pi} \frac{\partial W_5}{\partial p}(\zeta, \psi, r_{1n}(\zeta), r'_{1n}(\zeta))\nu_1^j(\zeta; r_{1n}, r_{2m})\varrho_{1n}(\zeta)\, d\zeta$$
$$+\sigma^* \int_0^{2\pi} \frac{\partial W_5}{\partial q}(\zeta, \psi, r_{1n}(\zeta), r'_{1n}(\zeta))\nu_1^j(\zeta; r_{1n}, r_{2m})\varrho'_{1n}(\zeta)\, d\zeta$$

and the functions $A^j(r_{1n}, r_{2m})(\psi)$ are obtained by applying the operators A^j to the functions $r_{1n}(\psi)$ and $r_{2m}(\psi)$.

Equations (2.16), together with system (2.13)–(2.15), define the linear operator equations

$$B^j[r_{1n}, r_{2m}]\varrho_{1n}(\psi) = h^j(\psi), \qquad 0 \le \psi \le 2\pi. \tag{2.17}$$

for the unknown function $\varrho_{1n}(\psi)$. The values of the operators $B^j[r_{1n}, r_{2m}]\varrho_{1n}(\psi)$ linear with respect to $\varrho_{1n}(\psi)$ are defined by the left-hand sides of Eqs. (2.16) with the functions $\hat{\mu}^j(\psi; r_{1n}, r_{2m})$, $\hat{\nu}_1^j(\psi; r_{1n}, r_{2m})$, and $\hat{\nu}_2^j(\psi; r_{1n}, r_{2m})$ found from system (2.13)–(2.15), and the functions $h^j(\psi)$ have the form $h^j(\psi) = g^j(\psi) - A^j(r_{1n}, r_{2m})(\psi)$.

The set of equations (2.17) for all $j = 1, \ldots, k$ can be represented in the form of a linear operator equation for the function $\varrho_{1n}(\psi)$,

$$\overline{B}[r_{1n}, r_{2m}]\varrho_{1n} = \overline{h}(\psi), \tag{2.18}$$

where

$$\overline{B}[r_{1n}, r_{2m}] = \{B^1[r_{1n}, r_{2m}], B^2[r_{1n}, r_{2m}], \ldots, B^k[r_{1n}, r_{2m}]\},$$
$$\overline{h}(\psi) = \{h^1(\psi), h^2(\psi), \ldots, h^k(\psi)\}.$$

By solving this equation and by finding the function $\varrho_{1n}(\psi)$, we obtain

$$r_{1n+1}(\psi) = r_{1n}(\psi) + \varrho_{1n}(\psi).$$

The linear equation for the unknown increment $\varrho_{2m}(\psi)$ of the function $r_{2m}(\psi)$ can be constructed in a similar way. For the initial approximation to the functions $r_{10}(\psi)$ and $r_{20}(\psi)$ one can take very simple curves, for example, circles. We have thereby completely described the iterative method.

3. IMPLEMENTATION OF THE METHOD AND COMPUTER EXPERIMENTS

Consider an implementation of the suggested iterative method. The definition of the initial approximation to $r_{10}(\psi)$ and $r_{20}(\psi)$ in the form of circles with fixed radius is an easy problem. After the definition of these circles, for their centers we take the points $M_{01}(x_{01}, y_{01})$ and $M_{02}(x_{02}, y_{02})$, respectively. Next, we introduce the above-mentioned polar coordinate systems, and on the interval $[0, 2\pi]$ we introduce grids and grid counterparts of all functions. After the replacement of integrals by quadrature formulas, the discrete analogs of integral equations are systems of linear algebraic equations. Therefore, the discrete analog of Eq. (2.18) is a system of linear algebraic equations. To solve this system of linear algebraic equations, we use the Tikhonov regularization method [15, p. 122]. The value of the regularization parameter is coordinated with the accuracy of the determination of the original information and with the step of the iterative process.

Consider examples of use of the suggested iterative method for the numerical solution of the electric impedance tomography problem. In the first numerical experiment, the boundary Γ_0 was given by a circle of radius 50. For the unknown boundary Γ_1 we take an ellipsoid with the principal axes 40 and 20, and for the unknown boundary Γ_2 we take a convex curve defined by a cubic spline (see Fig. 1). We use the constants $\sigma_0 = 5$ and $\sigma = 1$. The number of measurements on the external boundary is $k = 15$. The values of the functions $f^j(M) = f^j(x, y)$ on Γ_0 are defined in the polar coordinate system with origin coinciding with the center of the circle Γ_0 by the functions $f^j(50\cos\psi, 50\sin\psi) = f^j(\psi)$, $\psi \in [0, 2\pi]$ of the polar angle, and

$$f^j(\psi) = 50(\exp[-4\sin^2(\psi/2 - j\pi/15)] - \exp[-4\cos^2(\psi/2 - j\pi/15)]), \qquad j = 1, \ldots, 15.$$

The scheme of the numerical experiment was as follows. For given Γ_0, Γ_1, Γ_2, σ_0, σ, and $f^j(\psi)$, we solved the Dirichlet problems (1.1)–(1.4) and found the values of the functions $g^j(\psi)$, that is, the values of the normal derivatives of solutions of the Dirichlet problems (1.1)–(1.4) on the boundary Γ_0. These functions were perturbed, and we obtained the functions $g^j_\delta(\psi)$ such that

$$\frac{\|g^j(\psi) - g^j_\delta(\psi)\|_{L_2[0,2\pi]}}{\|g^j(\psi)\|_{L_2[0,2\pi]}} = 0.01, \qquad j = 1, \ldots, 15.$$

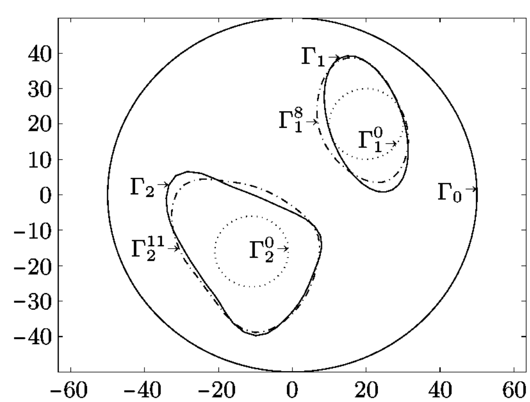

Fig. 1. Results of the first numerical experiment.

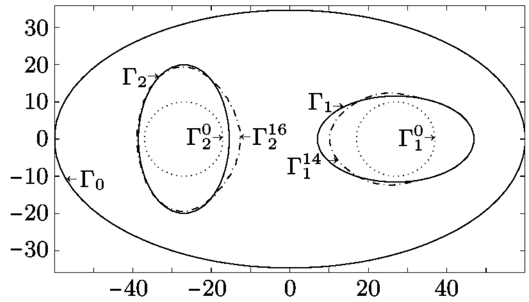

Fig. 2. Results of the second numerical experiment.

Next, for the functions $g_\delta^j(\psi)$ we solved the electric impedance tomography problem by the suggested iterative method. Figure 1 presents the results of the first numerical experiment. For the initial approximation to the boundaries Γ_1^0 and Γ_2^0 we took circles. On the curves Γ_0, Γ_1, and Γ_2, we chose uniform grids with 120, 60, and 60 nodes, respectively. The curves Γ_1^8 and Γ_2^{11} were obtained by the solution of the inverse problem after 8 iterations for the curve Γ_1 and 11 iterations for the curve Γ_2. The termination condition for the iterative process was given by the error level for the discrepancy.

In the second numerical experiment, the boundary Γ_0 was given by an ellipsoid with half-axes 120 and 70 (Fig. 2). For the boundaries Γ_1 and Γ_2 we took ellipsoids with half-axes 40 and 24 (Fig. 2). The constants σ_0 and σ and the number of measurements k were the same as in the preceding experiment. The values of the functions $f^j(M) = f^j(x, y)$ on Γ_0 were defined in the polar coordinate system with origin coinciding with the center of the ellipsoid Γ_0 by the functions

$$f^j(60\cos\psi, 35\sin\psi) = f^j(\psi), \qquad \psi \in [0, 2\pi],$$
$$f^j(\psi) = 50(\exp[-4\sin^2(\psi/2 - j\pi/15)] - \exp[-4\cos^2(\psi/2 - j\pi/15)]), \qquad j = 1, \dots, 15.$$

The scheme of the numerical experiment was similar to the previous experiment. For given Γ_0, Γ_1, Γ_2, σ_0, σ, and $f^j(\psi)$, we solved the Dirichlet problems (1.1)–(1.4) and found values of the functions $g^j(\psi)$. We perturbed these functions and obtained functions $g_\delta^j(\psi)$ such that

$$\frac{\|g^j(\psi) - g_\delta^j(\psi)\|_{L_2[0,2\pi]}}{\|g^j(\psi)\|_{L_2[0,2\pi]}} = 0.01, \qquad j = 1, \dots, 15.$$

Next, for the functions $g_\delta^j(\psi)$ we solved the electric impedance tomography problem. Figure 2 presents the results of the second numerical experiment. For the initial approximation to the boundaries Γ_1^0 and Γ_2^0 we took circles. On the curves Γ_0, Γ_1, and Γ_2, we chose uniform grids of 120, 60, and 60 nodes, respectively. The curves Γ_1^{14} and Γ_2^{16} were obtained by the solution of the inverse problem after 14 iterations for the curve Γ_1 and 16 iterations for the curve Γ_2. The termination condition for the iterative process was given by the error level for the discrepancy.

It follows from our numerical experiments that a sufficiently accurate determination of several boundaries requires a large number of Dirichlet–Neumann pairs.

ACKNOWLEDGMENTS

The research was supported by the Russian Foundation for Basic Research (project no. 14-01-00244).

REFERENCES

1. Alessandrini, G. and Isakov, V., Analyticity and Uniqueness for the Inverse Conductivity Problem, *Rend. Istit. Mat. Univ. Trieste. I, II*, 1996, vol. 28, pp. 351–369.
2. Barceo, B., Fabes, E., and Seo, J.K., The Inverse Conductivity Problem with One Measurement: Uniqueness for Convex Polyhedra, *Proc. Amer. Math. Soc.*, 1994, vol. 122, no. 1, pp. 183–189.

3. Bellout, H., Friedman, A., and Isakov, V., Stability for an Inverse Problem in Potential Theory, *Trans. Amer. Math. Soc.*, 1992, vol. 332, pp. 271–296.
4. Astala, K. and Paivarinta, L., Calderon's Inverse Conductivity Problem in the Plane, *Ann. Math.*, 2006, vol. 163, pp. 265–299.
5. Kang, H., Seo, J.K., and Sheen, D., Numerical Identification of Discontinuous Conductivity Coefficients, *Inverse Problems*, 1997, vol. 13, pp. 113–123.
6. Bruhl, M. and Hanke, M., Numerical Implementation of Two Noniterative Methods for Locating Inclusions by Impedance Tomography, *Inverse Problems*, 2000, vol. 16, pp. 1029–1042.
7. Eckel, H. and Kress, R., Nonlinear Integral Equations for the Inverse Electrical Impedance Problem, *Inverse Problems*, 2007, vol. 23, pp. 475–491.
8. Knudsen, K., Lassas, M., Mueller, J.L., and Siltanen, S., Regularized D-Bar Method for the Inverse Conductivity Problem, *Inverse Problems and Imaging*, 2009, no. 3, pp. 599–624.
9. Lee, E., Seo, J.K., Harrach, B., and Kim, S., Projective Electrical Impedance Reconstruction with Two Measurements, *SIAM J. Appl. Math.*, 2013, vol. 73, no. 4, pp. 1659–1675.
10. Denisov, A.M., Zakharov, E.V., Kalinin, A.V., and Kalinin, V.V., Numerical Methods for Solving Some Inverse Problems of Heart Electrophysiology, *Differ. Uravn.*, 2009, vol. 45, no. 7, pp. 1014–1022.
11. Gavrilov, S.V. and Denisov, A.M., Numerical Methods for Determining the Inhomogeneity Boundary in a Boundary Value Problem for the Laplace Equation in a Piecewise Homogeneous Medium, *Zh. Vychisl. Mat. Mat. Fiz.*, 2011, vol. 51, no. 8, pp. 1476–1489.
12. Gavrilov, S.V. and Denisov, A.M., Iterative Method for Solving a Three-Dimensional Electrical Impedance Tomography Problem in the Case of Piecewise Constant Conductivity and One Measurement on the Boundary, *Zh. Vychisl. Mat. Mat. Fiz.*, 2012, vol. 52, no. 8, pp. 1426–1436.
13. Gavrilov, S.V. and Denisov, A.M., Numerical Method for Solving a Two-Dimensional Electrical Impedance Tomography Problem in the Case of Measurements on Part of the Outer Boundary, *Zh. Vychisl. Mat. Mat. Fiz.*, 2014, vol. 54, no. 11, pp. 1756–1766.
14. Gavrilov, S.V., Iterative Method for Determining the Shape and Conductivity of Homogeneous Inclusion in the Two-Dimensional Electric Impedance Tomography Problem, *Vychisl. Metody i Program.*, 2015, vol. 16, pp. 501–505.
15. Tikhonov, A.N. and Arsenin, V.Ya., *Metody resheniya nekorrektnykh zadach* (Methods for Solving Ill-Posed Problems), Moscow: Nauka, 1979.
16. Samarskii, A.A. and Tikhonov, A.N., *Uravneniya matematicheskoi fiziki* (Equations of Mathematical Physics), Moscow: Moskov. Gos. Univ., 1999.