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ORDINARY DIFFERENTIAL EQUATIONS

Periodic Solutions of Functional-Differential Equations of Point Type

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Abstract— We study periodic solutions of a functional-differential equation of point type. We state conditions for the existence and uniqueness of an ω-periodic solution of the original nonlinear functional-differential equation of point type. An iterative process for constructing such a solution is described, and its convergence rate is estimated.

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1. INTRODUCTION. STATEMENT OF THE PROBLEM

Of basic methods for the study of the existence of periodic solutions of differential equations, we note the method of Poincaré–Andronov point mappings [1, p. 328; 2, p. 66], the topological method, the method of directing functions [3, p. 72; 4, p. 172], variational methods, etc. The approach suggested in the present paper is most close to the method of integral equations, which is presented in detail in the monograph [5, p. 146]; however, we substantially modify it. Such an approach was also used in the monograph [6, p. 26] and in a number of other publications for the study of periodic and bounded solutions of differential equations. These papers are characterized by the procedure of construction of a Green operator function, which is used for the construction of a periodic solution. The construction of a Green operator function and verification of conditions that should be satisfied for it are cumbersome procedures. The solution of each particular problem requires a nontrivial preliminary study. One should separately study whether the solution is classical.

The approach developed in the present paper permits one to avoid these difficulties. The conditions ensuring the existence and uniqueness of a classical ω-periodic solution are easy to verify and are stated in terms of characteristics of the right-hand side of the differential equation (the Lipschitz constant, the value of deviations in the case of a functional-differential equation of point type, and the coefficients of the linearized equation). The linearization of the right-hand side of the equation is one essential characteristic of the approach considered below. As a rule, the Taylor linearization is the most widespread method for the extraction of the linear part. There are examples that show that the Taylor linearization does not necessarily permit one to prove the existence of a periodic solution; but this is possible with other linearizations.

In the present paper, we consider the functional-differential equation of point type

$$\dot{x}(t) = g(t, x(t+\tau_1), \ldots, x(t+\tau_s)), \qquad t \in \mathbb{R}, \tag{1}$$

where the function $g(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R} \times \mathbb{R}^{n \times s}, \mathbb{R}^n)$, $k \in \{0, 1, \ldots\}$, is 2π-periodic with respect to time. A solution of Eq. (1) is defined as any absolutely continuous function $x(\cdot)$ satisfying the equation. Since the right-hand side of the equation belongs to the space $\mathbb{C}^{(k)}(\mathbb{R} \times \mathbb{R}^{n \times s}, \mathbb{R}^n)$, $k \in \{0, 1, \ldots\}$, it follows that any solution $x(\cdot)$ belongs to the space $\mathbb{C}^{(k+1)}(\mathbb{R}, \mathbb{R}^n)$. An equation of any period $\omega > 0$

can be obviously reduced to an equation of period 2π. We state conditions for the existence and uniqueness of a 2π-periodic solution $x(\cdot)$ of Eq. (1), describe an iterative algorithm for constructing such a solution, and estimate the convergence rate of the process.

Since we study 2π-periodic solutions, it follows that, without loss of generality, one can assume that all deviations $\tau_1, \dots, \tau_s$ belong to the interval $[0, 2\pi)$. Indeed, if one has deviations $\tau_j \in [2\pi k, 2\pi(k+1))$, $j \in \{1, \dots, s\}$, $k \in \mathbb{N}$, then, instead of them, one can take the deviations $\overline{\tau}_j = \tau_j - 2\pi k$. If one has deviations $\tau_j \in [-2\pi(k+1), -2\pi k)$, $k \in \mathbb{N}$, then, instead of them, one can take the deviations $\overline{\tau}_j = \tau_j + 2\pi(k+1)$. Obviously, the resulting equation has the same 2π-periodic solutions as the original one.

In addition, we assume that the deviations $\tau_1, \dots, \tau_s$ satisfy the condition of commensurability. This means that, for arbitrary τ_i and τ_j, $i, j \in \{1, \dots, s\}$, there exist numbers $n_1, n_2 \in \mathbb{N} \cup \{0\}$ satisfying the conditions $n_1 + n_2 \neq 0$ and $n_1|\tau_i| = n_2|\tau_j|$.

Equation (1) can be represented in the form

$$\dot{x}(t) = \sum_{j=1}^{s} a_j x(t + \tau_j) + f(t, x(t + \tau_1), \dots, x(t + \tau_s)), \qquad t \in \mathbb{R}, \tag{2}$$

where

$$a_j \in \mathbb{R}, \qquad \tau_j \in [0, 2\pi), \qquad j \in \{1, \dots, s\},$$

$$f(t, x(t + \tau_1), \dots, x(t + \tau_s)) = g(t, x(t + \tau_1), \dots, x(t + \tau_s)) - \sum_{j=1}^{s} a_j x(t + \tau_j),$$

and the deviations $\tau_1, \dots, \tau_s$ are commensurable. The present paper deals with the study of conditions imposed on a_j, τ_j, $j \in \{1, \dots, s\}$, and $f(\cdot)$ and ensuring the existence and uniqueness of a 2π-periodic solution.

This type of functional-differential equations was studied in the monograph [7, p. 37], where conditions were obtained ensuring the existence and uniqueness of a solution of the Cauchy problem

$$\dot{x}(t) = g(t, x(t + \tau_1), \dots, x(t + \tau_s)), \qquad t \in \mathbb{R}, \tag{3}$$

$$x(0) = \overline{x}, \qquad \overline{x} \in \mathbb{R}^n, \tag{4}$$

in a special function class. The function $g(\cdot)$ should satisfy the following conditions.

I. $g(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R} \times \mathbb{R}^{n \times s}, \mathbb{R}^n)$, $k \in \{0, 1, \dots\}$.

II. The inequalities

$$\|g(t, x_1, \dots, x_s)\|_{\mathbb{R}^n} \leq M_0(t) + M_1 \sum_{j=1}^{s} \|x_j\|_{\mathbb{R}^n}, \quad M_0(\cdot) \in \mathbb{C}^{(0)}(\mathbb{R}, \mathbb{R}),$$

$$\|g(t, x_1, \dots, x_s) - g(t, \overline{x}_1, \dots, \overline{x}_s)\|_{\mathbb{R}^n} \leq L_g \sum_{j=1}^{s} \|x_j - \overline{x}_j\|_{\mathbb{R}^n}$$

hold for all t, x_j, and $\overline{x}_j$, $j = 1, \dots, s$. Note that the second inequality is the Lipschitz condition.

III. There exists a $\mu^* \in \mathbb{R}$ such that the expression

$$\sup_{i \in \mathbb{Z}} M_0(t + i)(\mu^*)^{|i|}$$

has finite value for any $t \in \mathbb{R}$ and is a continuous function of the argument t.

IV. There exists a $\mu^* \in \mathbb{R}$ such that the family of functions

$$\tilde{g}_{i, z_1, \dots, z_s}(t) = g(t + i, z_1, \dots, z_s)(\mu^*)^{|i|}, \qquad i \in \mathbb{Z}, \qquad (z_1, \dots, z_s) \in \mathbb{R}^{n \times s},$$

is equicontinuous on any finite interval.

Obviously, by virtue of the periodicity of the right-hand side $g(\cdot)$ of Eq. (1), condition III is necessarily satisfied; therefore, throughout the following, we assume that $\mu^* = 1$.

We introduce the space

$$\mathcal{L}^n_\mu\mathbb{C}^{(k)}(\mathbb{R}) = \left\{ x(\cdot)|\ x(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R};\mathbb{R}^n),\ \max_{0\le r\le k} \sup_{t\in\mathbb{R}} \|x^{(r)}(t)e^{-\delta|t|}\|_{\mathbb{R}^n} < +\infty \right\},$$

where $k \in \{0, 1, \dots\}$ and $\mu = e^{-\delta}$.

The following assertion was proved in the monograph [7, p. 45].

Theorem 1. *If the function $g(\cdot)$ satisfies conditions* I–IV *and the inequality*

$$L_g \sum_{j=1}^{s} \mu^{-|\tau_j|} < \ln \mu^{-1} \tag{5}$$

holds for some $\mu \in (0,\mu^*) \cap (0,1)$, then for each $\overline{x} \in \mathbb{R}^n$, there exists a solution $x(\cdot) \in \mathcal{L}^n_\mu\mathbb{C}^{(k)}(\mathbb{R})$ of the Cauchy problem (3), (4). *This solution is unique and, moreover, belongs to the class $\mathcal{L}^n_\mu\mathbb{C}^{(k+1)}(\mathbb{R})$.*

If the function $g(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R} \times \mathbb{R}^{n\times s}, \mathbb{R}^n)$, $k \in \{0, 1, \dots\}$, occurring on the right-hand side in Eq. (3) is ω-periodic, one can state a corollary of this theorem.

Corollary 1. *Let the function $g(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R} \times \mathbb{R}^{n\times s}, \mathbb{R}^n)$, $k \in \{0, 1, \dots\}$, occurring in Eq.* (3) *be ω-periodic with respect to time. If it satisfies conditions* II *and* IV *and inequality* (5) *holds for some $\mu \in (0, 1)$, then for each $\overline{x} \in \mathbb{R}^n$, there exists a solution $x(\cdot) \in \mathcal{L}^n_\mu\mathbb{C}^{(k)}(\mathbb{R})$ of the Cauchy problem* (3), (4). *Such a solution is unique and, moreover, belongs to the class $\mathcal{L}^n_\mu\mathbb{C}^{(k+1)}(\mathbb{R})$.*

Consider the function space

$$V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n) = \{f(\cdot) :\ f(\cdot) \quad \text{satisfies the conditions I–III}\}.$$

For all functions in $V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n)$, the parameter $\mu^* \in \mathbb{R}_+$ coincides with the corresponding constant in condition III. In the space $V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n)$, one can introduce the Lipschitz norm

$$\begin{aligned}\|g(\cdot)\|_{L_{ip}} &= \sup_{t\in R} \|f(t, 0, \dots, 0)e^{-\delta^*|t|}\|_{R^n} \\ &+ \sup_{t,z_1,\dots,z_s,\bar z_1,\dots,\bar z_s \in \mathbb{R}\times\mathbb{R}^{ns}\times\mathbb{R}^{ns}} \|g(t, z_1, \dots, z_s) - g(t, \bar z_1, \dots, \bar z_s)\|_{R^n} \left(\sum_{j=1}^{s} \|z_j - \bar z_j\|_{R^n}\right)^{-1}, \\ \mu^* &= e^{-\delta^*}.\end{aligned}$$

Obviously, for a function $g(\cdot) \in V_{\mu^*}(\mathbb{R}\times\mathbb{R}^{ns}, \mathbb{R}^n)$, the least value of the constant L_g in the Lipschitz condition (condition II in this section) coincides with the value of the second term in the definition of the norm of $f(\cdot)$. Throughout the following, the Lipschitz constant L_g is treated in the sense of its minimum value. We consider the right-hand side of a functional-differential equation of point as an element of the Banach space $V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n)$.

To indicate the dependence of the solution of the Cauchy problem (3), (4) on the initial value $\bar x$ and on the right-hand side $g(\cdot)$ of the functional-differential equation itself, we use the notation $x(t; \bar t, \bar x, g)$. The continuous dependence of the solution $x(\cdot)$ is treated as its continuous dependence on the variables $\bar t, \bar x, g \in \mathbb{R}^1 \times \mathbb{R}^n \times V_{\mu^*}(\mathbb{R} \times \mathbb{R}^{ns}, \mathbb{R}^n)$.

The following assertion was proved in [7, p. 47].

Theorem 2 (the structural stability theorem). *Under the assumptions of Theorem* 1 *and Corollary* 1, *the solution of the main Cauchy problem* (3), (4) *depends continuously on the variables $\bar t$, $\bar x$, and g.*

Remark 1 [7, p. 47]. In Theorem 2, the solution treated as an element of the space $\mathcal{L}_\mu^n\mathbb{C}^{(0)}(\mathbb{R})$ depends continuously on $\bar{x}$ and $g(\cdot)$.

Since we consider only periodic functions in what follows, instead of the spaces $\mathcal{L}_\mu^n\mathbb{C}^{(k)}(\mathbb{R})$ we use the ordinary spaces $\mathbb{C}^{(k)}(\mathbb{R},\mathbb{R}^n)$, $k \in \{0,1,\ldots\}$ of continuous functions.

2. PROPERTIES OF PERIODIC SOLUTIONS OF THE LINEAR HOMOGENEOUS EQUATION

Let us prove some properties of linear functional-differential equations of point type which will be used in forthcoming considerations.

Consider the homogeneous functional-differential equation of point type

$$\dot{x}(t) = \sum_{j=1}^{s} a_j x(t+\tau_j), \qquad t \in \mathbb{R}, \tag{6}$$

where $a_j \in \mathbb{R}$, $\tau_j \in [0,2\pi)$, $j \in \{1,\ldots,s\}$.

Let us describe the domain of all $(a_1,\ldots,a_s,\tau_1,\ldots,\tau_s) \in \mathbb{R}\times\cdots\times\mathbb{R}\times[0,2\pi)\times\cdots\times[0,2\pi)$ for which the homogeneous equation (6) has only the zero 2π-periodic solution.

Lemma 1. *The homogeneous equation* (6) *has a unique* 2π*-periodic solution if and only if the parameter set* $(a_1,\ldots,a_s,\tau_1,\ldots,\tau_s) \in \mathbb{R}^s\times[0,2\pi)\times\cdots\times[0,2\pi)$ *simultaneously satisfies the conditions*

$$\sum_{j=1}^{s} a_j \neq 0; \qquad \left|\sum_{j=1}^{s} a_j \cos k\tau_j\right| + \left|k - \sum_{j=1}^{s} a_j \sin k\tau_j\right| \neq 0 \tag{7}$$

for all $k \in \mathbb{N}$. *Such a* 2π*-periodic solution is trivial. Otherwise*, *the homogeneous equation* (6) *has infinitely many* 2π*-periodic solutions.*

Proof. By taking into account the fact that, in particular, the solutions of the homogeneous equation (6) belong to the space $\mathbb{C}^{(1)}(\mathbb{R},\mathbb{R}^n)$, one can represent any ith coordinate, $i \in \{1,\ldots,n\}$, of an arbitrary 2π-periodic solution on the interval $[0,2\pi]$ in the form of the convergent Fourier series

$$x_i(t) = \alpha_{i,0} + \sum_{k=1}^{\infty} \alpha_{i,2k-1}\cos kt + \alpha_{i,2k}\sin kt, \qquad i \in \{1,\ldots,n\}.$$

By substituting which representation into Eq. (6) and by matching the coefficients of the corresponding basis functions, we find that the relations

$$\begin{aligned}
1:&\quad -\alpha_{i,0}\sum_{j=1}^{s} a_j = 0,\\
\cos kt:&\quad -\alpha_{i,2k-1}\sum_{j=1}^{s} a_j\cos k\tau_j + \alpha_{i,2k}\left(k - \sum_{j=1}^{s} a_j \sin k\tau_j\right) = 0,\\
\sin kt:&\quad \alpha_{i,2k-1}\left(-k + \sum_{j=1}^{s} a_j \sin k\tau_j\right) - \alpha_{i,2k}\sum_{j=1}^{s} a_j\cos k\tau_j = 0
\end{aligned}$$

should hold for arbitrary $i \in \{1,\ldots,n\}$ and $k = 1,2,\ldots$ Consequently, there exists a nonzero 2π-periodic solution of Eq. (6) if and only if either $\sum_{j=1}^{s} a_j = 0$ or the relation $\det A_k = 0$ holds for some $k \in \mathbb{N}$, where

$$A_k = \begin{pmatrix} -\sum_{j=1}^{s} a_j\cos k\tau_j & k - \sum_{j=1}^{s} a_j \sin k\tau_j \\ -k + \sum_{j=1}^{s} a_j \sin k\tau_j & -\sum_{j=1}^{s} a_j\cos k\tau_j \end{pmatrix}. \tag{8}$$

One can readily see that

$$\det A_k = \left(\sum_{j=1}^{s} a_j \cos k\tau_j\right)^2 + \left(k - \sum_{j=1}^{s} a_j \sin k\tau_j\right)^2,$$

which implies the assertion of the lemma.

If the right-hand side of Eq. (6) consists of a single term, i.e., has the form

$$\dot{x}(t) = ax(t+\tau), \qquad t \in \mathbb{R}, \tag{9}$$

then the result can be refined. In this case, we have $\det A_k = a^2 + k^2 - 2ak\sin k\tau$, $k \in \mathbb{N}$, and the represented expression can be zero if either $a = k$ and $\sin k\tau = 1$ or $a = -k$ and $\sin k\tau = -1$. By taking into account the above argument, we state one more assertion. To this end, we introduce the system of sets

$$H_k = \left\{(a,\tau) \,\middle|\, a = k,\ \tau = \frac{\pi}{2k} + 2\pi\frac{j}{k},\ j \in \{0,\dots,k-1\}\right\},$$
$$H_{-k} = \left\{(a,\tau) \,\middle|\, a = -k,\ \tau = \frac{3\pi}{2k} + 2\pi\frac{j}{k},\ j \in \{0,\dots,k-1\}\right\}, \quad k \in \mathbb{N}.$$

Lemma 2. *The homogeneous equation* (9) *has a unique* 2π*-periodic solution if and only if* $a \neq 0$ *and the set of parameters* $(a,\tau) \in \mathbb{R} \times [0, 2\pi)$ *does not belong to the countable set* $\bigcup_{k\in\mathbb{N}}(H_k \cup H_{-k})$. *Such a solution is trivial. Otherwise, the equation has infinitely many* 2π*-periodic solutions.*

3. PROPERTIES OF PERIODIC SOLUTIONS OF THE LINEAR INHOMOGENEOUS EQUATION

Let us present some general properties of periodic solutions, which are well-known for ordinary differential equations and are necessary for forthcoming considerations. Similar results for ordinary differential equations can be found in [8].

For an equation of the general form (1), we state a simple but very important assertion.

Proposition 1. *Let the assumptions of Corollary* 1 *be satisfied; then a solution* $x(\cdot)$ *of Eq.* (1) *is* 2π*-periodic if and only if it satisfies the relation* $x(0) = x(2\pi)$.

Proof. The desired assertion readily follows from the 2π-periodicity of the function $g(\cdot)$ with respect to t and the validity of the condition for the existence and uniqueness of the solution of the Cauchy problem (3), (4) (see Corollary 1). The proof of the assertion is complete.

Let us proceed to the study of the linear inhomogeneous equation

$$\dot{x}(t) = \sum_{j=1}^{s} a_j x(t+\tau_j) + \xi(t), \qquad t \in \mathbb{R}, \tag{10}$$

where $a_j \in \mathbb{R}$, $\tau_j \in [0, 2\pi)$, $j \in \{1,\dots,s\}$, and $\xi(\cdot) \in \mathbb{C}^{(1)}(\mathbb{R},\mathbb{R}^n)$ is a 2π-periodic function. Along with it, we consider the corresponding linear homogeneous equation (6).

Obviously, conditions I–IV hold for the right-hand sides of Eqs. (10) and (6). Let us introduce the constant

$$M = \max_{1\le j\le s} |a_j|.$$

Theorem 3. *Let inequality* (5), *which acquires the form*

$$M\sum_{j=1}^{s} \mu^{-|\tau_j|} < \ln \mu^{-1}, \tag{11}$$

hold for some $\mu \in (0,1)$. *Then Eq.* (10) *has a unique* 2π*-periodic solution if and only if the identical zero is the unique* 2π*-periodic solution of the homogeneous equation* (6).

Proof. Before proceeding to the proof, we introduce the fundamental solution matrix $\phi(t)$. It is the solution of the matrix equation

$$\dot{\phi}(t) = \sum_{j=1}^{s} a_j \phi(t+\tau_j), \qquad t \in \mathbb{R},$$

with the initial condition

$$\phi(0) = \mathbb{I}.$$

The existence of such a fundamental solution matrix follows from Corollary 1. By Corollary 1, any solution of the homogeneous equation (6) can be represented in the form $x(t) = \phi(t)x(0)$, and an arbitrary solution of the inhomogeneous equation (10) admits the representation

$$x(t) = \phi(t)x(0) + \psi(t),$$

where $\psi(t)$ is the particular solution of Eq. (10) with the initial condition $\psi(0) = 0$.

Sufficiency. Let the trivial solution be the unique 2π-periodic solution of the homogeneous equation (6). Then Proposition 1, together with the representation $x(2\pi) = \phi(2\pi)x(0)$ of solutions of the homogeneous equation (6), implies that $x = 0$ should be the unique solution of the equation $x = \phi(2\pi)x$. Consequently, $\det(\mathbb{I} - \phi(2\pi)) \neq 0$. On the other hand, an arbitrary solution of the inhomogeneous equation (10) satisfies the relation $x(2\pi) = \phi(2\pi)x(0) + \psi(2\pi)$. Since a periodic solution satisfies the condition $x(0) = x(2\pi)$, it follows that the equation can be reduced to the solution of the equation $(\mathbb{I} - \phi(2\pi))x = \psi(2\pi)$. Since $\det(\mathbb{I} - \phi(2\pi)) \neq 0$, we find that the 2π-periodic solution of the inhomogeneous equation (10) is unique.

Necessity. Assume that the inhomogeneous equation (10) has a unique 2π-periodic solution. We argue by contradiction. Suppose that the homogeneous equation (6), in addition to the zero solution, has at least one more 2π-periodic solution. Hence it follows that $\det(\mathbb{I} - \phi(2\pi)) = 0$. In this case, the equation $(\mathbb{I} - \phi(2\pi))x = \psi(2\pi)$ either has no solutions or has infinitely many solutions, which contradicts the uniqueness of the 2π-periodic solution of the inhomogeneous equation (10). The proof of the theorem is complete.

In the proof of the theorem, we have shown that if the homogeneous equation (6) has a nonzero periodic solution, then the corresponding inhomogeneous equation (10) either has infinitely many periodic solutions or does not have them. For illustration, consider the simplest one-dimensional ordinary differential equation $\dot{x}(t) = \xi(t)$ as an example. For this equation, we have $a_j \equiv 0$, $j \in \{1, \ldots, s\}$, and the corresponding linear equation acquires the form $\dot{x} = 0$; i.e., the linear equation has infinitely many periodic solutions $x(t) = C$, $C \in \mathbb{R}$. Then for $\xi(t)$, we take the function $\xi(t) \equiv 1$. In this case, the set of solutions has the form $x(t) = t + C$, $C \in \mathbb{R}$; i.e., this equation has no periodic solutions. On the other hand, if we set $\xi(t) = \cos t$, then the solutions of the equation acquire the form $x(t) = \sin t + C$, $C \in \mathbb{R}$; i.e., all solutions are periodic.

Now, on the basis of Theorem 3 and Lemma 1, one can state a corollary refining Theorem 3.

Corollary 2. *Let inequality* (11) *be satisfied for some* $\mu \in (0, 1)$. *The inhomogeneous equation* (10) *has a unique* 2π*-periodic solution if and only if the set of parameters*

$$(a_1, \ldots, a_s, \tau_1, \ldots, \tau_s) \in \mathbb{R}^s \times [0, 2\pi) \times \cdots \times [0, 2\pi)$$

simultaneously satisfies conditions (7) *for all* $k \in \mathbb{N}$.

We separately consider the case with a single term on the right-hand side. The inhomogeneous equation (10) is reduced to an equation of the form

$$\dot{x}(t) = ax(t+\tau) + \xi(t), \qquad t \in \mathbb{R}, \tag{12}$$

where $a \in \mathbb{R}$, $\tau \in [0, 2\pi)$, and $\xi(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R}, \mathbb{R}^n)$, $k \in \{0, 1, \ldots\}$, is a 2π-periodic function.

Corollary 3. *Let the inequality*

$$a\mu^{-|\tau|} < \ln \mu^{-1}$$

be satisfied for some $\mu \in (0,1)$. *Then the inhomogeneous equation* (12) *has a unique* 2π*-periodic solution if and only if* $a \neq 0$ *and the set of parameters* $(a,\tau) \in \mathbb{R} \times [0,2\pi)$ *does not belong to the countable set* $\bigcup_{k\in\mathbb{N}}(H_k \cup H_{-k})$.

4. OPERATOR OF PERIODIC SOLUTIONS

Let us return to the study of the linear inhomogeneous equation (10), where $a_j \in \mathbb{R}$, the deviations $\tau_j \in [0,2\pi)$, $j \in \{1,\ldots,s\}$, are commensurable, and $\xi(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R},\mathbb{R}^n)$, $k \in \{0,1,\ldots\}$, is a 2π-periodic function. Consider both the corresponding linear homogeneous equation (6) and homogeneous equations (6) for which the parameters

$$(a_1,\ldots,a_s,\tau_1,\ldots,\tau_s) \in \mathbb{R}^s \times [0,2\pi) \times \cdots \times [0,2\pi)$$

satisfy the assumptions of Corollary 2. Each homogeneous equation defines an operator $\mathbb{P}$ of periodic solutions as follows: by Corollary 2, for each 2π-periodic function $\xi(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R},\mathbb{R}^n)$, $k \in \{0,1,\ldots\}$, one should set $\mathbb{P}\xi(\cdot) = x(\cdot)$, where $x(\cdot)$ is a 2π-periodic solution of the corresponding linear inhomogeneous equation (10) [moreover, $x(\cdot) \in \mathbb{C}^{(k+1)}(\mathbb{R},\mathbb{R}^n)$]. For each $k = 0,1,\ldots,$ we introduce the spaces

$$\mathbb{C}_{2\pi}^{(k)}(\mathbb{R},\mathbb{R}^n) = \{x(\cdot) \in \mathbb{C}^{(k)}(\mathbb{R},\mathbb{R}^n)|\ x^{(j)}(t) = x^{(j)}(t+2\pi),\ j = 0,\ldots,k,\ t \in \mathbb{R}\}.$$

We have thereby defined the linear operator

$$\mathbb{P}:\ \mathbb{C}_{2\pi}^{(k)}(\mathbb{R},\mathbb{R}^n) \to \mathbb{C}_{2\pi}^{(k+1)}(\mathbb{R},\mathbb{R}^n), \qquad \mathbb{P}\xi(\cdot) = x(\cdot), \tag{13}$$

for each $k = 0,1,\ldots$ One can readily see that the action of the operator $\mathbb{P}$ is one-to-one. For the operator $\mathbb{P}$, we omit the index k since this does not lead to any misunderstanding. Moreover, the operator $\mathbb{P}$ for $k \in \{1,2,\ldots\}$ is a restriction of a similar operator for the index $k-1$.

For each $k = 0,1,\ldots,$ we introduce the spaces

$$\mathbb{C}_{2\pi}^{(k),n} = \{x(\cdot) \in \mathbb{C}^{(k)}([0,2\pi],\mathbb{R}^n)|,\ x^{(j)}(0) = x^{(j)}(2\pi),\ j = 0,\ldots,k\}.$$

We introduce the same norm on these spaces as on $\mathbb{C}^{(k)}([0,2\pi],\mathbb{R}^n)$. By virtue of Proposition 1 applied to the inhomogeneous equation (6), the operator $\mathbb{P}$ of periodic solutions is in a one-to-one correspondence with its restriction to the interval $[0,2\pi]$ and, for each $k = 0,1,\ldots,$ has the form

$$\hat{\mathbb{P}}:\ \mathbb{C}_{2\pi}^{(k),n} \to \mathbb{C}_{2\pi}^{(k+1),n}, \qquad \hat{\mathbb{P}}\hat{\xi}(\cdot) = \hat{x}(\cdot). \tag{14}$$

Let $\mathbb{J}:\ \mathbb{C}_{2\pi}^{(k+1),n} \to \mathbb{C}_{2\pi}^{(k),n}$, $k = 0,1,\ldots,$ be the natural embedding operator. In what follows, the operator of periodic solutions is treated as the linear operator

$$\mathbb{J}\,\hat{\mathbb{P}}:\ \mathbb{C}_{2\pi}^{(k),n} \to \mathbb{C}_{2\pi}^{(k),n}, \qquad k = 0,1,\ldots$$

Obviously, the action of the operator $\mathbb{J}\,\hat{\mathbb{P}}$ is one-to-one.

Theorem 2 and Corollary 1 implies the following assertion.

Proposition 2. *Let inequality* (11) *hold for some* $\mu \in (0,1)$, *and let conditions* (7) *be simultaneously satisfied for all* $k \in \mathbb{N}$. *Then the operator*

$$\mathbb{J}\,\hat{\mathbb{P}}:\ \mathbb{C}_{2\pi}^{(0),n} \to \mathbb{C}_{2\pi}^{(0),n}, \qquad \hat{\mathbb{P}}\hat{\xi}(\cdot) = \hat{x}(\cdot), \tag{15}$$

is continuous.

The continuity of the operator $\mathbb{J}\,\hat{\mathbb{P}}$ is insufficient. In addition, we need a sharp estimate for the norm of such an operator. However, the derivation of such an estimate is quite cumbersome. In fact, it suffices to have estimates obtained for the restriction of the considered operator to the subspace of 2π-periodic functions of the class $\mathbb{C}^{(1)}$. To this end, we introduce the quantities

$$\mathbb{A} = \left|\sum_{j=1}^{s} a_j\right|^{-1}, \qquad \mathbb{D} = \left(\sum_{k=1}^{\infty} \frac{1}{\det A_k}\right)^{1/2}, \tag{16}$$

$$\det A_k = \left(\sum_{j=1}^{s} a_j \cos k\tau_j\right)^2 + \left(k - \sum_{j=1}^{s} a_j \sin k\tau_j\right)^2, \qquad k = 1, 2, \ldots \tag{17}$$

Proposition 3. *Let the assumptions of Proposition* 2 *be satisfied. Then*

$$\sup_{\hat{\xi}(\cdot)\in\mathbb{C}_{2\pi}^{(1),n},\ \|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}}=1} \|\mathbb{J}\,\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}} \le \sqrt{\mathbb{A}^2 + 2\mathbb{D}^2}.$$

Proof. We split the proof into four stages and carry it out in the one-dimensional case. The desired assertion in the n-dimensional case follows from the fact that system (10) consists of n independent one-dimensional equations. A detailed proof will be given at the end of Section 4.

1. Construction of the operator $\mathbb{J}\,\hat{\mathbb{P}}$ in closed form ($n = 1$, the one-dimensional case). By using Fourier series, we construct the operator $(\mathbb{J}\,\hat{\mathbb{P}})^{-1}$. We expand the periodic solution $\hat{x}(\cdot)$ of Eq. (10) and the function $\hat{\xi}(\cdot)$ on the right-hand side in this equation in Fourier series,

$$\hat{x}(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_{2k-1} \cos kt + \alpha_{2k} \sin kt,$$
$$\hat{\xi}(t) = \beta_0 + \sum_{k=1}^{\infty} \beta_{2k-1} \cos kt + \beta_{2k} \sin kt.$$

Then

$$\hat{\xi}(t) = ((\mathbb{J}\,\hat{\mathbb{P}})^{-1}\hat{x}(\cdot))(t) = \dot{\hat{x}}(t) - \sum_{j=1}^{s} a_j \hat{x}((t+\tau_j)(\operatorname{mod} 2\pi));$$

or, by replacing the corresponding functions by their Fourier series expansions and by matching the coefficients of like basis functions, we obtain

$$\beta_0 = -\alpha_0 \sum_{j=1}^{s} a_j,$$
$$\beta_{2k-1} = -\alpha_{2k-1} \sum_{j=1}^{s} a_j \cos k\tau_j + \alpha_{2k}\left(k - \sum_{j=1}^{s} a_j \sin k\tau_j\right),$$
$$\beta_{2k} = \alpha_{2k-1}\left(-k + \sum_{j=1}^{s} a_j \sin k\tau_j\right) - \alpha_{2k} \sum_{j=1}^{s} a_j \cos k\tau_j, \quad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, the coefficients β_{2k-1} and β_{2k} are found from the matrix equation

$$\begin{pmatrix} \beta_{2k-1} \\ \beta_{2k} \end{pmatrix} = A_k \begin{pmatrix} \alpha_{2k-1} \\ \alpha_{2k} \end{pmatrix},$$

where, just as above, A_k is a matrix of the form (8). Consequently, to define the operator $\mathbb{J}\hat{\mathbb{P}}$ in closed form, one should invert the matrix A_k. It follows from the assumptions of the proposition that these matrices are nonsingular; therefore,

$$\alpha_0 = -\frac{\beta_0}{a},$$
$$\begin{pmatrix}\alpha_{2k-1}\\ \alpha_{2k}\end{pmatrix} = A_k^{-1}\begin{pmatrix}\beta_{2k-1}\\ \beta_{2k}\end{pmatrix}, \qquad k\in\mathbb{N},$$

where

$$A_k^{-1} = \frac{1}{\det A_k}\begin{pmatrix} -\sum_{j=1}^s a_j\cos k\tau_j & -k+\sum_{j=1}^s a_j\sin k\tau_j \\ k-\sum_{j=1}^s a_j\sin k\tau_j & -\sum_{j=1}^s a_j\cos k\tau_j\end{pmatrix}.$$

Consequently, the operator $\mathbb{J}\hat{\mathbb{P}}$ acquires the form

$$(\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot))(t) = -\beta_0\left(\sum_{j=1}^s a_j\right)^{-1} + \sum_{k=1}^{\infty}\frac{1}{\det A_k}\Bigg\{\Bigg(-\beta_{2k-1}\sum_{j=1}^s a_j\cos k\tau_j + \beta_{2k}\left(-k+\sum_{j=1}^s a_j\sin k\tau_j\right)\Bigg)\cos kt + \left(\beta_{2k-1}\left(k-\sum_{j=1}^s a_j\sin k\tau_j\right) - \beta_{2k}\sum_{j=1}^s a_j\cos k\tau_j\right)\sin kt\Bigg\}.$$

2. A majorant for the norm $\|\mathbb{J}\hat{\mathbb{P}}\hat{\boldsymbol{\xi}}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}$, $\hat{\boldsymbol{\xi}}(\cdot)\in\mathbb{C}_{2\pi}^{(1),1}$, $\|\hat{\boldsymbol{\xi}}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}\le 1$. To derive the estimates stated in Assertion 3, one should solve the extremal problem

$$\|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}} \to \sup_{\hat{\xi}(\cdot)} \tag{18}$$

under the condition

$$\hat{\xi}(\cdot)\in\mathbb{C}_{2\pi}^{(1),1}, \qquad \|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}\le 1. \tag{19}$$

To this end, we use the closed-form expression for the action of the operator $\mathbb{J}\hat{\mathbb{P}}$ on the functions $\hat{\xi}(\cdot)\in\mathbb{C}_{2\pi}^{(1),1}$, $\|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}} = 1$, which has been obtained in part 1 of the proof. First, we prove the estimate

$$\frac{1}{\det A_k}\Bigg\{\Bigg(-\beta_{2k-1}\sum_{j=1}^s a_j\cos k\tau_j + \beta_{2k}\left(-k+\sum_{j=1}^s a_j\sin k\tau_j\right)\Bigg)\cos kt + \left(\beta_{2k-1}\left(k-\sum_{j=1}^s a_j\sin k\tau_j\right) - \beta_{2k}\sum_{j=1}^s a_j\cos k\tau_j\right)\sin kt\Bigg\} \le \frac{\sqrt{\beta_{2k-1}^2+\beta_{2k}^2}}{\sqrt{\det A_k}}, \qquad k\in\mathbb{N}.$$

Indeed, let us introduce the notation

$$\Gamma_k = \left(-\beta_{2k-1}\sum_{j=1}^s a_j\cos k\tau_j + \beta_{2k}\left(-k+\sum_{j=1}^s a_j\sin k\tau_j\right)\right),$$
$$\Delta_k = \left(\beta_{2k-1}\left(k-\sum_{j=1}^s a_j\sin k\tau_j\right) - \beta_{2k}\sum_{j=1}^s a_j\cos k\tau_j\right), \qquad k\in\mathbb{N}.$$

One can readily see that

$$\Gamma_k^2 + \Delta_k^2 = \det A_k(\beta_{2k-1}^2 + \beta_{2k}^2), \qquad k \in \mathbb{N}.$$

In the new notation, the left-hand side of the considered inequality acquires the form

$$\frac{\Gamma_k \cos kt + \Delta_k \sin kt}{\det A_k}.$$

By transforming the resulting expression, we obtain

$$\begin{aligned}\frac{1}{\det A_k}(\Gamma_k \cos kt + \Delta_k \sin kt) &= \frac{\sqrt{\Gamma_k^2 + \Delta_k^2}}{\det A_k}(\cos\theta_k \cos kt + \sin\theta_k \sin kt)\\ &= \frac{\sqrt{\Gamma_k^2 + \Delta_k^2}}{\det A_k}\cos(\theta_k - kt) = \frac{\sqrt{\beta_{2k-1}^2 + \beta_{2k}^2}}{\sqrt{\det A_k}}\cos(\theta_k - kt),\end{aligned}$$

where

$$\cos\theta_k = \frac{\Gamma_k}{\sqrt{\Gamma_k^2 + \Delta_k^2}}, \qquad \sin\theta_k = \frac{\Delta_k}{\sqrt{\Gamma_k^2 + \Delta_k^2}}.$$

Obviously, this expression attains its maximum for t such that $\cos(\theta_k - kt) = 1$. This completes the proof of the estimate.

Thus, for any function $\hat{\xi}(\cdot) \in \mathbb{C}_{2\pi}^{(1),1}$, $\|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}} = 1$, the norm $\|\mathbb{J}\,\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}$ is majorized as follows:

$$\|\mathbb{J}\,\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}} \le \left|\beta_0\left(\sum_{j=1}^{s} a_j\right)^{-1}\right| + \sum_{k=1}^{\infty}\frac{\sqrt{\beta_{2k-1}^2 + \beta_{2k}^2}}{\sqrt{\det A_k}}. \tag{20}$$

In what follows, we show that the series on the right-hand side in inequality (20) is convergent.

3. An auxiliary extremal problem for estimating the norm $\|\mathbb{J}\,\hat{\mathbb{P}}\hat{\boldsymbol{\xi}}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}$. It is very difficult to estimate this norm. Therefore, we replace this problem by a simpler one. To this end, we replace the norm $\|\mathbb{J}\,\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}$ by a new function in the form of the right-hand side of inequality (20) and maximize the new function on a wider set of functions satisfying the condition $\|\hat{\xi}(\cdot)\|_{L_2([0,2\pi],\mathbb{R})} \le \sqrt{2\pi}$. This extremal problem is posed as follows:

$$\left|\beta_0\left(\sum_{j=1}^{s} a_j\right)^{-1}\right| + \sum_{k=1}^{\infty}\frac{\sqrt{\beta_{2k-1}^2 + \beta_{2k}^2}}{\sqrt{\det A_k}} \to \sup_{\beta_k,\, k\in\mathbb{N}\cup\{0\}} \tag{21}$$

under the condition

$$\|\hat{\xi}(\cdot)\|_{L_2([0,2\pi],\mathbb{R})} \le \sqrt{2\pi}. \tag{22}$$

Obviously, in this case, the value of the functional on the solution of problem (21), (22) exceeds the value of the functional on the solution of problem (18), (19). By the Parseval relation, for the orthogonal basis $\{1, \cos kt, \sin kt\}_{k\in\mathbb{N}}$ of the space $L_2([0, 2\pi], \mathbb{R})$, we have

$$\|\xi(\cdot)\|_{L_2[0,2\pi]}^2 = \int_0^{2\pi} \hat{\xi}^2(t)dt = 2\pi\beta_0^2 + \pi\sum_{k=1}^{\infty}\beta_k^2.$$

In this case, the extremal problem (21), (22) can be considered under the new condition

$$\beta_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\beta_k^2 \le 1. \tag{23}$$

4. Completion of the proof of the proposition. Let us introduce the space l_2 of numerical sequences. Consider the elements r_1 and r_2 of this space defined by the formulas

$$r_1 = \left\{ \frac{1}{\sqrt{2}} \left| \sum_{j=1}^{s} a_j \right|^{-1}, \frac{1}{\sqrt{\det A_1}}, \frac{1}{\sqrt{\det A_2}}, \ \dots \right\},$$
$$r_2 = \left\{ \sqrt{2}|\beta_0|, \sqrt{\beta_1^2 + \beta_2^2}, \sqrt{\beta_3^2 + \beta_4^2}, \ \dots \right\}.$$

One can readily see that r_1 belongs to the space l_2, because the relation $\dfrac{1}{\det A_k} = O\left(\dfrac{1}{k^2}\right)$ holds for sufficiently large k and $\|r_1\|_{l_2} < +\infty$. Then the optimization problem (21), (23) acquires the form

$$(r_1, r_2)_{l_2} \to \sup_{r_2 \in l_2}$$

provided that

$$\|r_2\|_{l_2}^2 \le 2.$$

By using the Cauchy–Schwarz inequality, we obtain the upper bound

$$(r_1, r_2)_{l_2} \le \|r_1\|_{l_2} \|r_2\|_{l_2} \le \sqrt{2}\|r_1\|_{l_2}.$$

Let us compute the norm of r_1,

$$\|r_1\|_{l_2}^2 = \frac{1}{2} \left(\sum_{j=1}^{s} a_j \right)^{-2} + \sum_{k=1}^{\infty} \frac{1}{\det A_k}.$$

It is well known that the equality in the Cauchy–Schwarz inequality is achieved for collinear vectors. Consequently, if we choose $\bar{\beta}_k$, $k \in \mathbb{N} \cup \{0\}$, for which the vector r_2 is collinear to the vector r_1 and the inequality $\|r_2\|_{l_2} \le \sqrt{2}$ holds, then this will solve the original maximization problem. Obviously, there exists such a $\bar{\beta}_k$, $k \in \mathbb{N} \cup \{0\}$. In this case, at the point of maximum, the objective functional (21) is equal to

$$\left| \bar{\beta}_0 \left(\sum_{j=1}^{s} a_j \right)^{-1} \right| + \sum_{k=1}^{\infty} \frac{\sqrt{\bar{\beta}_{2k-1}^2 + \bar{\beta}_{2k}^2}}{\sqrt{\det A_k}} = \left(\left(\sum_{j=1}^{s} a_j \right)^{-2} + 2 \sum_{k=1}^{\infty} \frac{1}{\det A_k} \right)^{1/2} = \sqrt{\mathbb{A}^2 + 2\mathbb{D}^2},$$

which completes the proof of the proposition in the one-dimensional case.

Let us present the proof of the proposition in the multidimensional case. Let

$$\hat{\xi}(\cdot) = (\hat{\xi}_1(\cdot), \dots, \hat{\xi}_n(\cdot))' \in \mathbb{C}_{2\pi}^{(1),n}.$$

The operator $\mathbb{J}\hat{\mathbb{P}}$ satisfies the estimate

$$\begin{aligned}
&\sup_{\hat{\xi}(\cdot) \in \mathbb{C}_{2\pi}^{(1),n},\, \|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}} \le 1} \|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}}^2 \\
&= \sup_{\hat{\xi}(\cdot) \in \mathbb{C}_{2\pi}^{(1),n},\, \|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}} \le 1} (\|\mathbb{J}\hat{\mathbb{P}}_1\hat{\xi}_1(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}^2 + \dots + \|\mathbb{J}\hat{\mathbb{P}}_1\hat{\xi}_n(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}^2) \\
&\le \sup_{\hat{\xi}(\cdot) \in \mathbb{C}_{2\pi}^{(1),n},\, \|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}} \le 1} ((\mathbb{A}^2 + 2\mathbb{D}^2)\|\hat{\xi}_1(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}^2 + \dots + (\mathbb{A}^2 + 2\mathbb{D}^2)\|\hat{\xi}_n(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),1}}^2) = \mathbb{A}^2 + 2\mathbb{D}^2,
\end{aligned}$$

where $\mathbb{J}\hat{\mathbb{P}}_1$ stands for the operator of periodic solutions in the one-dimensional case. Consequently, for the multidimensional case, we obtain the estimate

$$\sup_{\hat{\xi}(\cdot) \in \mathbb{C}_{2\pi}^{(1),n},\, \|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}} \le 1} \|\mathbb{J}\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}}^2 \le \mathbb{A}^2 + 2\mathbb{D}^2.$$

The proof of the proposition is complete.

In the case of the simplest linearization with a unique term in the linear part, one can obtain a refined result. Consider the homogeneous linear equation (12) and the corresponding inhomogeneous linear equation (9). In this case, we have

$$\mathbb{A} = \frac{1}{|a|}, \qquad \mathbb{D} = \left(\sum_{k=1}^{\infty} \frac{1}{\det A_k} \right)^{1/2}, \qquad \det A_k = (a^2 + k^2 - 2ak \sin k\tau).$$

Corollary 4. *Let the inequality*

$$a\mu^{-|\tau|} < \ln \mu^{-1}$$

be satisfied for some $\mu \in (0,1)$, *and let*

$$a \neq 0, \qquad (a,\tau) \notin \bigcup_{k\in\mathbb{N}} (H_k \cup H_{-k}).$$

Then

$$\sup_{\hat{\xi}(\cdot)\in\mathbb{C}_{2\pi}^{(1),n},\, \|\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}}=1} \|\mathbb{J}\,\hat{\mathbb{P}}\hat{\xi}(\cdot)\|_{\mathbb{C}_{2\pi}^{(0),n}} \leq \left(\frac{1}{a^2} + 2\mathbb{D}^2 \right)^{1/2}.$$

Remark 2. If $\tau = 0$, then the linear inhomogeneous functional-differential equation (9) becomes an ordinary differential equation. In this case, instead of Proposition 3, one can use a result in [8], where it was shown that the operator $\mathbb{J}\,\hat{\mathbb{P}} :\ \mathbb{C}_{2\pi}^{(0),n} \to \mathbb{C}_{2\pi}^{(0),n}$ satisfies the relation $\|\mathbb{J}\,\hat{\mathbb{P}}\| = 1/|a|$.

5. EXISTENCE AND UNIQUENESS OF A 2π-PERIODIC SOLUTION OF NONLINEAR EQUATION. CASE OF SIMPLEST LINEARIZATION

We obtain conditions providing the existence and uniqueness of periodic solutions of the nonlinear functional-differential equation of point type (1), where $g(\cdot) \in \mathbb{C}^{(1)}(\mathbb{R}\times\mathbb{R}^{n\times s}, \mathbb{R}^n)$ is a 2π-periodic function. Along with Eq. (1), consider Eq. (2) obtained by the linearization of Eq. (1). If the function $g(\cdot)$ occurring in Eq. (1) satisfies the Lipschitz condition with constant L_g, i.e., if

$$\|g(t, x_1, \ldots, x_s) - g(t, \overline{x}_1, \ldots, \overline{x}_s)\|_{\mathbb{R}^n} \leq L_g \sum_{j=1}^{s} \|x_j - \overline{x}_j\|_{\mathbb{R}^n}, \tag{24}$$

then the function f in Eq. (2), that is,

$$f(\cdot) = g(t, x(t+\tau_1), \ldots, x(t+\tau_s)) - \sum_{j=1}^{s} a_j x(t+\tau_j)$$

satisfies the Lipschitz condition with some constant L_f as well. Each linearization of Eq. (1) is related to a linear inhomogeneous system of the form (8). In turn, if the set $(a_1, \ldots, a_s, \tau_1, \ldots, \tau_s) \in \mathbb{R}^s \times [0, 2\pi) \times \cdots \times [0, 2\pi)$ satisfies the assumptions of Corollary 2, then the operator $\mathbb{J}\,\hat{\mathbb{P}}$ is well defined. Let us introduce the operator

$$\begin{aligned} &\mathbb{F} :\ \mathbb{C}_{2\pi}^{(k)}(\mathbb{R}, \mathbb{R}^n) \to \mathbb{C}_{2\pi}^{(k)}(\mathbb{R}, \mathbb{R}^n), \qquad k = 0, 1, \\ &\mathbb{F}[x(\cdot)](t) = f(t, x(t+\tau_1), \ldots, x(t+\tau_s)), \qquad t \in \mathbb{R}. \end{aligned}$$

The restriction of this operator to functions defined on the interval $[0, 2\pi]$ is denoted by $\hat{\mathbb{F}}$,

$$\begin{aligned} &\hat{\mathbb{F}} :\ \mathbb{C}_{2\pi}^{(k),n} \to \mathbb{C}_{2\pi}^{(k),n}, \qquad k = 0, 1, \\ &\hat{\mathbb{F}}[\hat{x}(\cdot)](t) = f(t, \hat{x}((t+\tau_1)(\operatorname{mod} 2\pi)), \ldots, \hat{x}(\,(t+\tau_s)(\operatorname{mod} 2\pi))), \qquad t \in [0, 2\pi]. \end{aligned}$$

Theorem 4. *Let the function* $g(\cdot) \in \mathbb{C}^{(1)}(\mathbb{R} \times \mathbb{R}^{n\times s}, \mathbb{R}^n)$ *occurring in the nonlinear equation* (1) *be* 2π*-periodic and satisfy the Lipschitz condition* (24); *let* L_f *be the Lipschitz constant for the function* $f(\cdot)$; *let the inequality*

$$M \sum_{j=1}^{s} \mu^{-|\tau_j|} < \ln \mu^{-1}, \qquad M = \max_{1\le j\le s} |a_j|,$$

hold for some $\mu \in (0, 1)$, *and let condition* (7) *be simultaneously valid for all* $k \in \mathbb{N}$. *If the inequality*

$$sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2} < 1 \tag{25}$$

holds, then Eq. (1) *has a* 2π*-periodic solution. This solution is unique and belongs to the space* $\mathbb{C}^{(2)}(\mathbb{R}, \mathbb{R}^n)$.

Moreover, for any initial function $\hat{x}^0(\cdot) \in \mathbb{C}^{(2),n}_{2\pi}$, *the sequence*

$$\hat{x}^k(\cdot) = (\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}})^k[\hat{x}^0(\cdot)]$$

tends to a unique function $\hat{x}(\cdot) \in \mathbb{C}^{(2),n}_{2\pi}$, *and the convergence rate can be estimated as*

$$\|(\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}})^k[\hat{x}^0(\cdot)](\cdot) - \hat{x}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}} \le (sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2})^k \|\hat{x}^0(\cdot) - \hat{x}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}}.$$

The periodic solution $x(\cdot) \in \mathbb{C}^{(2)}(\mathbb{R}, \mathbb{R}^n)$ *is induced by the function* $\hat{x}(\cdot)$ *by its* 2π*-periodic extension to the entire numerical line* $\mathbb{R}$.

Proof. In the space $\mathbb{C}^{(0),n}_{2\pi}$, we define the operator equation

$$(\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{x}(\cdot)])(\cdot) = \hat{x}(\cdot). \tag{26}$$

By Corollary 2, the 2π-periodic extension of any solution of Eq. (26) to the entire numerical line gives a periodic solution of Eq. (2) [respectively, (1)], and vice versa. Since $g(\cdot) \in \mathbb{C}^{(1)}(\mathbb{R}\times\mathbb{R}^{n\times s}, \mathbb{R}^n)$, it follows that each solution of Eq. (26) belongs to the space $\mathbb{C}^{(2),n}_{2\pi}$.

The Lipschitz condition for the function $f(\cdot)$ implies the inequality

$$\|\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{\mathbb{C}^{(0),n}_{2\pi}} \le sL_f\|\hat{y}(\cdot) - \hat{z}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}}.$$

By Proposition 3, the following chain of inequalities holds for arbitrary $\hat{y}(\cdot), \hat{z}(\cdot) \in \mathbb{C}^{(1),n}_{2\pi}$:

$$\begin{aligned}\|\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{y}(\cdot)] - \mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{\mathbb{C}^{(0),n}_{2\pi}} &= \|\mathbb{J}\,\hat{\mathbb{P}}(\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)])\|_{\mathbb{C}^{(0),n}_{2\pi}} \\ &= \left\|\mathbb{J}\,\hat{\mathbb{P}}\left(\frac{\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]}{\|\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]\|}\right)\right\|_{\mathbb{C}^{(0),n}_{2\pi}} \\ &\le \sqrt{\mathbb{A}^2 + 2\mathbb{D}^2}\,\|\hat{\mathbb{F}}[\hat{y}(\cdot)] - \hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{\mathbb{C}^{(0),n}_{2\pi}} \\ &\le sL_f\sqrt{\mathbb{A}^2 + 2\mathbb{D}^2}\,\|\hat{y}(\cdot) - \hat{z}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}}.\end{aligned} \tag{27}$$

From inequality (27), one can readily obtain the Cauchy property of the sequence

$$\hat{x}^k(\cdot) = (\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}})^k[\hat{x}^0(\cdot)](\cdot), \qquad k \in \mathbb{N},$$

for any function $\hat{x}^0(\cdot) \in \mathbb{C}^{(1),n}_{2\pi}$. By Proposition 2, the operator $\mathbb{J}\,\hat{\mathbb{P}}$ is continuous; hence so is $\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}}$. Therefore, each Cauchy sequence $(\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}})^k[\hat{x}^0(\cdot)](\cdot)$ converges to a fixed point of Eq. (26). It remains to show that the fixed point of Eq. (26) is unique. As was mentioned above, each fixed point of

Eq. (26) belongs to the space $\mathbb{C}^{(2),n}_{2\pi}$. If $\hat{y}(\cdot)$ and $\hat{z}(\cdot)$ are distinct fixed points of Eq. (26), then, by virtue of inequality (27), the relation

$$\|\hat{y}(\cdot)-\hat{z}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}} = \|\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{y}(\cdot)] - \mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}}[\hat{z}(\cdot)]\|_{\mathbb{C}^{(0),n}_{2\pi}} < \|\hat{y}(\cdot)-\hat{z}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}}$$

should hold, which is impossible. The proof of the theorem is complete.

In Theorem 4, for the choice of the linearization in the linear part, we take only deviations occurring on the right-hand side in the original functional-differential equation (1), and this is important. If, for the choice of the linearization, it turns out that the function $f(\cdot)$ contains at least one deviation $\bar{\tau}$ that does not coincide with any deviation $\tau_1,\dots,\tau_s$, then, as a rule, inequality (25) fails.

We separately consider the case of the simplest linearization with a single term in the linear part. In this case, Eq. (2) acquires the form

$$\dot{x}(t) = a_j x(t+\tau_j) + f(t, x(t+\tau_1),\dots,x(t+\tau_s)), \qquad t\in\mathbb{R}, \tag{28}$$

where $(a_j,\tau_j)\in\mathbb{R}\backslash\{0\}\times[0,2\pi)$, $j\in\{1,\dots,s\}$.

Corollary 5. *Let $g(\cdot)\in\mathbb{C}^{(1)}(\mathbb{R}\times\mathbb{R}^{n\times s},\mathbb{R}^n)$ occurring in the nonlinear equation* (1) *be a 2π-periodic function and satisfy the Lipschitz condition* (24); *let L_f be the Lipschitz constant of the function $f(\cdot)$; let the inequality*

$$|a_j|\mu^{-|\tau_j|} < \ln\mu^{-1}$$

hold for some $\mu\in(0,1)$, and let the condition

$$(a_j,\tau_j)\notin\bigcup_{k\in\mathbb{N}}(H_k\cup H_{-k})$$

be satisfied. If the inequality

$$sL_f\left(\frac{1}{a_j^2}+2\mathbb{D}^2\right)^{1/2} < 1$$

holds, then Eq. (1) *has a 2π-periodic solution. Such a solution is unique and belongs to the space $\mathbb{C}^{(2)}(\mathbb{R},\mathbb{R}^n)$.*

Moreover, for any initial function $\hat{x}^0(\cdot)\in\mathbb{C}^{(2),n}_{2\pi}$, the sequence

$$\hat{x}^k(\cdot) = (\mathbb{J}\hat{\mathbb{P}}\hat{\mathbb{F}})^k[\hat{x}^0(\cdot)]$$

converges to the unique function $\hat{x}(\cdot)\in\mathbb{C}^{(2),n}_{2\pi}$, and the convergence rate can be estimated as

$$\|(\mathbb{J}\,\hat{\mathbb{P}}\hat{\mathbb{F}})^k[\hat{x}^0(\cdot)](\cdot)-\hat{x}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}} \le \left(sL_f\left(\frac{1}{a_j^2}+2\mathbb{D}^2\right)^{1/2}\right)^k \|\hat{x}^0(\cdot)-\hat{x}(\cdot)\|_{\mathbb{C}^{(0),n}_{2\pi}}.$$

The periodic solution $x(\cdot)\in\mathbb{C}^{(2)}(\mathbb{R},\mathbb{R}^n)$ is induced by the function $\hat{x}(\cdot)$ as its 2π-periodic extension to the entire numerical line $\mathbb{R}$.

Remark 3. If, among the deviations $\tau_1,\dots,\tau_s$, there is a zero deviation $\tau_j=0$, $j\in\{1,\dots,s\}$, then, for the linear part of Eq. (28), one can take $a_jx(t)$. Then, by Remark 2, it suffices to satisfy the condition $sL_f/|a_j|<1$ for the existence of a unique 2π-periodic solution.

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