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PARTIAL DIFFERENTIAL EQUATIONS

Nonexistence of Global Solutions of the Cauchy Problem for Systems of Klein–Gordon Equations with Positive Initial Energy

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Abstract—We study the Cauchy problem for systems of weakly coupled Klein–Gordon equations with dissipations. We prove a theorem on the nonexistence of global solutions with positive initial energy.

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Consider the Cauchy problem for systems of weakly coupled Klein–Gordon equations with dissipations

$$u_{itt} - \Delta u_i + u_i + \gamma u_{it} = \sum_{\substack{j=1\\ j\neq i}}^{m} |u_j|^{p_j+1}|u_i|^{p_i-1}u_i, \qquad i = 1, \dots, m, \tag{1}$$

$$u_i(0, x) = u_{i0}(x), \qquad u_{it}(0, x) = u_{i1}(x), \qquad x \in R^n, \qquad i = 1, \dots, m, \tag{2}$$

where $(u_1, \dots, u_m)$ are real functions depending on $t \in R_+$ and $x \in R^n$,

$$n \geq 2, \qquad p_j \geq 0, \qquad j = 1, \dots, m, \tag{3}$$

and in addition,

$$0 < p_i + p_j \leq \frac{2}{n-2}, \quad i, j = 1, \dots, m \quad \text{if} \quad n \geq 3. \tag{4}$$

In the present paper, we study the nonexistence of global solutions with positive initial energy.

The nonexistence of global solutions was studied in [1] for nonlinear wave equations with negative energy and in [2] for a class of abstract equations that, in particular, contains nonlinear wave equations. The nonexistence of global solutions of nonlinear wave equations with positive initial energy was considered in [3]. It was shown in the study of nonlinear wave equations in [4] that there exist initial data with fixed initial energy such that the corresponding Cauchy problem does not have a global solution. This result was improved in [5]. A mixed problem for systems of two semilinear wave equations with viscosity and with memory was studied in [6], where the nonexistence of global solutions with positive initial energy was proved. The nonexistence of global solutions of problem (1), (2) with negative initial energy was studied in [7] for $m = 2$ and in [8] for $m = 2$ and $p_1 = p_2$. The nonexistence of global solutions of a generalized fourth-order Klein–Gordon equation with positive initial energy was analyzed in [9]. A fairly comprehensive picture of the studies in this direction can be gained from the monograph [10].

This problem with $m = 2$ and with distinct values of p_1 and p_2 was not considered in the abovementioned papers. For $m > 2$, each equation contains a sum of nonlinear terms of distinct growth, which take into account the interaction of various fields [11].

In what follows, we denote the norm on the space $L_2(R^n)$ by $|\cdot|$, the inner product on $L_2(R^n)$ by $\langle\cdot\,,\cdot\rangle$, and the norm on the Sobolev space $H^1 = W_2^1(R^n)$ by $\|\cdot\|$; i.e., $\|u\| = [|\nabla u|^2 + |u|^2|]^{1/2}$, where ∇ is the gradient. Let $E(t)$ be the energy function

$$E(t) = \sum_{j=1}^{m} \frac{p_j+1}{2}\left[|\dot{u}_{jt}(t,\cdot)|^2 + \|u_j(t,\cdot)\|^2 + 2\gamma\int_0^t |\dot{u}_{jt}(s,\cdot)|^2\,ds\right] - \sum_{\substack{i,j=1\\ i<j}}^{m} \int_{R_n} |u_i(t,x)|^{p_i+1}|u_j(t,x)|^{p_j+1}\,dx.$$

In addition, we introduce the notation

$$I(\phi_1,\ldots,\phi_m) = \sum_{j=1}^{m}\|\phi_j\|^2 - 2\sum_{\substack{i,j=1\\ i<j}}^{m}\int_{R_n}|\phi_i|^{p_i+1}|\phi_j|^{p_j+1}\,dx.$$

The main result of the present paper is stated in the following assertion.

Theorem 1. *Let conditions* (3) *and* (4) *be satisfied, let* $u_{i0}(\cdot) \in H^1$ *and* $u_{i1}(\cdot) \in L_2(R^n)$, $i = 1,\ldots,m$, *and in addition, let the following conditions be satisfied*:

$$E(0) > 0, \tag{5}$$

$$I(u_{10},\ldots,u_{m\,0}) < 0, \tag{6}$$

$$\sum_{j=1}^{m}\langle u_{j0}, u_{j1}\rangle \geq 0, \tag{7}$$

$$\sum_{j=1}^{m}|u_{j0}|^2 > 2E(0). \tag{8}$$

Then the solution of the Cauchy problem (1), (2) *blows up in finite time.*

Note that, using the notation $\mathcal{H} = L_2(R_n)\times\cdots\times L_2(R_n)$ and

$$w = [u_1,\ldots,u_m]^{\mathrm{T}}, \qquad A = \operatorname{diag}(-\Delta+1,\ldots,-\Delta+1),$$
$$D(A) = \mathcal{H}_2 = H^2\times\cdots\times H^2, \qquad H^2 = W_2^2(R^n),$$
$$B = \operatorname{diag}(\gamma,\ldots,\gamma), \qquad D(B) = L_2(R_n)\times\cdots\times L_2(R_n),$$
$$F(w) = \left[\sum_{\substack{j=1\\ j\neq 1}}^{m}|u_j|^{p_j+1}|u_1|^{p_1-1}u_1,\ldots,\sum_{\substack{j=1\\ j\neq m}}^{m}|u_j|^{p_j+1}|u_m|^{p_m-1}u_m\right]^{\mathrm{T}},$$

one can rewrite problem (1), (2) in the matrix form

$$\ddot{w} + B\dot{w} + Aw = F(w), \tag{9}$$

$$w(0) = w_0, \qquad \dot{w}_t(0) = w_1 \tag{10}$$

in the Hilbert space $\mathcal{H}$, where

$$\dot{w} = [\dot{u}_1,\ldots,\dot{u}_m]^{\mathrm{T}}, \qquad \ddot{w} = [\ddot{u}_1,\ldots,\ddot{u}_m]^{\mathrm{T}},$$
$$w_0 = [u_{10}(x),\ldots,u_{m0}(x)]^{\mathrm{T}}, \qquad w_1 = [u_{11}(x),\ldots,u_{m1}(x)]^{\mathrm{T}}.$$

Obviously, A is a self-adjoint positive definite operator. By using the embedding theorem and conditions (3) and (4), one can show that the nonlinear operator $F(w)$ from $\mathcal{H}_1 = D(A^{1/2}) = H^1\times\cdots\times H^1$ to $\mathcal{H}$ satisfies the local Lipschitz condition.

By using the solvability theorem for the Cauchy problem for nonlinear differential equations in a Hilbert space (see [12]), one can prove the following assertion.

Theorem 2. *Let conditions* (3) *and* (4) *be satisfied. Then there exists a* $T' > 0$ *such that, for arbitrary* $w_0 \in D(A^{1/2})$ *and* $w_1 \in \mathcal{H}$, *problem* (9), (10) *has a unique solution*

$$w(\cdot) \in C([0, T_{\max}); \mathcal{H}_1) \cap C^1([0, T_{\max}); \mathcal{H}).$$

If $T_{\max} = \sup T'$ (*i.e.*, $T_{\max}$ *is the length of the maximal existence interval of the solution* $w(\cdot) \in C([0, T_{\max}); \mathcal{H}_1) \cap C^1([0, T_{\max}); \mathcal{H})$), *then either*

(i) $T_{\max} = +\infty$, *or*

(ii) $\limsup_{t \to T_{\max} - 0}[\|w(t)\|_{\mathcal{H}_1} + \|\dot{w}(t)\|_{\mathcal{H}}] = +\infty$.

Remark 1. If $w_0 = D(A)$ and $w_1 \in D(A^{1/2})$, then

$$w(\cdot) \in C([0, T_{\max}); \mathcal{H}_2) \cap C^1([0, T_{\max}); \mathcal{H}_1) \cap ([0, T_{\max}); \mathcal{H}),$$

where $\mathcal{H}_2 = H^2 \times \cdots \times H^2$.

Proof of Theorem 1. First, we assume that $u_{i0}(\cdot) \in H^2$ and $u_{i1}(\cdot) \in H^1$, $i = 1, \ldots, m$. Let us show that $T_{\max} < +\infty$.

Suppose the contrary: $T_{\max} = +\infty$. Let $T_2 > 0$, $T_3 > 0$, and $k > 0$ be some positive numbers. Consider the functional

$$R(t) = \sum_{j=1}^{m} \frac{1}{2}\left[|u_j(t, \cdot)|^2 + \gamma \int_0^t |u_j(s, \cdot)|^2\, ds + \gamma |u_{j0}|^2 (T_1 - t)\right] + k(T_2 + t)^2. \tag{11}$$

Hence it follows that

$$\dot{R}(t) = \sum_{j=1}^{m} \frac{1}{2}[2\langle u_j(t, \cdot), \dot{u}_j(t, \cdot)\rangle + \gamma |u_j(t, \cdot)|^2 - \gamma |u_{j0}|^2] + 2k(t + T_2). \tag{12}$$

Next, by using relations (1) and (2), from (11), we obtain

$$\ddot{R}(t) = \sum_{j=1}^{m}[|\dot{u}_j(t, \cdot)|^2 - \|u_j(t, \cdot)\|^2] + 2 \sum_{\substack{i,j=1\\ i<j}}^{m} \int_{R^n} |u_i|^{p_i+1} |u_j|^{p_j+1}\, dx + 2k. \tag{13}$$

It follows from (1) and (2) that

$$\sum_{\substack{i,j=1\\ i<j}}^{m} \int_{R^n} |u_i|^{p_i+1} |u_j|^{p_j+1}\, dx = -2E(0) + \sum_{j=1}^{m} (p_j + 1)\left[|\dot{u}_j(t, \cdot)|^2 + \|u_j(t, \cdot)\|^2 + 2\gamma \int_0^t |\dot{u}_j(s, \cdot)|^2\, ds\right],$$

and by taking into account this relation in (13), we obtain

$$\ddot{R}(t) = \sum_{j=1}^{m}(p_j + 2)|\dot{u}_j(t, \cdot)|^2 + \sum_{j=1}^{m} p_j \|u_j(t, \cdot)\|^2 + 2\gamma \sum_{j=1}^{m}(p_j + 1) \int_0^t |\dot{u}_j(s, \cdot)|^2\, ds - 2E(0) + 2k. \tag{14}$$

Lemma 1. *Let the assumptions of Theorem* 1 *be satisfied. Then*

$$I(u_1(t, \cdot), \ldots, u_m(t, \cdot)) < 0, \qquad t \in [0, T_{\max}).$$

Lemma 2. *Let the assumptions of Theorem* 1 *be satisfied. Then*

$$\psi(t)=\sum_{j=1}^{m}|u_j(t,\cdot)|^2-2E(0)>0,$$

the function $\psi(t)$ *is monotone increasing*, *and*

$$\ddot{R}(t)>0,\qquad t\in[0,T_{\max}).$$

The proof of these lemmas will be given after the proof of Theorem 1.

By using the Hölder inequality, from (12), we obtain the estimate

$$\dot{R}^2(t)\le\left[\sum_{j=1}^{m}\left(|u_j(t,\cdot)|^2+\gamma\int_0^t|u_j(s,\cdot)|^2\,ds\right)+2k(t+T_2)^2\right]$$
$$\times\left[\sum_{j=1}^{2}(p_j+1)\left(|\dot{u}_j(t,\cdot)|^2+\gamma\int_0^t|\dot{u}_j(s,\cdot)|^2\,ds\right)+2k\right].\tag{15}$$

By choosing a sufficiently large T_2, from Lemma 2 and relations (11), (14), and (15), we obtain

$$R(t)\cdot\ddot{R}(t)-\mu\dot{R}^2(t)\ge R(t)\cdot\ddot{R}(t)-\mu\left[2R(t)-2\gamma(T_1-t)\sum_{j=1}^{m}(p_j+1)|u_{j0}|^2\right]$$
$$\times\left[\sum_{j=1}^{m}(p_j+1)\left(|\dot{u}_j(t,\cdot)|^2+\gamma\int_0^t|\dot{u}_j(s,\cdot)|^2\,ds\right)+2k\right]$$
$$\ge R(t)\left[\ddot{R}-2\mu\left(\sum_{j=1}^{m}|\dot{u}_j(t,\cdot)|^2\right)-\gamma\sum_{j=1}^{m}\int_0^t|\dot{u}_j(s,\cdot)|^2\,ds+2k\right]$$
$$=R(t)\left[\sum_{j=1}^{m}(p_j+2-2\mu)|\dot{u}_j(t,\cdot)|^2+\sum_{j=1}^{m}p_j\|u_j(t,\cdot)\|^2\right.$$
$$\left.+2\gamma\sum_{j=1}^{m}(p_j+1-\mu)\int_0^t|\dot{u}_j(s,\cdot)|^2\,ds-2E(0)+2k(1-2\mu)\right].\tag{16}$$

Now, by setting $\mu=\min_{j=1,\dots,m}p_j+1$, we obtain

$$R(t)\ddot{R}(t)+(1+\lambda)\dot{R}^2(t)\ge R(t)\cdot y(t),$$

where $\lambda=\min_{j=1,\dots,m}p_j$ and $y(t)=\psi(t)-k(1+2\lambda)$.

By using Lemma 2 and by choosing a sufficiently small k, we obtain the inequality $y(t)\ge 0$.

Therefore, the inequality

$$R(t)\cdot\ddot{R}(t)-(1+\lambda)\dot{R}^2(t)\ge 0\tag{17}$$

holds for sufficiently large $T_1,T_2>0$ and for sufficiently small $k>0$.

On the other hand, since

$$\dot{R}(0)=\sum_{j=1}^{m}\langle u_{j0},u_{j1}\rangle+2kT_2,$$

we have

$$\dot{R}(0)>0\tag{18}$$

for sufficiently large $T_2 > 0$. Next, by using inequalities (17) and (18), by the standard scheme (see [1, 2]) we find that there exists a T^*, $0 < T^* < +\infty$, such that $\lim_{t\to T^*-0} R(t) = +\infty$. The resulting contradiction implies that $T_{\max} < +\infty$.

Remark 2. If $u_{i0}(\cdot) \in H^1$ and $u_{i1}(\cdot) \in L_2(R^n)$, $i = 1, \dots, m$, then the justification can be carried out in a standard way, by approximation of the initial data by smoother functions.

Proof of Lemma 1. By condition (6), there exists a $T_1 > 0$ such that

$$I(u_1(t,\cdot), \dots, u_m(t,\cdot)) < 0, \qquad t \in [0, T_1). \tag{19}$$

Let us show that $T_1 = T_{\max}$. Let $T_1 < T_{\max}$; then

$$I(u_1(T_1,\cdot), \dots, u_m(T_1,\cdot)) = 0. \tag{20}$$

We introduce the functional $F(t) = \sum_{j=1}^m |u_j(t,\cdot)|^2$. By using Remark 2, we obtain

$$\begin{aligned}
\dot F(t) &= 2\sum_{j=1}^m \langle u_j(t,\cdot), \dot u_j(t,\cdot)\rangle,\\
\ddot F(t) &= 2\sum_{j=1}^m |\dot u_j(t,\cdot)|^2 + 2\sum_{j=1}^m [\langle u_j(t,\cdot), \Delta u_j(t,\cdot) - u_j(t,\cdot) - \gamma \dot u_j(t,\cdot)\rangle]\\
&\quad + 2\sum_{\substack{i,j=1\\ i<j}}^m \int_{R_n} |u_i(t,x)|^{p_i+1} |u_j(t,x)|^{p_j+1}\,dx.
\end{aligned}$$

Therefore,

$$\ddot F(t) + \gamma \dot F(t) = \varphi(t), \tag{21}$$

where

$$\varphi(t) = 2\sum_{j=1}^m |\dot u_j(t,\cdot)|^2 - I(u_1(t,\cdot), \dots, u_m(t,\cdot)).$$

By inequality (19), we have

$$\varphi(t) > 0, \qquad t \in [0, T_1). \tag{22}$$

It follows from condition (7) and relations (21) and (22) that

$$\dot F(t) > 0, \qquad t \in [0, T_1).$$

Therefore, the function $F(t)$ is monotone increasing on $[0, T_1)$; consequently,

$$F(t) > F(0) = \sum_{j=1}^m |u_{j0}|^2. \tag{23}$$

By taking into account the continuity of the function $F(t)$, from inequalities (8) and (23), we obtain

$$F(T_1) > 2E(0). \tag{24}$$

On the other hand, relation (20) and the definition of the function $E(t)$ imply the inequality

$$\sum_{j=1}^m |u_j(T_1,\cdot)|^2 \le 2E(0). \tag{25}$$

The resulting contradiction (24), (25) shows that our assumption fails. Therefore, $T_1 = T_{\max}$.

Proof of Lemma 2. By taking into account Lemma 1, from inequalities (8) and (23), we obtain

$$F(t) - 2E(0) > 0, \qquad t \in [0, T_{\max});$$

i.e., the function $\psi(t)$ is positive and monotone increasing. Then, by virtue of the representation (14), we obtain the desired inequality. The proof of the lemma is complete.

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REFERENCES

1. Levine, H.A., Some Additional Remarks on the Nonexistence of Global Solutions to Nonlinear Wave Equations, *SIAM J. Math. Anal.*, 1974, vol. 5, pp. 138–146.
2. Levine, H.A and Serrin, J., Global Nonexistence Theorems for Quasilinear Evolution Equation with Dissipation, *Arch. Ration Mech. Anal.*, 1997, vol. 137, pp. 341–361.
3. Pucci, P. and Serrin, J., Global Nonexistence for Abstract Evolution Equation with Positive Initial Energy, *J. Differential Equations*, 1998, vol. 150, pp. 203–214.
4. Levine, H.A. and Todorova, G., Blow up of Solutions of the Cauchy Problem for a Wave Equation with Nonlinear Damping and Source Terms and Positive Initial Energy, *Proc. Amer. Math. Soc.*, 2001, vol. 129, pp. 793–805.
5. Todorova, G. and Vitillaro, E., Blow-up for Nonlinear Dissipative Wave Equations, *J. Math. Anal. Appl.*, 2005, vol. 303, pp. 242–257.
6. Ma, J., Mu, C., and Zeng, R., A Blow-up Result for Viscoelastic Equations with Arbitrary Positive Initial Energy, *Boundary Value Problems*, 2011, vol. 6, pp. 1–10.
7. Aliev, A.B. and Kazimov, A.A., Global Solvability and Behavior of Solutions of the Cauchy Problem for a System of Two Semilinear Hyperbolic Equations with Dissipation, *Differ. Uravn.*, 2013, vol. 49, no. 4, pp. 476–486.
8. Wenjun Liu, Global Existence, Asymptotic Behavior and Blow-up of Solutions for Coupled Klein–Gordon Equations with Damping Terms, *Nonlinear Anal.*, 2010, vol. 73, pp. 244–255.
9. Korpusov, M.O., Non-Existence of Global Solutions to Generalized Dissipative Klein–Gordon Equations with Positive Energy, *Electron. J. Differential Equations*, 2012, vol. 2012, no. 119, pp. 1–10.
10. Mitidieri, E. and Pokhozhaev, S.I., A Priori Estimates and the Nonexistence of Solutions of Nonlinear Partial Differential Equations and Inequalities, *Tr. Mat. Inst. Steklova*, 2001, vol. 234.
11. Cocco, S., Barbi, M., and Peyard, M., Vector Nonlinear Klein–Gordon Lattices: General Derivation of Small Amplitude Envelope Solution Solutions, *Phys. Lett. A*, 1999, no. 253, pp. 161–167.
12. Aliev, A.B., Solvability "in the Large" of the Cauchy Problem for Quasilinear Equations of Hyperbolic Type, *Dokl. Akad. Nauk SSSR*, 1978, vol. 240, no. 2, pp. 249–252.