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PARTIAL DIFFERENTIAL EQUATIONS

Characteristic Boundary Value Problem for a Third-Order Functional-Differential Equation with the Bianchi Operator

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Abstract—We consider a version of the Goursat problem for a previously unstudied third-order equation and prove the existence and uniqueness of the solution of this problem.

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Ordinary differential equations in which the argument of the unknown function undergoes some shift have no established name. They are called functional-differential equations as well as equations with deviating argument (see the bibliography in [1–4]). There is no well-developed theory for the case of partial differential equations. Let us only indicate some related results (e.g., see [5–9]).

In the present paper, we extend the approach used in [6, 9] for a hyperbolic equation with a shift in the arguments of the unknown function and for an equation with a pseudo-parabolic third-order operator to an equation that is a generalization of the Bianchi equation [10] in three-dimensional space. This equation is used when studying processes related to vibration phenomena [11, p. 63] and other problems in mechanics and mathematical physics.

1. In the domain $D = \{0 < x, y, z < 1\}$, let us define the operators

$$L_s\theta \equiv a^s\theta_{xyz} + b^s\theta_{xy} + c^s\theta_{yz} + d^s\theta_{xz} + e^s\theta_x + f^s\theta_y + g^s\theta_z + h^s\theta \qquad (s = 0, 1, 2),$$

where $a^s \in C^{1+1+1}(\overline{D})$, $b^s \in C^{1+1+0}(\overline{D})$, $c^s \in C^{0+1+1}(\overline{D})$, $d^s \in C^{1+0+1}(\overline{D})$, $e^s \in C^{1+0+0}(\overline{D})$, $f^s \in C^{0+1+0}(\overline{D})$, $g^s \in C^{0+0+1}(\overline{D})$, and $h^s \in C(\overline{D})$.

Problem G. For the equation

$$L_0u + L_1v_1 + L_2v_2 = f, \qquad f \in C(\overline{D}), \tag{1}$$

find a regular solution in domain D that can be continuously extended to ∂D as $x, y, z \to 0$ and satisfies the conditions

$$\begin{aligned} u(0, y, z) &= \varphi_1(y, z), \qquad u(x, 0, z) = \varphi_2(x, z), \qquad u(x, y, 0) = \varphi_3(x, y),\\ \varphi_1(y, 0) &= \varphi_3(0, y), \qquad \varphi_2(x, 0) = \varphi_3(x, 0),\\ \varphi_1(0, z) &= \varphi_2(0, z), \qquad \varphi_1, \varphi_2, \varphi_3 \in C^{(1+1)}([0, 1] \times [0, 1]). \end{aligned} \tag{2}$$

Here the functions $v_i(x, y, z)$ $(i = 1, 2)$ are expressed via $u(x, y, z)$ by the formulas

$$v_1(x, y, z) \equiv u[g_3^1(z), g_1^1(x), g_2^1(y)], \qquad v_2(x, y, z) \equiv u[g_2^2(y), g_3^2(z), g_1^2(x)]. \tag{3}$$

Each of the functions $g_i^j \in C^1[0, 1]$ $(i = 1, 2, 3;\ j = 1, 2)$ is a sense-preserving mapping of the closed interval $[0, 1]$ onto itself. In particular,

$$g_i^j(0) = 0, \qquad g_i^j(1) = 1. \tag{4}$$

The pairs of functions (g_3^1, g_1^2), (g_1^1, g_2^2), and (g_2^1, g_3^2) are mutually inverse and satisfy the relations

$$\begin{aligned} v_1(g_3^1(z), g_1^1(x), g_2^1(y)) &= u(g_3^1(g_2^1(y)), g_1^1(g_3^1(z)), g_2^1(g_1^1(x))) = v_2(x,y,z), \\ v_2(g_2^2(y), g_3^2(z), g_1^2(x)) &= u(g_2^2(g_3^2(z)), g_3^2(g_1^2(x)), g_1^2(g_2^2(y))) = v_1(x,y,z). \end{aligned} \tag{5}$$

This statement generalizes the Goursat problem for the equation $L_1 u = f$, which was considered in [12].

2. Set $x = t_1$, $y = t_2$, and $z = t_3$ in Eq. (1) and integrate the resulting relation with respect to t_1 from 0 to x, with respect to t_2 from 0 to y, and with respect to t_3 from 0 to z. By taking into account conditions (2) and the relations

$$\begin{aligned} &v_1(x,0,z) = \varphi_3[g_3^1(z), g_1^1(x)], \quad v_1(x,y,0) = \varphi_1[g_1^1(x), g_2^1(y)], \quad v_1(0,y,z) = \varphi_2[g_3^1(z), g_2^1(y)], \\ &v_2(x,0,z) = \varphi_1[g_3^2(z), g_1^2(x)], \quad v_2(x,y,0) = \varphi_2[g_2^2(y), g_1^2(x)], \quad v_2(0,y,z) = \varphi_3[g_2^2(y), g_3^2(z)], \end{aligned}$$

which follow from relations (3) and (4), we obtain

$$\begin{aligned} &(a^1 u)(x,y,z) + (a^2 v_1)(x,y,z) + (a^3 v_2)(x,y,z) \\ &\quad = \int_0^x \sum_{i=1}^3 A_{100}^i v_{i-1}(t_1,y,z)\,dt_1 + \int_0^y \sum_{i=1}^3 A_{010}^i v_{i-1}(x,t_2,z)\,dt_2 + \int_0^z \sum_{i=1}^3 A_{001}^i v_{i-1}(x,y,t_3)\,dt_3 \\ &\quad + \int_0^x\!\!\int_0^y \sum_{i=1}^3 A_{110}^i v_{i-1}(t_1,t_2,z)\,dt_2\,dt_1 + \int_0^x\!\!\int_0^z \sum_{i=1}^3 A_{101}^i v_{i-1}(t_1,y,t_3)\,dt_3\,dt_1 \\ &\quad + \int_0^y\!\!\int_0^z \sum_{i=1}^3 A_{011}^i v_{i-1}(x,t_2,t_3)\,dt_3\,dt_2 + \int_0^x\!\!\int_0^y\!\!\int_0^z \sum_{i=1}^3 A_{111}^i v_{i-1}(t_1,t_2,t_3)\,dt_3\,dt_2\,dt_1 + \Omega(x,y,z). \end{aligned} \tag{6}$$

Here $v_0 \equiv u$, and the function $\Omega(x,y,z)$ is completely determined and depends on the coefficients of Eq. (1), the boundary values (2), and the functions g_i^j $(i = 1,2,3;\ j = 1,2)$; one has $A_{100}^i = a_x^i - c^i$, $A_{010}^i = a_y^i - d^i$, $A_{001}^i = a_z^i - b^i$, $A_{110}^i = a_{xy}^i - d_x^i - c_y^i + g^i$, $A_{101}^i = a_{xz}^i - c_z^i - b_x^i + f^i$, $A_{011}^i = a_{yz}^i - d_z^i - b_y^i + e^i$, and $A_{111}^i = A_{011x}^i - c_{yz}^i + f_y^i + g_z^i - e^i$.

The smoothness conditions for the coefficients of Eq. (1) permit one to carry out transformations used for the derivation of Eq. (6) and ensure that the functions $A^i_{p_1p_2p_3}$ $(p_1,p_2,p_3 = 0,1;$ $p_1+p_2+p_3 > 0;\ i = 1,2,3)$ and Ω belong to the class $C(\overline{D})$.

We denote the right-hand side of Eq. (6) by $k(x,y,z)$. Then it acquires the form

$$(a^1 u)(x,y,z) + (a^2 v_1)(x,y,z) + (a^3 v_2)(x,y,z) = k(x,y,z). \tag{7}$$

By setting $x = g_3^1(z)$, $y = g_1^1(x)$, and $z = g_2^1(y)$ in Eq. (7) and by taking into account property (5), we obtain

$$\begin{aligned} &a^3(g_3^1(z), g_1^1(x), g_2^1(y))u(x,y,z) + a^1(g_3^1(z), g_1^1(x), g_2^1(y))v_1(x,y,z) \\ &\quad + a^2(g_3^1(z), g_1^1(x), g_2^1(y))v_2(x,y,z) = k(g_3^1(z), g_1^1(x), g_2^1(y)). \end{aligned} \tag{8}$$

Now we set $x = g_2^2(y)$, $y = g_3^2(z)$, and $z = g_1^2(x)$ in Eq. (7) and take into account property (5); then we obtain the relation

$$\begin{aligned} &a^2(g_2^2(y), g_3^2(z), g_1^2(x))u(x,y,z) + a^3(g_2^2(y), g_3^2(z), g_1^2(x))v_1(x,y,z) \\ &\quad + a^1(g_2^2(y), g_3^2(z), g_1^2(x))v_2(x,y,z) = k(g_2^2(y), g_3^2(z), g_1^2(x)). \end{aligned} \tag{9}$$

Relations (7)–(9) form a system of linear algebraic equations for u, v_1, and v_2, whose determinant is equal to

$$\Delta(x,y,z)=\begin{vmatrix} a^1(x,y,z) & a^2(x,y,z) & a^3(x,y,z)\\ a^3(g_3^1(z),g_1^1(x),g_2^1(y)) & a^1(g_3^1(z),g_1^1(x),g_2^1(y)) & a^2(g_3^1(z),g_1^1(x),g_2^1(y))\\ a^2(g_2^2(y),g_3^2(z),g_1^2(x)) & a^3(g_2^2(y),g_3^2(z),g_1^2(x)) & a^1(g_2^2(y),g_3^2(z),g_1^2(x))\end{vmatrix}. \quad (10)$$

Set $B_1(x,y,z)=\begin{vmatrix} a^1(g_3^1(z),g_1^1(x),g_2^1(y)) & a^2(g_3^1(z),g_1^1(x),g_2^1(y))\\ a^3(g_2^2(y),g_3^2(z),g_1^2(x)) & a^1(g_2^2(y),g_3^2(z),g_1^2(x))\end{vmatrix}$ and

$$B_2(x,y,z)=\begin{vmatrix} a^2(x,y,z) & a^3(x,y,z)\\ a^3(g_2^2(y),g_3^2(z),g_1^2(x)) & a^1(g_2^2(y),g_3^2(z),g_1^2(x))\end{vmatrix},$$

$$B_3(x,y,z)=\begin{vmatrix} a^2(x,y,z) & a^3(x,y,z)\\ a^1(g_3^1(z),g_1^1(x),g_2^1(y)) & a^2(g_3^1(z),g_1^1(x),g_2^1(y))\end{vmatrix}.$$

If $\Delta(x,y,z)\neq 0$, then, by the Cramer formulas, we obtain

$$\begin{aligned}&(\Delta u)(x,y,z)\\ &=k(x,y,z)B_1(x,y,z)-k(g_3^1(z),g_1^1(x),g_2^1(y))B_2(x,y,z)+k(g_2^2(y),g_3^2(z),g_1^2(x))B_3(x,y,z), \quad (11)\\ &(\Delta v_1)(x,y,z)\\ &=k(x,y,z)B_3(g_3^1(z),g_1^1(x),g_2^1(y))-k(g_2^2(y),g_3^2(z),g_1^2(x))B_2(g_3^1(z),g_1^1(x),g_2^1(y))\\ &\quad+k(g_3^1(z),g_1^1(x),g_2^1(y))B_1(g_2^2(y),g_3^2(z),g_1^2(x)), \quad (12)\\ &(\Delta v_2)(x,y,z)\\ &=-k(x,y,z)B_2(g_2^2(y),g_3^2(z),g_1^2(x))+k(g_3^1(z),g_1^1(x),g_2^1(y))B_3(g_2^2(y),g_3^2(z),g_1^2(x))\\ &\quad+k(g_2^2(y),g_3^2(z),g_1^2(x))B_1(g_3^1(z),g_1^1(x),g_2^1(y)). \quad (13)\end{aligned}$$

Relation (10), together with property (5), implies that

$$\Delta(x,y,z)=\Delta(g_3^1(z),g_1^1(x),g_2^1(y))=\Delta(g_2^2(y),g_3^2(z),g_1^2(x)).$$

Therefore, if we set $x=g_3^1(z)$, $y=g_1^1(x)$, and $z=g_2^1(y)$ in (11) and use property (5) and the rule (3), then we obtain relation (12). If we set $x=g_2^2(y)$, $y=g_3^2(z)$, and $z=g_1^2(x)$ in (11) and take into account property (5) and the rule (3), then we obtain relation (13). Now we substitute the right-hand side $k(x,y,z)$ of Eq. (6) into (11),

$$\begin{aligned}(\Delta u)(x,y,z)=\Bigg(&\int_0^x\sum_{i=1}^3 A_{100}^i v_{i-1}(t_1,y,z)\,dt_1+\int_0^y\sum_{i=1}^3 A_{010}^i v_{i-1}(x,t_2,z)\,dt_2\\ &+\int_0^z\sum_{i=1}^3 A_{001}^i v_{i-1}(x,y,t_3)\,dt_3+\int_0^x\int_0^y\sum_{i=1}^3 A_{110}^i v_{i-1}(t_1,t_2,z)\,dt_2\,dt_1\\ &+\int_0^x\int_0^z\sum_{i=1}^3 A_{101}^i v_{i-1}(t_1,y,t_3)\,dt_3\,dt_1+\int_0^y\int_0^z\sum_{i=1}^3 A_{011}^i v_{i-1}(x,t_2,t_3)\,dt_3\,dt_2\\ &+\int_0^x\int_0^y\int_0^z\sum_{i=1}^3 A_{111}^i v_{i-1}(t_1,t_2,t_3)\,dt_3\,dt_2\,dt_1\Bigg)B_1(x,y,z)\end{aligned}$$

$$
\begin{aligned}
&- \Bigg(\int_0^{g_3^1(z)} \sum_{i=1}^3 A_{100}^i v_{i-1}(t_1, g_1^1(x), g_2^1(y))\, dt_1 + \int_0^{g_1^1(x)} \sum_{i=1}^3 A_{010}^i v_{i-1}(g_3^1(z), t_2, g_2^1(y))\, dt_2 \\
&+ \int_0^{g_2^1(y)} \sum_{i=1}^3 A_{001}^i v_{i-1}(g_3^1(z), g_1^1(x), t_3)\, dt_3 + \int_0^{g_3^1(z)} \int_0^{g_1^1(x)} \sum_{i=1}^3 A_{110}^i v_{i-1}(t_1, t_2, g_2^1(y))\, dt_2\, dt_1 \\
&+ \int_0^{g_3^1(z)} \int_0^{g_2^1(y)} \sum_{i=1}^3 A_{101}^i v_{i-1}(t_1, g_1^1(x), t_3)\, dt_3\, dt_1 + \int_0^{g_1^1(x)} \int_0^{g_2^1(y)} \sum_{i=1}^3 A_{011}^i v_{i-1}(g_3^1(z), t_2, t_3)\, dt_3\, dt_2 \\
&+ \int_0^{g_3^1(z)} \int_0^{g_1^1(x)} \int_0^{g_2^1(y)} \sum_{i=1}^3 A_{111}^i v_{i-1}(t_1, t_2, t_3)\, dt_3\, dt_2\, dt_1 \Bigg) B_2(x, y, z) \\
&+ \Bigg(\int_0^{g_2^2(y)} \sum_{i=1}^3 A_{100}^i v_{i-1}(t_1, g_3^2(z), g_1^2(x))\, dt_1 + \int_0^{g_3^2(z)} \sum_{i=1}^3 A_{010}^i v_{i-1}(g_2^2(y), t_2, g_1^2(x))\, dt_2 \\
&+ \int_0^{g_1^2(x)} \sum_{i=1}^3 A_{001}^i v_{i-1}(g_2^2(y), g_3^2(z), t_3)\, dt_3 + \int_0^{g_2^2(y)} \int_0^{g_3^2(z)} \sum_{i=1}^3 A_{110}^i v_{i-1}(t_1, t_2, g_1^2(x))\, dt_2\, dt_1 \\
&+ \int_0^{g_2^2(y)} \int_0^{g_1^2(x)} \sum_{i=1}^3 A_{101}^i v_{i-1}(t_1, g_3^2(z), t_3)\, dt_3\, dt_1 + \int_0^{g_3^2(z)} \int_0^{g_1^2(x)} \sum_{i=1}^3 A_{011}^i v_{i-1}(g_2^2(y), t_2, t_3)\, dt_3\, dt_2 \\
&+ \int_0^{g_2^2(y)} \int_0^{g_3^2(z)} \int_0^{g_1^2(x)} \sum_{i=1}^3 A_{111}^i v_{i-1}(t_1, t_2, t_3)\, dt_3\, dt_2\, dt_1 \Bigg) B_3(x, y, z) + \Omega_1(x, y, z), \qquad (14)
\end{aligned}
$$

$$\Omega_1(x, y, z) = (\Omega B_1)(x, y, z) - \Omega(g_3^1(z), g_1^1(x), g_2^1(y)) B_2(x, y, z) + \Omega(g_2^2(y), g_3^2(z), g_1^2(x)) B_3(x, y, z).$$

We divide Eq. (14) by $\Delta(x, y, z)$ and represent it in the operator form

$$u = Au + \Omega_1(x, y, z)/\Delta(x, y, z). \qquad (15)$$

Here the operator A acts by the rule

$$
\begin{aligned}
(Au)(x, y, z) = \frac{1}{\Delta(x, y, z)} &(B_1(x, y, z) I - B_2(x, y, z) \Lambda_1 \\
&+ B_3(x, y, z) \Lambda_2)(K_1 + K_2 \Lambda_1 + K_3 \Lambda_2) u(x, y, z),
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 u(x, y, z) = &\int_0^x A_{100}^1 u(t_1, y, z)\, dt_1 + \int_0^y A_{010}^1 u(x, t_2, z)\, dt_2 + \int_0^z A_{001}^1 u(x, y, t_3)\, dt_3 \\
&+ \int_0^x \int_0^y A_{110}^1 u(t_1, t_2, z)\, dt_2\, dt_1 + \int_0^x \int_0^y A_{101}^1 u(t_1, y, t_3)\, dt_3\, dt_1 \\
&+ \int_0^y \int_0^z A_{011}^1 u(x, t_2, t_3)\, dt_3\, dt_2 + \int_0^x \int_0^y \int_0^z A_{111}^1 u(t_1, t_2, t_3)\, dt_3\, dt_2\, dt_1,
\end{aligned}
$$

$$K_i\Lambda_{i-1}u(x,y,z) = \int\limits_0^x A^i_{100}v_{i-1}(t_1,y,z)\,dt_1 + \int\limits_0^y A^i_{010}v_{i-1}(x,t_2,z)\,dt_2 + \int\limits_0^z A^i_{001}v_{i-1}(x,y,t_3)\,dt_3$$
$$+ \int\limits_0^x\int\limits_0^y A^i_{110}v_{i-1}(t_1,t_2,z)\,dt_2\,dt_1 + \int\limits_0^x\int\limits_0^y A^i_{101}v_{i-1}(t_1,y,t_3)\,dt_3\,dt_1$$
$$+ \int\limits_0^y\int\limits_0^z A^i_{011}v_{i-1}(x,t_2,t_3)\,dt_3\,dt_2 + \int\limits_0^x\int\limits_0^y\int\limits_0^z A^i_{111}v_{i-1}(t_1,t_2,t_3)\,dt_3\,dt_2\,dt_1,$$

$i = 2, 3$, I is the operator of the identity transformation, and

$$\Lambda_1 u(x,y,z) = u(g^1_3(z), g^1_1(x), g^1_2(y)), \qquad \Lambda_2 u(x,y,z) = u(g^2_2(y), g^2_3(z), g^2_1(x)).$$

The operators K_1, K_2, and K_3 are Volterra operators, and Λ_1 and Λ_2 are operators such that

$$\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = I, \qquad \Lambda_1^2 = \Lambda_2, \qquad \Lambda_2^2 = \Lambda_1, \qquad \|\Lambda_1\| = \|\Lambda_2\| = 1.$$

Let $|\Delta(x,y,z)| \ge m > 0$, and let $|a^iB_j|$, $|b^iB_j|$, $|c^iB_j|$, $|d^iB_j|$, $|e^iB_j|$, $|f^iB_j|$, $|g^iB_j|$, and $|h^iB_j|$, $i, j = 1, 2, 3$, be bounded by the number M. By the continuity of these variables, there exists a number M satisfying this condition, and one can assume that these variables are strictly less than M. Assume that $u_1, u_2 \in C(\overline{D})$. Set

$$\sigma(x,y,z) = x + y + z + 2xy + 2xz + 2yz + 4xyz.$$

Then

$$|(Au_1 - Au_2)(x,y,z)| \le \frac{M}{m}\|u_1 - u_2\| \times 6[\sigma(x,y,z) + \Lambda_1\sigma(x,y,z) + \Lambda_2\sigma(x,y,z)].$$

Consequently,

$$\|Au_1 - Au_2\| \le 234\frac{M}{m}\|u_1 - u_2\|.$$

Hence it follows that A is a continuous operator in the domain $\overline{D}$. One can show that $(K_1 + K_2\Lambda_1 + K_3\Lambda_2)$ is a Volterra operator and hence is invertible. Then one can show that the operator

$$(B_1(x,y,z)I - B_2(x,y,z)\Lambda_1 + B_3(x,y,z)\Lambda_2)$$

is invertible as well. Indeed, should we assume the contrary, we would arrive at a contradiction with the condition $\Delta(x,y,z) \ne 0$. Consequently, A is an invertible operator as a product of invertible operators. It follows that Eq. (15) is uniquely solvable.

The substitution of this solution into Eq. (14) makes it an identity, which, together with its derivation method, implies that the relation (6) obtained by the straightforward integration of Eq. (1) with regard of condition (2) is identically true. This justifies the application of the inverse operation $\partial^3/\partial x\,\partial y\,\partial z$ to relation (6), which leads to the identical validity of Eq. (1). We have thereby proved the following assertion.

Theorem. *If the function* $\Delta(x,y,z)$ *defined by relation* (10) *does not vanish for* $(x,y,z) \in \overline{D}$, *then there exists a unique solution of Problem G.*

All considerations can also be carried out in the case where the domain D is an arbitrary parallelepiped with sides parallel to coordinate axes. A domain of the cubic form has been chosen to simplify the formulas.

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