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PARTIAL DIFFERENTIAL EQUATIONS

Periodic Solutions of the Wave Equation with Nonconstant Coefficients and with Homogeneous Dirichlet and Neumann Boundary Conditions

I. A. Rudakov
Bauman Moscow State Technical University, Moscow, Russia
e-mail: rudakov_bgu@mail.ru

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Abstract—We prove theorems on the existence and regularization of periodic solutions of the wave equation with variable coefficients on an interval with homogeneous Dirichlet and Neumann boundary conditions. The nonlinear term has a power-law growth or satisfies the nonresonance condition at infinity.

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1. INTRODUCTION

Consider the problem on the periodic solutions of the wave equation

$$p(x)u_{tt} - (p(x)u_x)_x = g(x,t,u) + f(x,t), \qquad 0 < x < \pi, \qquad t \in \mathbf{R}, \tag{1}$$

$$u(x,t+T) = u(x,t), \qquad 0 < x < \pi, \qquad t \in \mathbf{R}. \tag{2}$$

The boundary conditions have one of the forms

$$u(0,t) = u(\pi,t) = 0, \qquad t \in \mathbf{R}, \tag{3}$$

$$u(0,t) = u'(\pi,t) = 0, \qquad t \in \mathbf{R}, \tag{4}$$

$$u'(0,t) = u'(\pi,t) = 0, \qquad t \in \mathbf{R}. \tag{5}$$

The more general equation

$$\varrho(z)u_{tt} - (\mu(z)u_z)_z = h(z,t,u) + F(z,t),$$

which describes the propagation of seismic waves, can be reduced to Eq. (1) by the change of variables [1] $x = \int_0^z \sqrt{\varrho(s)/\mu(s)}\, ds$.

The function $p(x)$ satisfies the conditions

$$p(x) \in C^2[0,\pi], \qquad p(x) > 0, \qquad x \in [0,\pi]. \tag{6}$$

Set $\Omega = [0,\pi] \times \mathbf{R}/(T\mathbf{Z})$, $\eta_p(x) = 2^{-1}p''/p - 4^{-1}(p'/p)^2$, and $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$.

The problem on the periodic solutions of a quasilinear wave equation with constant coefficients was studied in numerous papers (e.g., see [2–7]). It was proved in [1, 8–11] that there exist time-periodic solutions of the wave equation with variable coefficients for the case in which the function $\eta_p(x)$ is of constant sign ($\eta_p(x) > 0$, $x \in [0,\pi]$, in [1, 8–10] and $\eta_p(x) < 0$, $x \in [0,\pi]$, in [11]). The aim of the present paper is to prove theorems on the existence of time-periodic solutions of the wave equation (1) with one of the boundary conditions (3), (4), and (5) for the case in which the function $\eta_p(x)$ can change sign on the interval $[0,\pi]$.

2. PROPERTIES OF THE LINEAR PART OF THE EQUATION

We seek solutions of problems (1)–(3); (1), (2), (4); and (1), (2), (5) as sums of Fourier series. To construct the corresponding orthonormal systems, consider the Sturm–Liouville problems

$$-(p(x)\varphi'(x))' = \lambda p(x)\varphi(x), \qquad 0 < x < \pi, \tag{7}$$

$$\varphi(0) = \varphi(\pi) = 0, \tag{8}$$

$$\varphi(0) = \varphi'(\pi) = 0, \tag{9}$$

$$\varphi'(0) = \varphi'(\pi) = 0. \tag{10}$$

Consider the spaces $L_2(0,\pi)$ and $L_2(\Omega)$ in which the inner product is defined by the formulas

$$(\varphi,\psi) = \int_0^\pi \varphi(x)\psi(x)p(x)\,dx, \qquad \varphi,\psi \in L_2(0,\pi),$$

$$(u,v) = \int_\Omega u(x,t)v(x,t)p(x)\,dx\,dt, \qquad u,v \in L_2(\Omega).$$

It follows from problem (7), (8) [or (7), (9), or (7), (10)] that

$$\lambda \int_0^\pi \varphi^2(x)p(x)\,dx = \int_0^\pi (\varphi'(x))^2 p(x)\,dx. \tag{11}$$

Therefore, problems (7), (8); (7), (9); and (7), (10) have nonnegative simple [12, pp. 220–222 of the Russian translation] eigenvalues $\lambda = \lambda_n^2$, $n \in \mathbf{N}$ ($\lambda_n \ge 0$), with the corresponding eigenfunctions $\varphi_n(x)$. Here the eigenvalues are numbered in ascending order, and for all three Sturm–Liouville problems, we introduce the same notation for the eigenvalues and eigenfunctions. Note that the inequality $\lambda_n > 0$, $n \in \mathbf{N}$, holds for problems (7), (8) and (7), (9), and, for problem (7), (10), we have $\lambda_1 = 0$, $\lambda_n > 0$, $n \ge 2$, and φ_1 is a constant function.

We assume that the functions $\varphi_n(x)$ are normalized in $L_2(0,\pi)$. By the Steklov theorem, the function system $\{\varphi_n(x)\}$ is complete and orthonormal in $L_2(0,\pi)$. Note that relations (8)–(11) imply that the function system

$$\{\varphi_n'(x)/\lambda_n\} \tag{12}$$

[$n \ge 2$ for problem (7), (10)] is orthonormal in $L_2(0,\pi)$ as well.

The following asymptotic representation of eigenvalues of the Sturm–Liouville problem (7), (8) was proved in the monograph [12, pp. 220–222 of the Russian translation] :

$$\lambda_n = n + \frac{B}{2\pi}\frac{1}{n} + \alpha_n, \tag{13}$$

where $B = \int_0^\pi \eta_p(x)\,dx$ and $\alpha_n = O(1/n^2)$, $n \in \mathbf{N}$. For problem (7), (9), we have the relation

$$\lambda_n = n - \frac{1}{2} + \frac{B}{2\pi}\frac{1}{n} + \beta_n, \tag{14}$$

where $B = \int_0^\pi \eta_p(x)\,dx - p'(\pi)/p(\pi)$ and $\beta_n = O(1/n^2)$, $n \in \mathbf{N}$, and, for problem (7), (10), we have $\lambda_1 = 0$ and

$$\lambda_n = n - 1 + \frac{B}{2\pi}\frac{1}{n} + \gamma_n, \tag{15}$$

where $B = \int_0^\pi \eta_p(x)\,dx + p'(0)/p(0) - p'(\pi)/p(\pi)$ and $\gamma_n = O(1/n^2)$, $n \in \mathbf{N}$, $n \ge 2$.

Let $H_1(\Omega)$ be the Sobolev space obtained by the closure of the space $C^\infty(\Omega)$ in the norm $\|u\|_1 = (\int_\Omega (u^2 + u_x^2 + u_t^2)p(x)\,dx\,dt)^{1/2}$, and let $H_1^0(\Omega)$ be the closure, in the norm $\|\cdot\|_1$, of the space of functions infinitely differentiable in Ω and compactly supported with respect to x on $[0,\pi]$ for each t. The function system

$$\Lambda = \left\{ \frac{1}{\sqrt{T}}\varphi_n(x), \frac{\sqrt{2}}{\sqrt{T}}\varphi_n(x)\cos\frac{2\pi}{T}mt, \frac{\sqrt{2}}{\sqrt{T}}\varphi_n(x)\sin\frac{2\pi}{T}mt \right\}_{n,m\in N}$$

is complete and orthonormal in $L_2(\Omega)$.

By D we denote the set of finite linear combinations of functions in the system Λ. We define the operator $A_0 : L_2(\Omega) \to L_2(\Omega)$ such that $D(A_0) = D$ and $A_0\varphi = p\varphi_{tt} - (p\varphi_x)_x$, $\varphi \in D(A_0)$. Let $\overline{A}_0\varphi = p^{-1}A_0\varphi$, $\varphi \in D(A_0)$. Set $A = (\overline{A}_0)^*$ in $L_2(\Omega)$. The functions in the system Λ are the eigenfunctions of the operators $\overline{A}_0$ and A with eigenvalues $\mu_{nm} = \lambda_n^2 - \left(\frac{2\pi}{T}m\right)^2$, $n \in N$, $m \in Z_+$, which correspond to the eigenfunctions $T_m\varphi_n(x)\cos\frac{2\pi}{T}mt$, $n \in \mathbf{N}$, $m \in Z_+$, $T_m\varphi_n(x)\sin\frac{2\pi}{T}mt$, $n, m \in \mathbf{N}$. Here $T_m = \frac{1}{\sqrt{T}}$ for $m = 0$ and $T_m = \sqrt{\frac{2}{T}}$ for $m > 0$.

We seek periodic solutions for which the time period has the form

$$T = 2\pi\frac{b}{a}, \qquad a, b \in \mathbf{N}, \qquad (a, b) = 1. \tag{16}$$

The function

$$u = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} T_m\varphi_n(x)\left(a_{nm}\cos\frac{a}{b}mt + b_{nm}\sin\frac{a}{b}mt\right) \tag{17}$$

belongs to $D(A)$ if and only if the series $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\mu_{nm}^2(a_{nm}^2 + b_{nm}^2)$ is convergent. In addition,

$$Au = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\mu_{nm}T_m\varphi_n(x)\left(a_{nm}\cos\frac{a}{b}mt + b_{nm}\sin\frac{a}{b}mt\right).$$

Set $\sigma(A) = \{\mu_{nm} |\ (n, m) \in N \times Z_+\}$. It follows from relations (13)–(15) that $\mu_{nm} = 0$ if and only if the relations

$$am - nb = \frac{Bb}{2\pi n} + b\alpha_n, \quad 2am - (2n-1)b = \frac{Bb}{\pi n} + 2b\beta_n, \quad am - (n-1)b = \frac{Bb}{2\pi n} + b\gamma_n, \tag{18}$$

hold for problem (1)–(3), problem (1), (2), (4), and problem (1), (2), (5), respectively. If $B \neq 0$, then, for sufficiently large n, the right-hand sides of relations (18) belong to the interval $(-1, 1)$ and are nonzero. Therefore, Eq. (18) has at most finitely many solution pairs (n, m) for $B \neq 0$. Consequently, in this case, the space $\operatorname{Ker} A$ is finite-dimensional.

From relations (13)–(15), we obtain the following representations for μ_{nm} corresponding to problem (1)–(3), problem (1), (2), (4), and problem (1), (2), (5), respectively:

$$\mu_{nm} = \frac{1}{b^2}(nb - am)(nb + am) + \frac{B}{\pi} + \bar{\alpha}_n, \tag{19}$$

$$\mu_{nm} = \frac{1}{4b^2}((2n-1)b - 2am)((2n-1)b + 2am) + \frac{B}{\pi} + \bar{\beta}_n, \tag{20}$$

$$\mu_{nm} = \frac{1}{b^2}((n-1)b - am)((n-1)b + am) + \frac{B}{\pi} + \bar{\gamma}_n, \tag{21}$$

where $\bar{\alpha}_n \to 0$, $\bar{\beta}_n \to 0$, and $\bar{\gamma}_n \to 0$ as $n \to \infty$.

Note that the equations $nb - am = 0$ and $(n-1)b - am = 0$ have the solutions $n = ar$ and $m = br$, $r \in \mathbf{N}$, respectively; $n = ar + 1$ and $m = br$, $r \in \mathbf{N}$. Therefore, for problems (1)–(3) and (1), (2), (5), we have the relations

$$\lim_{r\to\infty} \mu_{(ar)(br)} = \frac{B}{\pi}, \qquad \lim_{r\to\infty} \mu_{(ar+1)(br)} = \frac{B}{\pi}. \tag{22}$$

If b is even (accordingly, a is odd), then the equation

$$(2n-1)b - 2am = 0 \tag{23}$$

has the solutions $n = ar - (a-1)/2$, $m = b(2r-1)/2$, $r \in \mathbf{N}$, and for problem (1), (2), (4), there exists a limit

$$\lim_{r\to\infty} \mu_{(ar-a_1)(b(2r-1)/2)} = \frac{B}{\pi}, \tag{24}$$

where $a_1 = (a-1)/2$.

Therefore, for problems (1)–(3) and (1), (2), (5), the set $\sigma(A)$ has the unique limit point B/π. For problem (1), (2), (4), the set $\sigma(A)$ has the unique limit point B/π for even b, while for odd b Eq. (23) has no integer solution, and $\sigma(A)$ is a discrete unbounded set without finite limit points.

One can readily see that there exists a positive constant C_0 such that

$$|\mu_{nm}| \ge C_0(n+m) \tag{25}$$

if $\mu_{nm} \ne 0$; $bn \ne am$ for problem (1)–(3), $b(n-1) \ne am$ for problem (1), (2), (4), and either b is odd or $b(2n-1) \ne 2am$ for problem (1), (2), (5).

In a standard way (see [1]), one can prove the following properties of the operator A: (a) the operator A is self-adjoint in $L_2(\Omega)$; (b) $R(A)$ is closed in $L_2(\Omega)$; (c) $L_2(\Omega) = \operatorname{Ker} A \oplus R(A)$.

3. QUASILINEAR EQUATION

First, we assume that $f \in L_2(\Omega)$ and the nonlinear term g is continuous with respect to all variables and satisfies the following condition: there exist constants $\alpha, \beta \in \mathbf{R}$ and $C \in (0, +\infty)$ such that

$$\alpha \le \frac{g(x,t,u)}{p(x)u} \le \beta, \qquad u \in (-\infty, -C) \cup (C, +\infty), \qquad (x,t) \in \Omega. \tag{26}$$

Definition 1. A *generalized solution* of problems (1)–(3); (1), (2), (4); and (1), (2), (5) is a function $u \in L_2(\Omega)$ such that

$$\int_\Omega u(p\varphi_{tt} - (p\varphi_x)_x)\,dx\,dt = \int_\Omega (g(x,t,u) + f)\varphi\,dx\,dt, \qquad \varphi \in D.$$

The following assertion holds for problem (1), (2), (4).

Theorem 1. *Let the function g be continuous with respect to all variables and T-periodic in t, and let conditions* (6), (16), *and* (26) *be satisfied, where b is odd and $[\alpha, \beta] \cap \sigma(A) = \varnothing$. Then for each function $f(x,t) \in L_2(\Omega)$, problem* (1), (2), (4) *has a generalized solution*

$$u \in H_1(\Omega) \cap C(\Omega).$$

If, in addition, the function $g(x,t,u)$ satisfies the condition

$$\alpha(u-v)^2 \le \frac{1}{p(x)}(g(x,t,u) - g(x,t,v))(u-v) \le \beta(u-v)^2, \qquad u, v \in \mathbf{R}, \qquad (x,t) \in \Omega, \tag{27}$$

that this generalized solution is unique.

Proof. Set $B(u)=\dfrac{1}{p(x)}g(x,t,u)$. Then B is an operator from $L_2(\Omega)$ to $L_2(\Omega)$. The function $u\in L_2(\Omega)$ is a generalized solution of problem (1), (2), (4) if and only if

$$Au-B(u)=\frac{1}{p}f. \tag{28}$$

We prove the existence of a solution of Eq. (28) with the use of Theorem 3.1 in [10]. In a standard way (see [4]), one can prove the convergence of the series $\sum_{\mu_{nm}\neq 0}1/\mu_{nm}^2<\infty$. Therefore, the operator $A^{-1}:\ R(A)\to R(A)$ is compact.

It follows from condition (26) and the assumptions of the theorem that there exist constants C_1, C_2, α_1, β_1, d, and λ such that $[\alpha,\beta]\subseteq(\alpha_1,\beta_1)$, $[\alpha_1,\beta_1]\cap\sigma(A)=\varnothing$, $\lambda\in(\alpha_1,\beta_1)$, $d\in(0,\beta_1-\lambda)$, $C_1,C_2\in(0,+\infty)$, and $g(x,t,u)/p(x)=\lambda u+h(x,t,u)$, where

$$h(x,t,u)u\ge -C_1,\qquad |h(x,t,u)|\le d|u|+C_2,\qquad (x,t,u)\in\Omega\times\mathbf{R}. \tag{29}$$

Consequently,

$$\begin{aligned}(B(u)-\lambda u,u)&=\int\limits_\Omega h(x,t,u)up(x)\,dx\,dt=\int\limits_\Omega |h(x,t,u)u+C_1|p(x)\,dx\,dt-C_1\int\limits_\Omega p(x)\,dx\,dt\\
&\ge\int\limits_\Omega |h(x,t,u)|\,|u|p(x)\,dx\,dt-2C_1\int\limits_\Omega p(x)\,dx\,dt\\
&\ge\frac{1}{d}\int\limits_\Omega h^2(x,t,u)p(x)\,dx\,dt-\frac{C_2}{d}\int\limits_\Omega |h(x,t,u)|p(x)\,dx\,dt-C_3\\
&\ge\frac{1}{d}\|h(x,t,u)\|^2-C_4\|h(x,t,u)\|-C_3\ge\left(\frac{1}{d}-\varepsilon^2\right)\|h(x,t,u)\|^2-\frac{C_4^2}{4\varepsilon^2}-C_3.\end{aligned}$$

Here C_3 and C_4 are positive constants independent of u, and ε is an arbitrary positive constant. The assumptions of Theorem 3.1 in [10] are satisfied for sufficiently small ε. This implies the existence of a solution $u\in L_2(\Omega)$ of Eq. (28).

Set $h=B(u)+p^{-1}f\in L_2(\Omega)$. We expand the function h in a Fourier series in the system Λ,

$$h=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}T_m\varphi_n(x)\left(\bar a_{nm}\cos\frac{a}{b}mt+\bar b_{nm}\sin\frac{a}{b}mt\right).$$

Let $u=u_1+u_2$, where $u_1\in\operatorname{Ker}A$ and $u_2\in R(A)$. Then

$$u_2=\sum_{\mu_{nm}\neq 0}T_m\varphi_n(x)\frac{1}{\mu_{nm}}\left(\bar a_{nm}\cos\frac{a}{b}mt+\bar b_{nm}\sin\frac{a}{b}mt\right).$$

Since $|\varphi_n(x)|\le C$ for arbitrary n and $x\in[0,\pi]$ (see [12, pp. 220–222 of the Russian translation]) and

$$\sum_{\mu_{nm}\neq 0}\frac{1}{|\mu_{nm}|}(|\bar a_{nm}|+|\bar b_{nm}|)\le\left(\sum_{\mu_{nm}\neq 0}\frac{1}{\mu_{nm}^2}\right)^{1/2}\|h\|<\infty,$$

we have $u_2\in C_2(\Omega)$. It follows from inequality (25) that the sequence $\left\{\dfrac{m}{\mu_{nm}}\right\}$ is bounded. Consequently, $(u_2)_t\in L_2(\Omega)$. It follows from relations (25) and (14) that the sequence $\left\{\dfrac{\lambda_n}{\mu_{nm}}\right\}$ is bounded. Since system (12) is orthonormal, we have the inclusion $(u_2)_x\in L_2(\Omega)$. Therefore, $u_2\in H_1(\Omega)\cap C(\Omega)$.

The uniqueness of the solution under condition (27) can be proved by analogy with Assertion 1.1 in [13]. The proof of the theorem is complete.

Remark 1. Condition (27) is satisfied if, say, $g \in C^1(\Omega \times \mathbf{R})$ and $\alpha \le \frac{1}{p(x)} g'_u \le \beta$, $(x,t,u) \in \Omega \times \mathbf{R}$.

Consider problems (1)–(3) and (1), (2), (5) and problem (1), (2), (4) in the case of even b.

Theorem 2. *Let the function $g \in C^1(\Omega \times \mathbf{R})$ be T-periodic in t, and let conditions* (6) *and* (16) *be satisfied. In the case of problem* (1), (2), (4), *assume that b is even and there exist positive constants $M_1, M_2 > 0$ such that*

$$|g_t(x,t,u)| \le M_1|u| + M_2, \qquad (u,x,t) \in \mathbf{R} \times \Omega. \tag{30}$$

In addition, assume that either $B < 0$ and condition (26) *is satisfied, where*

$$\alpha > B/\pi, \qquad [\alpha, \beta] \cap \sigma(A) = \varnothing$$

and

$$-\gamma \le \frac{g_u(x,t,u)}{p(x)} \le M_3, \qquad (u,x,t) \in \mathbf{R} \times \Omega,$$

or $B > 0$ and condition (26) *is satisfied with $\beta < B/\pi$, $[-\beta, -\alpha] \cap \sigma(A) = \varnothing$, and*

$$-\gamma \le -\frac{g_u(x,t,u)}{p(x)} \le M_3, \qquad (u,x,t) \in \mathbf{R} \times \Omega,$$

where $M_3 > 0$ and $\gamma \in (0, |B|/\pi)$. Then, for each $f(x,t) \in H_1(\Omega)$, problems (1)–(3); (1), (2), (4); *and* (1), (2), (5) *have a generalized solution $u \in H_1(\Omega) \cap C(\Omega)$. In the case of problem* (1)–(3), *the solution u belongs to $H_1^0(\Omega) \cap C(\Omega)$.*

Proof. Consider the case in which $B < 0$. (The case of $B > 0$ can be considered in a similar way.) We prove the existence of a solution of Eq. (28) by using Theorem 3.2 in [10] and by noting that the assertion of this theorem remains valid if the condition $\underline{\lambda} \ge 0$ is replaced by the condition $\underline{\lambda} \ge -a$. In this case, its proof remains the same.

We rewrite Eq. (28) in the form

$$-Au + B(u) = -\frac{1}{p} f. \tag{31}$$

Set $a_0 = \frac{1}{2}\left(\gamma + \frac{|B|}{\pi}\right)$ and $b_0 = \frac{1}{2}\left(-\gamma + \frac{3|B|}{\pi}\right)$. It follows from relations (22) and (24) that there exists a number $r_0 \in \mathbf{N}$ such that the inclusions

$$-\mu_{(ar)(br)} \in [a_0, b_0], \qquad -\mu_{(ar+1)(br)} \in [a_0, b_0], \qquad -\mu_{(ar-a_1)(b(2r-1)/2)} \in [a_0, b_0]$$

hold for the respective three problems for all $r \ge r_0$.

For each of the considered problems, by M and L we denote the sets

$$\begin{aligned} M &= \{(n,m) \in \mathbf{N} \times \mathbf{Z}_+ |\ \mu_{nm} \ne 0,\ nb \ne ma\} \\ &\quad \cup \{(n,m)|\ n = ar,\ m = br,\ r \in \mathbf{N},\ r < r_0,\ \mu_{(ar)(br)} \ne 0\}, \\ L &= \{(n,m)|\ n = ar,\ m = br,\ r \in \mathbf{N},\ r \ge r_0\} \end{aligned}$$

for problem (1)–(3),

$$\begin{aligned} M &= \{(n,m) \in \mathbf{N} \times \mathbf{Z}_+ |\ \mu_{nm} \ne 0,\ (2n-1)b \ne 2ma\} \\ &\quad \bigcup \left\{(n,m)|\ n = ar - \frac{a-1}{2},\ m = \frac{b}{2}(2r-1),\ r \in \mathbf{N},\ r < r_0,\ \mu_{(ar-a_1)(b(r-1)/2)} \ne 0\right\}, \\ L &= \left\{(n,m)|\ n = ar - \frac{a-1}{2},\ m = \frac{b}{2}(2r-1),\ r \in \mathbf{N},\ r \ge r_0\right\} \end{aligned}$$

for problem (1), (2), (4), and

$$M = \{(n,m) \in \mathbf{N} \times \mathbf{Z}_+ |\ \mu_{nm} \neq 0,\ (n-1)b \neq ma\}$$
$$\cup \{(n,m)|\ n = ar+1,\ m = br,\ r \in \mathbf{N},\ r < r_0,\ \mu_{(ar+1)(br)} \neq 0\},$$
$$L = \{(n,m)|\ n = ar+1,\ m = br,\ r \in \mathbf{N},\ r \geq r_0\}$$

for problem (1), (2), (5). Here $a_1 = (a-1)/2$. In addition, we introduce the sets

$$\Lambda_1 = \{u \in \Lambda|\ Au = 0\}, \qquad \Lambda_2 = \Big\{\varphi_n(x) \cos \frac{a}{b}mt,\ \varphi_n(x) \sin \frac{a}{b}mt|\ (n,m) \in L\Big\},$$
$$\Lambda_3 = \Big\{\varphi_n(x) \cos \frac{a}{b}mt,\ \varphi_n(x) \sin \frac{a}{b}mt|\ (n,m) \in M\Big\}.$$

Let N_1, N_2, and N_3 be the closures of the sets of finite linear combinations of Λ_1, Λ_2, and Λ_3, respectively, in $L_2(\Omega)$. Note that $N_1 = \operatorname{Ker} A$.

In a standard way [4], one can show that $\sum_{(n,m)\in M} 1/\mu_{nm}^2 < \infty$. Therefore, properties I and II in [10] hold for the operator $-A$.

It follows from the assumptions of the theorem that there exists a number $\lambda < \alpha$ such that $[\lambda, \alpha] \cap \sigma(A) = \varnothing$. Set $h(x,t,u) = g(x,t,u)/p(x) - \lambda u$. It follows from condition (26) that there exist constants $C_1, C_2 \in (0,\infty)$ and $d \in (0, \beta - \lambda)$ such that inequalities (29) are true. Just as in Theorem 1, one can prove the existence of a constant C_5 such that

$$(B(u) - \lambda u, u) \geq \gamma_1^{-1} \|B(u) - \lambda u\|^2 - C_5, \qquad u \in L_2(\Omega).$$

Here $\gamma_1 \in (0, \bar{\lambda} - \lambda)$, and $\bar{\lambda}$ is the least eigenvalue of A exceeding β. Therefore, the assumptions of Theorem 3.2 in [10] are satisfied. This implies the existence of a generalized solution $u \in L_2(\Omega)$ of problems (1)–(3); (1), (2), (4); and (1), (2), (5).

Let us show that if u is a generalized solution of problem (1)–(3), then $u \in H_1^0(\Omega) \cap C(\Omega)$. The inclusion $u \in H_1(\Omega) \cap C(\Omega)$ for problems (1), (2), (4) and (1), (2), (5) can be proved in a similar way.

By P_1, P_2, and P_3 we denote the orthogonal projections in $L_2(\Omega)$ onto the subspaces N_1, N_2, and N_3, respectively. Then $u = u_1 + u_2 + u_3$, where $u_i = P_i u$, $i \in \{1,2,3\}$. We project Eq. (31) onto N_1, N_2, and N_3,

$$P_1\left(\frac{1}{p(x)} g(x,t,u)\right) + P_1\left(\frac{1}{p(x)} f(x,t)\right) = 0,$$
$$Au_2 = P_2\left(\frac{1}{p(x)} g(x,t,u)\right) + P_2\left(\frac{1}{p(x)} f(x,t)\right), \tag{32}$$
$$Au_3 = P_3\left(\frac{1}{p(x)} g(x,t,u)\right) + P_3\left(\frac{1}{p(x)} f(x,t)\right). \tag{33}$$

We expand the function $(1/p(x))f(x,t)$ in a Fourier series in the system Λ,

$$\frac{1}{p} f = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} T_m \varphi_n(x) \Big(a_{nm} \cos \frac{a}{b}mt + b_{nm} \sin \frac{a}{b}mt\Big).$$

Since the function $p^{-1}f \in H_1(\Omega)$ is T-periodic in t, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m^2 (a_{nm}^2 + b_{nm}^2) < \infty. \tag{34}$$

Consequently,

$$\sum_{k=1}^{\infty} k^2 (\alpha_k^2 + \beta_k^2) < \infty, \tag{35}$$

where $\alpha_k = a_{(ak)(bk)}$ and $\beta_k = b_{(ak)(bk)}$. Set $f_i = P_i(p^{-1}f)$, $i \in \{1, 2, 3\}$. Then

$$f_2 = \sum_{k=r_0}^{\infty} T_{bk}\varphi_{ak}(x)(\alpha_k \cos(akt) + \beta_k \sin(akt)). \tag{36}$$

It follows from inequality (34) that $(f_2)_t \in L_2(\Omega)$. In addition,

$$\begin{aligned}\sum_{k=r_0}^{\infty} |T_{bk}\varphi_{ak}(x)(\alpha_k \cos(akt) + \beta_k \sin(akt))| &\le \sum_{k=r_0}^{\infty} |T_{bk}|\varphi_{ak}(x)|(|\alpha_k| + |\beta_k|)| \\ &\le C_6 \left(\sum_{k=r_0}^{\infty} \frac{1}{k^2}\right)^{1/2} \left(\sum_{k=r_0}^{\infty} k^2(\alpha_k^2 + \beta_k^2)\right)^{1/2} < \infty,\end{aligned}$$

because $T_m|\varphi_n(x)| \le C_6$ for arbitrary n, m, and $x \in [0, \pi]$ (see [12]). Consequently, $f_2 \in C(\Omega)$. Consider the series

$$\sum_{k=r_0}^{\infty} T_{bk}\varphi'_{ak}(x)(\alpha_k \cos(akt) + \beta_k \sin(akt)). \tag{37}$$

Since system (12) is orthonormal, it follows from relations (13) and (35) that the series (37) is convergent in $L_2(\Omega)$. Consequently, $(f_2)_x \in L_2(\Omega)$ and $f_2 \in H_1(\Omega)$. Since N_1 is finite-dimensional, we have the inclusion $f_3 \in H_1(\Omega)$.

Relation (33), inequality (25), and the convergence of the series $\sum_{(n,m)\in M} 1/\mu_{nm}^2$ imply the inclusion $u_3 \in H_1^0(\Omega) \cap C(\Omega)$ (see [11]). By using the Rabinovich method [16], we show that $u_2 \in H_1^0(\Omega)$.

For a function $F \in L_2(\Omega)$, let $F^h = h^{-1}(F(x, t+h) - F(x, t))$ for $h \ne 0$. Take the inner product of relation (32) by $(u_2^h)^{-h} \in N_2$ in $L_2(\Omega)$,

$$(Au_2^h, u_2^h) = \int_\Omega (g(x, t, h))^h u_2^h \, dx \, dt + \int_\Omega f^h u_2^h \, dx \, dt.$$

Let us transform the integrated function as follows:

$$\begin{aligned}(g(x, t, u))^h &= \frac{1}{h}(g(x, t+h, u(x, t+h)) - g(x, t, u(x, t+h))) \\ &\quad + \frac{1}{h}(g(x, t, u(x, t+h)) - g(x, t, u(x, t))) \\ &= g_t'(x, \tau(x, t, h), u(x, t+h)) + g_u(x, t, \theta(x, t, h))u^h.\end{aligned}$$

Consequently,

$$\begin{aligned}&((-A)u_2^h, u_2^h) + \int_\Omega g_u(x, t, \theta(x, t, h))(u_2^h)^2 \, dx \, dt \\ &\quad = -\int_\Omega (g_t(x, \tau(x, t, h), u(x, t+h)) + f^h)u_2^h \, dx \, dt - \int_\Omega g_u(x, t, \theta(x, t, h))(u_1^h + u_3^h)u_2^h \, dx \, dt.\end{aligned}$$

From inequality (30), the assumptions of the theorem, and the definition of N_2, we obtain the estimate

$$\alpha_0 \|u_2^h\|^2 \le C_7(1 + \|u_1^h + u_3^h\| + \|f^h\|)\|u_2^h\|,$$

where $\alpha_0 = \dfrac{1}{2}\left(\dfrac{|B|}{\pi} - \gamma\right)$ and C_7 is some positive constant.

Since $f, u_1, u_3 \in H_1(\Omega)$, we have

$$\|u_1^h + u_2^h\| + \|f\|^h \le C_8$$

for some positive constant C_8 independent of h. Therefore,

$$\|u_2^h\| \le \alpha_0^{-1} C_7 (1 + C_8)$$

for all h; consequently, there exists $(u_2)_t \in L_2(\Omega)$. It follows from the definition of N_2 that

$$u_2 = \sum_{r=r_0}^{\infty} T_{br} \varphi_{ar}(x)(\alpha_r \cos(art) + \beta_r \sin(art)).$$

Since u_2 is a periodic function, we have the relation

$$(u_2)_t = a \sum_{r=r_0}^{\infty} T_{br} \varphi_{ar}(x) r(-\alpha_r \sin(art) + \beta_r \cos(art)).$$

Since $(u_2)_t \in L_2(\Omega)$, it follows that

$$\sum_{r=r_0}^{\infty} r^2(\alpha_r^2 + \beta_r^2) < \infty. \tag{38}$$

Since

$$\sum_{r=r_0}^{\infty} (|\alpha_r| + |\beta_r|) \le \left(\sum_{r=r_0}^{\infty} \frac{1}{r^2} \right)^{1/2} \left(\sum_{r=r_0}^{\infty} r^2(\alpha_r^2 + \beta_r^2) \right)^{1/2} < \infty,$$

it follows that $u_2 \in C(\Omega)$. Since the function system (12) is orthonormal, it follows from relations (13) and (38) that $(u_2)_x \in L_2(\Omega)$. Consequently, $u_2 \in H_1^0(\Omega) \cap C(\Omega)$. The proof of the theorem is complete.

Remark 2. The solution found in Theorem 2 is unique, if, in addition to assumptions of the theorem, condition (27) holds for $B < 0$ and the condition

$$\alpha(u - v)^2 \le (1/p(x))(g(x,t,v) - g(x,t,u))(u - v) \le \beta(u - v)^2,$$

$u, v \in \mathbf{R}$, $(x,t) \in \Omega \times \mathbf{R}$, is satisfied for $B > 0$.

4. WAVE EQUATION WITH NONLINEAR TERM OF POWER-LAW GROWTH

We write out the wave equation in the form

$$p(x)u_{tt} - (p(x)u_x)_x + g(x,t,u) = 0, \qquad 0 < x < \pi, \qquad t \in \mathbf{R}. \tag{39}$$

Suppose that there exist positive constants A_1, A_2, A_3, A_4, and r such that the inequality

$$A_3|u|^{r-1} - A_4 \le |g(x,t,u)| \le A_1|u|^{r-1} + A_2, \tag{40}$$

where

$$r > 2, \qquad \frac{2}{r} A_1 < A_3 \le A_1, \tag{41}$$

holds for all $(x,t,u) \in \Omega \times \mathbf{R}$.

Definition 2. A *generalized solution of problems* (38), (2), (3); (39), (2), (4); and (39), (2), (5) is defined as a function $u \in L_r(\Omega)$ such that

$$\int_{\Omega} u(p\varphi_{tt} - (p\varphi_x)_x)\, dx\, dt + \int_{\Omega} g(x,t,u)\varphi\, dx\, dt = 0, \qquad \varphi \in D.$$

Theorem 3. *Let conditions* (6) *and* (16) *be satisfied, let the function* g *be continuous on* $\Omega \times \mathbf{R}$, *be* T*-periodic with respect to* t, *and satisfy conditions* (40) *and* (41), *and let either* g *be independent of* t *or* $g(x,t,-u) = -g(x,t,u)$ *for all* $(x,t,u) \in \Omega \times \mathbf{R}$. *In addition, suppose that either* $B > 0$ *and the function* g *is nondecreasing with respect to* u *for all* $(x,t) \in \Omega$ *or* $B < 0$ *and the function* g *is nonincreasing with respect to* u *for all* $(x,t) \in \Omega$. *Then for each* $d > 0$, *there exists a generalized solution* $u \in L_r(\Omega)$ *of problems* (39), (2), (3); (39), (2), (4); *and* (39), (2), (5) *such that* $\|u\|_r \geq d$. *For odd* b, *the generalized solution* u *of problem* (38), (2), (4) *satisfies the inclusion* $u \in H_1(\Omega) \cap C(\Omega)$.

The proof of the theorem reproduces that of Theorem 3.1 in [15] with the use of the Feirisl method [17].

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