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PARTIAL DIFFERENTIAL EQUATIONS

On the Solvability of Boundary Value Problems for the Stationary Schrödinger Equation in Unbounded Domains on Riemannian Manifolds

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Abstract—We study the solutions of the stationary Schrödinger equation in unbounded domains on Riemannian manifolds with noncompact boundary. Our approach to the statement of boundary value problems is based on the notion of a class of equivalent functions. We obtain sufficient conditions for the solvability of boundary value problems and prove the solvability of the Dirichlet problem on the cone of a model manifold with continuous boundary data on the boundary.

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1. INTRODUCTION

The study of elliptic equations on Riemannian manifolds is a topical trend in modern mathematics and belong in the rapidly developing field of geometric nonlinear analysis and qualitative theory of differential equations on noncompact Riemannian manifolds.

One main problem in this direction is to study the solvability of various classes of boundary value problems for elliptic differential equations, including problems in unbounded domains of Euclidean space and on noncompact Riemannian manifolds. In particular, the solvability of the Dirichlet problem on the reconstruction of a harmonic function from continuous boundary data on noncompact Riemannian manifolds that admit a natural compactification was studied. For example, the solvability of the Dirichlet problem on connected Riemannian manifolds M with negative sectional curvature bounded away from zero and infinity was studied in [1, 2]. The compactification of M by adding the sphere $S(\infty)$ at infinity was used there to prove the unique solvability of the Dirichlet problem on $\overline{M} = M \cup S(\infty)$ on the reconstruction of a harmonic function from continuous boundary data on $S(\infty)$. Another class of noncompact Riemannian manifolds that admit a natural compactification is the class of spherically symmetric or, more generally, model and quasimodel manifolds. Solvability conditions for various boundary value problems for elliptic linear equations on these manifolds were obtained in [3–8].

On the other hand, the statement of the Dirichlet problem on an arbitrary noncompact Riemannian manifold is very complicated. The use of classes of equivalent functions, originally suggested in [9], is one possible approach to coping with this difficulty. This approach permits one to pose boundary value problems on manifolds without a natural geometric compactification. We obtain sufficient conditions for the solvability of some boundary value problems on such manifolds. The above-mentioned approach was developed in [10–13] and other papers. However, the results obtained there dealt with solutions of linear and quasilinear equations on manifolds with empty boundary or solutions of such equations on manifolds with compact boundary. But the solvability of boundary value problems on manifolds with noncompact boundary and in unbounded domains on Riemannian manifolds remained unstudied.

In the present paper, we study the solutions of the stationary Schrödinger equation

$$Lu \equiv \Delta u - c(x)u = 0 \tag{1}$$

in unbounded domains on Riemannian manifolds with noncompact boundary. Here $c(x)$ is a smooth nonnegative function; moreover, $c(x) \not\equiv 0$. Throughout the following, the solutions of Eq. (1) are referred to as L-harmonic functions. The aim of the present paper is to obtain solvability conditions for boundary value problems for L-harmonic functions in the considered domains.

Let us proceed to precise statements. Let M be a connected noncompact smooth Riemannian manifold without boundary, and let Ω be a connected bounded domain in M with smooth boundary $\partial\Omega$. Let $\{B_k\}_{k=1}^{\infty}$ be a smooth exhaustion of M, that is, a sequence of precompact open subsets of M with smooth boundaries ∂B_k such that $M = \bigcup_{k=1}^{\infty} B_k$ and $\overline{B}_k \subset B_{k+1}$ for all k. Throughout the following, we assume that the exhaustion satisfies the following conditions: all sets $B_k \cap \Omega$ are nonempty and simply connected, and ∂B_k is transversal to $\partial\Omega$ for each k. In the present paper, we consider L-harmonic functions $u(x)$ (on M or on Ω).

Let f_1 and f_2 be continuous functions on M (or on Ω, or on $\partial\Omega$). We say that f_1 and f_2 *are equivalent* on M (or on Ω, or on $\partial\Omega$) and write $f_1 \overset{M}{\sim} f_2$ (respectively, $f_1 \overset{\Omega}{\sim} f_2$ or $f_1 \overset{\partial\Omega}{\sim} f_2$) if the relation

$$\lim_{k\to\infty} \sup_{M\setminus B_k} |f_1 - f_2| = 0$$

(respectively, $\lim_{k\to\infty} \sup_{\Omega\setminus B_k} |f_1 - f_2| = 0$ or $\lim_{k\to\infty} \sup_{\partial\Omega\setminus B_k} |f_1 - f_2| = 0$) holds for some smooth exhaustion $\{B_k\}_{k=1}^{\infty}$ of the manifold M. The relation "$\sim$" is an equivalence relation and is independent of the choice of the exhaustion M (see [9, 13]).

We say that a continuous function f on Ω (respectively, on M) belongs to the *class of admissible functions* on Ω (respectively, on M) and write $f \in K(\Omega)$ [respectively, $f \in K(M)$] if, for some compact set B on $\Omega\setminus B$ (respectively, on $M\setminus B$), there exists an L-harmonic function u with the property $u \overset{\Omega}{\sim} f$ (respectively, $u \overset{M}{\sim} f$; see also [10–12]).

Let us introduce the notion of L-potential of the manifold M with respect to some compact set $B \subset M$. Without loss of generality, we assume that $B \subset B_k$ for all k. Let $\{v_k\}_{k=1}^{\infty}$ be the sequence of solutions of the following Dirichlet problems in $B_k\setminus B$:

$$Lv_k = 0 \quad \text{on} \quad B_k\setminus B, \quad v_k = 1 \quad \text{on} \quad \partial B, \quad v_k = 0 \quad \text{on} \quad \partial B_k.$$

By the maximum principle, the function sequence $\{v_k\}_{k=1}^{\infty}$ is monotone increasing and converges to a limit function $v_{M\setminus B}(x) = \lim_{k\to\infty} v_k(x)$, which is L-harmonic on $M\setminus B$ and satisfies $v_{M\setminus B}|_{\partial B} = 1$ and $0 \le v_{M\setminus B}(x) \le 1$ on $M\setminus B$. The function $v_{M\setminus B}(x)$ is called the *L-potential of the manifold M with respect to the compact set B* (see also [9]).

Following [9], we say that a manifold M is *L-strict* if the L-potential of M with respect to some compact set $B \subset M$ is equivalent to zero. Note that the property of being L-strict is independent of the choice of the compact set B (e.g., see [9]).

We define the L-potential of an unbounded domain Ω as follows. Set $B_k' = B_k\setminus\Omega$. Let $v_{M\setminus B_k'}$ be the L-potential of M with respect to B_k'. By the maximum principle, the sequence $\{v_{M\setminus B_k'}\}_{k=1}^{\infty}$ is monotone increasing and bounded; therefore, there exists a limit function v_Ω, which is L-harmonic in Ω and satisfies $0 \le v_\Omega \le 1$; moreover, $v_\Omega|_{\partial\Omega} = 1$. The function v_Ω will be called the L-potential of the set Ω.

The following assertion is the main result of the present paper.

Theorem 1. *Let M be an L-strict manifold, and let $f \in K(\Omega)$. Then for each function φ continuous on $\partial\Omega$ such that $\varphi \overset{\partial\Omega}{\sim} f$, there exists a unique solution of the problem*

$$Lu = 0 \quad \text{on} \quad \Omega,$$
$$u(x)|_{\partial\Omega} = \varphi, \quad u \overset{\Omega}{\sim} f.$$

Let us present a corollary of this theorem for cones of model manifolds. Let $\hat{M}$ be a connected noncompact Riemannian manifold without boundary that can be represented in the form

$\hat{M} = \hat{B} \cup \hat{D}$, where $\hat{B}$ is some compact set and $\hat{D}$ is isometric to the direct product $(r_0, +\infty) \times S$ with the metric

$$ds^2 = dr^2 + g^2(r)d\theta^2,$$

where S is a compact Riemannian manifold without boundary, $g(r)$ is a positive smooth function on $(r_0, +\infty)$, and $d\theta^2$ is a metric on S. Such a manifold $\hat{M}$ is said to be model (e.g., see [4, 14]).

Next, let $M^* \subset \hat{M}$ be a cone of the model manifold $\hat{M}$, that is, a connected noncompact Riemannian manifold with noncompact boundary ∂M^* that can be represented in the form $M^* = B \cup D$, where $B \subset \hat{B}$ is some compact set and $D \subset \hat{D}$ is isometric to the direct product $(r_0, +\infty) \times G$ with the metric $ds^2 = dr^2 + g^2(r)d\theta^2$. Here G is a simply connected domain in the compact set S ($\partial G \neq \varnothing$) with smooth boundary ∂G, and $d\theta^2$ is a metric on S.

On M^*, consider the solutions of the stationary Schrödinger equation (1); throughout the following, we assume that $c(x) \equiv c(r) \not\equiv 0$ on D.

Let $n = \dim M^*$. We introduce the notation

$$I = \int\limits_{r_0}^{\infty} \frac{1}{g^{n-1}(t)} \left(\int\limits_{r_0}^{t} c(z) g^{n-1}(z)\, dz \right) dt, \qquad J = \int\limits_{r_0}^{\infty} \frac{1}{g^{n-1}(t)} \left(\int\limits_{r_0}^{t} g^{n-3}(z)\, dz \right) dt.$$

We say that the Dirichlet problem with continuous boundary data is uniquely solvable on M^* if, for any continuous function $f(\theta)$ on $\overline{G}$ and any continuous function $\varphi(y)$ on ∂M^* such that

$$\lim_{r\to\infty} \sup_{\partial G} |\varphi(r,\theta) - f(\theta)| = 0,$$

there exists a unique solution of the problem

$$\begin{aligned} &Lu = 0 \quad \text{in} \quad M^*, \\ &u(y) = \varphi(y) \quad \text{for all} \quad y \in \partial M^*, \quad \lim_{r\to\infty} \sup_{G} |u(r,\theta) - f(\theta)| = 0. \end{aligned}$$

We say that the boundary value problem

$$\begin{aligned} &Lu = 0 \quad \text{in} \quad M^*, \\ &u(y) = \varphi(y) \quad \text{for all} \quad y \in \partial M^*, \quad \lim_{r\to\infty} \sup_{G} |u(r,\theta) - C| = 0 \end{aligned} \tag{2}$$

with continuous boundary data is uniquely solvable on M^* if, for any constant C and any continuous function $\varphi(y)$ on ∂M^* such that

$$\lim_{r\to\infty} \sup_{\partial G} |\varphi(r,\theta) - C| = 0,$$

there exists a unique solution of this problem.

Theorem 2. 1. *If $I < \infty$ and $J < \infty$, then the Dirichlet problem with continuous boundary data is uniquely solvable on M^*.*

2. *If $I < \infty$ and $J = \infty$, then the boundary value problem* (2) *with continuous boundary data is uniquely solvable on M^*.*

Note that, in the present paper, we consider the case in which $c(x) \not\equiv 0$. If $c(x) \equiv 0$, then the stationary Schrödinger equation becomes the Beltrami–Laplace equation. Note that the case of L-harmonic functions is somewhat different from the case of harmonic functions (i.e., solutions of the Beltrami–Laplace equation). For example, the triviality of the space of bounded harmonic functions on a manifold without boundary is equivalent to the triviality of the space of nonnegative harmonic functions on such manifolds (e.g., see [14]). For L-harmonic functions, this property is not true (e.g., see [12]). Results on the solvability of some boundary value problems for harmonic functions in unbounded domains of Riemannian manifolds and on cones of model manifolds can be found, e.g., in [15].

2. PROOF OF THEOREM 1

In the proof, we need the following assertion.

Lemma 1. *Let B be a compact set, and let $u(x)$ be an L-harmonic function on $\Omega\backslash B$. Then there exist a constant C and an L-harmonic function f on Ω such that*

$$|f-u| \le C v_{M\backslash B} \quad \text{on} \quad \Omega\backslash B.$$

Proof. Let $B' \subset B$; moreover, $\operatorname{dist}(\partial B', \partial B) \ne 0$ and $B' \cap \Omega \ne \varnothing$. We extend the function u by continuity by zero everywhere on $\Omega \cap B'$ so as to ensure that $|u| < \max_{\partial B \cap \Omega} |u|$ on $\partial\Omega \cap (B\backslash B')$.

Let $\{B_k\}_{k=1}^{\infty}$ be a smooth exhaustion of M such that $\Omega \cap B_k \ne \varnothing$, $\partial\Omega$ and ∂B_k are transversal for all k, and $B \subset B_k$ for all k.

Set $\Omega(k) = \partial(B_k \cap \Omega)$ and $\Omega(0) = \partial \cap \Omega$. Consider the sequence of functions $\{\varphi_k\}_{k=1}^{\infty}$ that are the solutions of the problem

$$L\varphi_k = 0 \quad \text{on} \quad B_k \cap \Omega,$$
$$\varphi_k|_{\Omega(k)} = u|_{\Omega(k)}.$$

First, let us show that this sequence is uniformly bounded on $\Omega(0)$. Suppose the contrary: there exists a subsequence $\{k_n\}$ such that $a_{k_n} = \max_{\Omega(0)} |\varphi_{k_n}| \to \infty$ as $n \to \infty$. Set $k_n = k$ and $\Phi_k = \varphi_k / a_k$ on $B_k \cap \Omega$. Then

$$\Phi_k = u/a_k \quad \text{on} \quad \partial(B_k \cap \Omega)\backslash B,$$
$$\Phi_k = 0 \quad \text{on} \quad \partial\Omega \cap B',$$
$$\max_{\Omega(0)} |\Phi_k| = 1, \quad \Phi_k = u/a_k \quad \text{on} \quad \partial\Omega \cap (B\backslash B').$$

Then, by using the maximum principle for the function $\Phi_k - u/a_k$ firstly on $(B_k \cap \Omega)\backslash B$ and then on $B \cap \Omega$, we obtain

$$-1 - \frac{\max_{\Omega(0)} |u|}{a_k} + \frac{u}{a_k} \le \Phi_k \le 1 + \frac{\max_{\Omega(0)} |u|}{a_k} + \frac{u}{a_k} \quad \text{on} \quad B_k \cap \Omega. \tag{3}$$

Indeed, since $\max_{\Omega(0)} |\Phi_k| = 1$, we have $-1 \le \Phi_k \le 1$ on $\Omega(0)$. Hence it follows that

$$-1 - \frac{\max_{\Omega(0)} |u|}{a_k} \le \Phi_k - \frac{u}{a_k} \le 1 + \frac{\max_{\Omega(0)} |u|}{a_k} \quad \text{on} \quad \Omega(0).$$

Then, by taking into account the relations

$$\Phi_k - \frac{u}{a_k} = \frac{\varphi_k}{a_k} - \frac{u}{a_k} = \frac{u}{a_k} - \frac{u}{a_k} = 0 \quad \text{on} \quad (\partial\Omega\backslash B) \cup (\partial B_k \cap \Omega),$$

we obtain

$$-1 - \frac{\max_{\Omega(0)} |u|}{a_k} \le \Phi_k - \frac{u}{a_k} \le 1 + \frac{\max_{\Omega(0)} |u|}{a_k} \quad \text{on} \quad (B_k \cap \Omega)\backslash B. \tag{4}$$

On the other hand, $\Phi_k = u/a_k$ on $\partial\Omega \cap B$. Since $a_k \to \infty$ as $k \to \infty$, it follows that $|\Phi_k| < 1/2$ on $\partial\Omega \cap B$ for sufficiently large k. By taking into account the relation $\max_{\Omega(0)} |\Phi_k| = 1$ and by using the maximum principle, we obtain

$$-1 \le \Phi_k \le 1 \quad \text{on} \quad B \cap \Omega \tag{5}$$

for sufficiently large k.

By combining the estimates (4) and (5), we arrive at relations (3).

It follows from (3) that the sequence $\{\Phi_k\}_{k=1}^{\infty}$ is locally uniformly bounded in Ω. Therefore, there exists a subsequence of $\{\Phi_k\}$ converged uniformly to some limit function Φ on any compact subset Ω; moreover, $L\Phi = 0$ in Ω, $|\Phi||_{\partial\Omega\cap B} < 1/2$, and $-1 \le \Phi \le 1$ on Ω. Note also that, by choosing an appropriate subsequence of $\{\Phi_k\}$, one can assume that $\max_{\Omega(0)} |\Phi| = 1$. We have obtained a contradiction with the maximum principle.

Therefore, the assumption that $a_{k_n} = \max_{\Omega(0)} \varphi_{k_n} \to \infty$ as $n \to \infty$ is invalid; thus, the sequence φ_k is uniformly bounded on $\Omega(0)$. Hence we find that the sequence $\{\varphi_k - u\}_{k=1}^{\infty}$ is uniformly bounded on Ω, which implies that there exists a function $f = \lim_{k\to\infty} \varphi_k$, $Lf = 0$.

As was shown above, the supremum $\overline{a} = \sup_k \max_{\Omega(0)} |\varphi_k| < \infty$ exists. By virtue of the maximum principle and the relation $\varphi_k|_{\Omega(k)} = u|_{\Omega(k)}$, we have

$$u - \left(\overline{a} + \max_{\Omega(0)} |u|\right) v_{M\backslash B} \le \varphi_k \le u + \left(\overline{a} + \max_{\Omega(0)} |u|\right) v_{M\backslash B} \quad \text{on} \quad (B_k \cap \Omega)\backslash B.$$

By passing to the limit as $k \to \infty$ in the last estimate, we obtain the desired assertion. The proof of Lemma 1 is complete.

Let us proceed to the proof of Theorem 1. We split the proof of Theorem 1 into two stages. At the first stage, we prove Theorem 1 for the case in which $f \equiv 0$ and $\varphi \overset{\partial\Omega}{\sim} 0$ is a continuous bounded function on $\partial\Omega$. At the second stage, we prove Theorem 1 for arbitrary continuous functions f and φ.

Stage I. Let $\hat{f}$ be a continuous bounded continuation of the function φ from $\partial\Omega$ to the entire manifold M such that $\hat{f} \overset{M}{\sim} f \equiv 0$.

Let $B'_k = B_k\backslash\Omega$ and $\Omega_k = B_k \cap \Omega$. By [9], since M is an L-strict manifold, the following problem is solvable in $M\backslash B'_k$:

$$\begin{gathered} Lw_k = 0 \quad \text{in} \quad M\backslash B'_k, \\ w_k = \hat{f} \quad \text{on} \quad \partial B'_k, \quad w_k \overset{M\backslash B'_k}{\sim} \hat{f} \overset{M\backslash B'_k}{\sim} 0. \end{gathered} \tag{6}$$

By the maximum principle for the function w_k in $M\backslash B'_k$, we have

$$|w_k| \le \max\left\{\sup_{\partial B'_k} |\hat{f}|, \lim_{n\to\infty} \sup_{M\backslash B_n} |\hat{f}|\right\} = \text{const} \le \sup_M |\hat{f}|,$$

and hence the sequence $\{w_k\}_{k=1}^{\infty}$ is uniformly bounded in Ω.

The uniform boundedness of the sequence $\{w_k\}_{k=1}^{\infty}$ in Ω implies that there exists a subsequence $\{w_{l_k}\}_{k=1}^{\infty}$ (which is also denoted by $\{w_k\}_{k=1}^{\infty}$ in what follows) converging uniformly on Ω to some limit L-harmonic function w. Hence it follows that

$$\lim_{k\to\infty} \sup_{\Omega} |w_k - w| = 0, \tag{7}$$

and consequently,

$$\lim_{k\to\infty} \sup_{\Omega\backslash B_k} |w| \le \lim_{k\to\infty} \sup_{\Omega\backslash B_k} |w - w_k| + \lim_{k\to\infty} \sup_{\Omega\backslash B_k} |w_k - w_n| + \lim_{k\to\infty} \sup_{\Omega\backslash B_k} |w_n|$$

for all n. This, together with condition (7), the inclusion $M\backslash B_k \supset \Omega\backslash B_k$, and the equivalence $w_n \overset{M\backslash B'_k}{\sim} 0$ for all n, implies the inequality

$$\lim_{k\to\infty} \sup_{\Omega\backslash B_k} |w| \le \lim_{k\to\infty} \sup_{\Omega\backslash B_k} |w_k - w_n|$$

for all n. By passing to the limit as $n \to \infty$ and by taking into account the uniform convergence of the sequence $\{w_k\}_{k=1}^{\infty}$ on Ω, we obtain $\lim_{k\to\infty} \sup_{\Omega\backslash B_k} |w| = 0$; i.e., $w \overset{\Omega}{\sim} 0$.

Let us show that $w(y) = \varphi(y)$ for all $y \in \partial\Omega$. Let $y \in \partial\Omega$. Then there exists an R such that $y \in \overline{B'_k}$ for all $k > R$ (since $\{B_k\}_{k=1}^{\infty}$ is an exhaustion of M and $B'_k = B_k\backslash\Omega$). Then $w_k(y) = \hat{f}(y)$

for all $k > R$ by virtue of relations (6). It follows from the condition $\hat{f}|_{\partial\Omega} = \varphi$ and the inclusion $y \in \partial\Omega$ that $w_k(y) = \varphi(y)$ for all $k > R$. By passing to the limit as $k \to \infty$, we obtain the desired relation $w(y) = \varphi(y)$.

The uniqueness of the constructed function w follows from the maximum principle.

Stage II. Now let f and φ be arbitrary continuous functions on Ω and $\partial\Omega$, respectively.

Since $f \in K(\Omega)$ and M is L-strict, it follows from Lemma 1 that on Ω there exists a function v such that

$$Lv = 0 \quad \text{in} \quad \Omega, \quad v \overset{\Omega}{\sim} f.$$

Then $(\varphi - v) \overset{\partial\Omega}{\sim} 0$, because $v \overset{\partial\Omega}{\sim} f \overset{\partial\Omega}{\sim} \varphi$. As was shown at Stage I of the proof, there exists a unique function w that is a solution of the problem

$$Lw = 0 \quad \text{in} \quad \Omega,$$
$$w|_{\partial\Omega} = \varphi - v|_{\partial\Omega}, \qquad w \overset{\Omega}{\sim} 0.$$

Set $u = w + v$. Then $Lu = 0$, $u \overset{\Omega}{\sim} v \overset{\Omega}{\sim} f$, and $u|_{\partial\Omega} = w|_{\partial\Omega} + v|_{\partial\Omega} = \varphi - v|_{\partial\Omega} + v|_{\partial\Omega} = \varphi$. The proof of Theorem 1 is complete.

3. PROOF OF THEOREM 2

As was shown in [6], the condition $I < \infty$ implies the existence of a function $\hat{v}$ on $\hat{D}$ such that

$$L\hat{v} = 0 \quad \text{on} \quad \hat{D}, \quad \hat{v} = 1 \quad \text{on} \quad \partial\hat{D}, \quad \lim_{r\to\infty} \sup_S |\hat{v}(r,\theta)| = 0.$$

By the maximum principle, the function $\hat{v}$ is an L-potential of the manifold $\hat{M}$ with respect to the compact set $\hat{B}$. Note that $\hat{v} \overset{\hat{M}\setminus\hat{B}}{\sim} 0$, which implies that $\hat{M}$ is an L-strict manifold.

Let us prove the first assertion of Theorem 2. Let $f(\theta)$ be a continuous function on $\overline{G}$, and let $\varphi(y)$ be a continuous function on ∂M^* such that

$$\lim_{r\to\infty} \sup_{\partial G} |\varphi(r,\theta) - f(\theta)| = 0. \tag{8}$$

Let us continuously extend the function f from $\overline{G}$ to the manifold $\hat{M}$ so as to ensure that the resulting continuation f^* has the property

$$\lim_{r\to\infty} \sup_{G} |f^*(r,\theta) - f(\theta)| = 0. \tag{9}$$

It follows from [6] and the inequality $J < \infty$ that

$$f^* \in K(\hat{M}). \tag{10}$$

By taking into account condition (8), we obtain

$$\varphi \overset{\partial M^*}{\sim} f^*. \tag{11}$$

By using the fact that $\hat{M}$ is an L-strict manifold, relations (10) and (11), and Theorem 1, we find that on M^* there exists a unique L-harmonic function u such that $u|_{\partial M^*} = \varphi$ and $u \overset{M^*}{\sim} f^*$. This, together with condition (9), implies that $\lim_{r\to\infty} \sup_G |u(r,\theta) - f(\theta)| = 0$. The proof of the first part of Theorem 2 is complete.

Let us prove the second part of Theorem 2. Let C be an arbitrary constant, and let $\varphi(y)$ be a continuous function on ∂M^* such that $\lim_{r\to\infty} \sup_{\partial G} |\varphi(r,\theta) - C| = 0$. Hence we readily obtain

$$\varphi \overset{\partial M^*}{\sim} C. \tag{12}$$

The conditions $I < \infty$ and $J = \infty$ and the results in [6] imply that the constant C is an admissible function on $\hat{M}$. Then, since $\hat{M}$ is an L-strict manifold, it follows from condition (12) and Theorem 1 that there exists a unique L-harmonic function u on M^* satisfying the relations $u|_{\partial M^*} = \varphi$ and $u \overset{M^*}{\sim} C$. Hence we obtain $\lim_{r\to\infty} \sup_G |u(r,\theta) - C| = 0$. The proof of Theorem 2 is complete.

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