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ORDINARY DIFFERENTIAL EQUATIONS

Solvability Theorems for an Inverse Nonself-Adjoint Sturm–Liouville Problem with Nonseparated Boundary Conditions

V. A. Sadovnichii, Ya. T. Sultanaev, and A. M. Akhtyamov
Lomonosov Moscow State University, Moscow, Russia
M. Akmullah Bashkir State Pedagogical University, Ufa, Bashkortostan, Russia
Institute of Mechanics, Ufa Scientific Center of the Russian Academy of Sciences, Ufa, Bashkortostan, Russia
Bashkir State University, Ufa, Bashkortostan, Russia
e-mail: rector@msu.ru, sultanaevyt@gmail.com, AkhtyamovAM@mail.ru

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Abstract—We prove theorems on the solvability of the inverse Sturm–Liouville problem with nonseparated conditions by two spectra and one eigenvalue and theorems on the unique solvability by two spectra and three eigenvalues. We find exact and approximate solutions of the inverse problems. Related examples and counterexample are given.

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1. INTRODUCTION

The inverse Sturm–Liouville problem was considered in numerous papers (for details, see [1–25]). The analysis of the inverse nonself-adjoint problem Sturm–Liouville with nonseparated boundary conditions was initiated in [5]. It was shown there that three spectra and two sets of weight numbers and residues of certain functions are sufficient for the unique reconstruction of a nonself-adjoint Sturm–Liouville problem with nonseparated boundary conditions. Moreover, these spectral data were used essentially [6]. Later, there were attempts to choose the problem to be reconstructed or auxiliary problems so as to use less spectral data for the reconstruction [7–11]. In particular, in [8, 9] a nonself-adjoint problem was replaced by a self-adjoint one, and it was shown that, for its unique reconstruction, as spectral data it suffices to use three spectra, some sequence of signs, and some real number. In [10], an auxiliary problem was chosen so as to reduce the number of spectral data required for the reconstruction of a self-adjoint problem by one spectrum; i.e., only two spectra, some sequence of signs, and some real number were used as spectral data. In the present paper, we consider a nonself-adjoint Sturm–Liouville problem with nonseparated boundary conditions. We show that, for its unique reconstruction, one can use even less spectral data as compared with the reconstruction of a self-adjoint problem in [8–10]; more precisely, we need two spectra and, in addition, three eigenvalues. Moreover, we show that the result obtained in the present paper generalizes the Levitan–Gasymov criterion [12].

2. STATEMENT OF THE PROBLEM

By L_0 we denote the following Sturm–Liouville spectral problem.

Problem $\mathbf{L_0}$:

$$ly = -y'' + q(x)y = \lambda y = s^2 y, \tag{1}$$

$$U_1(y) = y'(0) + a_{11}y(0) + a_{12}y(\pi) = 0, \tag{2}$$

$$U_2(y) = y'(\pi) + a_{21}y(0) + a_{22}y(\pi) = 0 \tag{3}$$

($x \in [0,\pi]$, $y = y(x) \in C^2[0,\pi]$, $q(x)$ is an integrable function, and the a_{ij}, $i,j = 1,2$, are real constants).

Along with problem L_0, we consider the following two problems with separated boundary conditions.

Problem $\mathbf{L_1}$:

$$\begin{aligned} ly &= -y'' + q(x)y = \lambda y, \\ U_{1,1}(y) &= y'(0) + a_{11}y(0) = 0, \\ U_{2,1}(y) &= y'(\pi) + (a_{21} + a_{22})y(\pi) = 0. \end{aligned}$$

Problem $\mathbf{L_2}$:

$$\begin{aligned} ly &= -y'' + q(x)y = \lambda y, \\ U_{1,2}(y) &= y'(0) + ay(0) = 0, \\ U_{2,2}(y) &= y'(\pi) + (a_{21} + a_{22})y(\pi) = 0, \end{aligned}$$

where a is some number different from a_{11}.

For problem L_0, we pose the inverse problem.

Inverse problem. Let the potential function $q(x)$ and the coefficients in the boundary conditions of problems L_j $(j = 0, 1, 2)$ be unknown. The spectra of problems L_j $(j = 0, 1, 2)$ are known. Find a function $q(x)$ and boundary conditions of problems L_j $(j = 0, 1, 2)$ on the basis of their spectra.

In the following, we denote a problem of the type of L_j with different coefficients in the equation and different parameters in the boundary forms by $\widetilde{\mathrm{L}}_j$. We assume that if some symbol stands for an object related to problem L_j, then the same symbol with tilde stands for the corresponding object related to problem $\widetilde{\mathrm{L}}_j$.

The uniqueness of the solution of this inverse problem was justified in [18]; more precisely, the following assertion was proved.

Theorem 1. *Let $a_{11} \neq a$ and $\widetilde{a}_{11} \neq \widetilde{a}$. If the eigenvalues of problems L_j and $\widetilde{\mathrm{L}}_j$ coincide for $j = 0, 1, 2$ with regard to their algebraic multiplicities, then the coefficients in the equations and the constants in the boundary conditions of problems L_j and $\widetilde{\mathrm{L}}_j$ $(j = 0, 1, 2)$ coincide as well; i.e.,*

$$q(x) = \widetilde{q}(x), \qquad a = \widetilde{a}, \qquad a_{ij} = \widetilde{a}_{ij}, \qquad i, j = 1, 2.$$

The Borg uniqueness theorem is a special case of Theorem 1. Indeed, in the case of separated conditions ($a_{12} = a_{21} = 0$ and $\widetilde{a}_{12} = \widetilde{a}_{21} = 0$), problem L_0 coincides with problem L_1, and problem $\widetilde{\mathrm{L}}_0$ coincides with problem $\widetilde{\mathrm{L}}_1$. Therefore, only two spectra (the spectra of problems $\mathrm{L}_0 = \mathrm{L}_1$ and L_2) are used for the unique reconstruction of problems $\mathrm{L}_0 = \mathrm{L}_1$ and L_2. A separate condition concerning the coincidence of the spectra of problems L_0 and $\widetilde{L}_0$ (in the assumptions of Theorem 1) is unnecessary in this case because the coincidence follows from that of the eigenvalues of problems L_1 and $\widetilde{\mathrm{L}}_1$.

Next, we study the following problem.

Unique solvability of the inverse problem. Given three sequences of real numbers λ_k, μ_k, and ν_k, do there exist an absolutely continuous function $q(x)$ and numbers a and a_{ij}, $i, j = 1, 2$, such that $\{\lambda_k\}$ is the spectrum of problem L_0, $\{\mu_k\}$ is the spectrum of problem L_1, and $\{\nu_k\}$ is the spectrum of problem L_2?

In the present paper, we show that if the sequences λ_k, μ_k, and ν_k satisfy certain conditions, then there exists an absolutely continuous function $q(x)$ and numbers a and a_{ij}, $i, j = 1, 2$, with these properties. Moreover, we show that, for such an identification, from the whole sequence of numbers $\{\lambda_k\}$, it suffices to choose three numbers; i.e., knowledge of the whole sequence of numbers $\{\lambda_k\}$ for the identification of an absolutely continuous function $q(x)$ and numbers a and a_{ij}, $i, j = 1, 2$ is a superfluous requirement.

3. SOLVABILITY OF THE INVERSE PROBLEM WITH RESPECT TO TWO SPECTRA AND ONE EIGENVALUE

Before proving the unique solvability of the inverse problem with respect to two spectra and three eigenvalues, we pose the problem on the (not necessarily unique) solvability with respect to two spectra and one eigenvalue.

Solvability of the inverse problem. Given a real number λ_1 and two sequences of real numbers μ_k and ν_k, do there exist an absolutely continuous function $q(x)$ and numbers a and a_{ij}, $i, j = 1, 2$, such that $\{\mu_k\}$ is the spectrum of problem L_1, $\{\nu_k\}$ is the spectrum of problem L_2, and λ_1 is an eigenvalue of problem L_0?

Suppose that the sequences μ_k and ν_k of real numbers satisfy the following two conditions.

Condition 1. The numbers μ_k and ν_k alternate; i.e., $\mu_0 < \nu_0 < \mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots$ (or $\nu_0 < \mu_0 < \nu_1 < \mu_1 < \nu_2 < \mu_2 < \cdots$).

Condition 2. The following asymptotic relations hold:

$$\mu_k = k^2 + b_0 + o(1), \qquad \nu_k = k^2 + b_0' + o(1), \qquad b_0' \neq b_0.$$

By applying the Levitan solvability theorem [20, pp. 64–65] to problems L_1 and L_2, we obtain the following assertion.

Lemma 1. *If two sequences μ_k and ν_k of real numbers satisfy Conditions* 1 *and* 2, *then there exists an absolutely continuous function $q(x)$ and numbers a, a_{11}, and $b = a_{21} + a_{22}$ such that $\{\mu_k\}$ is the spectrum of problem L_1 and $\{\nu_k\}$ is the spectrum of problem L_2.*

If Condition 2 contains k exact terms of the asymptotics (except for the first one), then the function $q(x)$ is continuously differentiable $k-2$ times. In particular, there exists an infinite classical asymptotics for the numbers μ_k and ν_k if and only if the function $q(x)$ is infinitely differentiable.

Lemma 1 contains only sufficient conditions for the solvability. To state a solvability criterion, we need one more condition.

Condition 3. The function

$$\Phi(x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{\nu_k - \mu_k}{b_0' - b_0}\cos\sqrt{\mu_k} - \cos kx\right)$$

has an integrable derivative.

By applying Theorem 3.4.2 in [12, p. 58] to problems L_1 and L_2, we obtain a solvability criterion.

Lemma 2. *Two sequences μ_k and ν_k of real numbers are the eigenvalues of problems L_1 and L_2 if and only if Conditions* 1–3 *are satisfied.*

For the solvability of the posed inverse problem, it remains to prove the existence of the coefficients a_{12} and a_{22} ($a_{21} = b - a_{22}$). Let us prove this fact.

Since the possibility to find the function $q(x)$ has been already proved, one can deal with solutions of Eq. (1). Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be linearly independent solutions of Eq. (1) satisfying the conditions

$$y_1(0,\lambda) = 1, \qquad y_1'(0,\lambda) = 0, \qquad y_2(0,\lambda) = 0, \qquad y_2'(0,\lambda) = 1. \tag{4}$$

The eigenvalues of problem L_0 are the zeros of the characteristic function

$$\begin{aligned}\Delta(\lambda) = {} & a_{12}(1 - by_2(\pi,\lambda)) + a_{22}(a_{11}y_2(\pi,\lambda) - y_1(\pi,\lambda) + 1)\\ & + a_{12}a_{22}(y_2(\pi,\lambda)) - (b + y_1'(\pi,\lambda) - a_{11}y_2'(\pi,\lambda))\end{aligned} \tag{5}$$

(where $b = a_{21} + a_{22}$); moreover, the algebraic multiplicity of an eigenvalue coincides with the multiplicity of the corresponding zero of the function $\Delta(\lambda)$.

Note that for $\lambda = \lambda_1$ the functions $(1 - by_2(\pi, \lambda))$ and $(y_2(\pi, \lambda))$ do not vanish simultaneously. Therefore, the equation $\Delta(\lambda_1) = 0$ is always solvable for the unknown coefficients a_{12} and a_{22}.

By using Lemmas 1 and 2, we obtain the following two theorems (a theorem and a solvability criterion for the inverse problem).

Theorem 2. *Let* λ_1 *and* μ_k, ν_k *be a real number and two sequences of real numbers, respectively, satisfying Conditions* 1 *and* 2. *Under these conditions, there exists an absolutely continuous function* $q(x)$ *and numbers* a *and* a_{ij}, $i, j = 1, 2$, *such that* $\{\mu_k\}$ *is the spectrum of problem* L_1, $\{\nu_k\}$ *is the spectrum of problem* L_2, *and the number* λ_1 *is an eigenvalue of problem* L_0.

Theorem 3. *A number* λ_1 *is an eigenvalue of problem* L_0 *with a function* $q(x)$ *integrable on the interval* $(0, \pi)$, *and two sequences of real numbers* μ_k *and* ν_k *are eigenvalues of problems* L_1 *and* L_2, *respectively, if and only if Conditions* 1–3 *are satisfied.*

Remark 1. If problem L_0 is a spectral problem with separated boundary conditions ($a_{12} = a_{21} = 0$), then it coincides with problem L_1. By taking the value μ_1 for λ_1, we find that the Levitan solvability theorem [12, pp. 64–65] is a special case of Theorem 2. Theorem 3 generalizes the criterion proved by Levitan and Gasymov in [12, Th. 3.4.2, p. 58]. Indeed, in the special case where $a_{12} = a_{21} = 0$ ($\mathrm{L}_0 = \mathrm{L}_1$), the number λ_1 coincides with some term of the numerical sequence μ_k; therefore, for $a_{12} = a_{21} = 0$ Theorem 3 coincides with the criterion by Levitan and Gasymov.

4. UNIQUE SOLVABILITY OF THE INVERSE PROBLEM

If λ_1 is a simple root of Eq. (5), then a_{12} and a_{22} are nonuniquely determined. The number λ_1 alone is insufficient to determine the coefficients a_{12} and a_{22}. One needs either more eigenvalues or a higher multiplicity of the "eigenvalue" λ_1 uniquely.

Let λ_i ($i = 1, 2, 3$) be roots of the equation $\Delta(\lambda_i) = 0$. Here and in the following, the function $\Delta(\lambda)$ is understood as the entire function defined by relation (5).

The following condition is a key requirement for the solvability of the posed inverse problem.

Condition 4. Two or three equations in the finite set of equations

$$\Delta(\lambda_i) = 0, \qquad \left[\frac{d^k}{d\lambda^k}\,\Delta(\lambda)\right]_{\lambda=\lambda_i} = 0, \qquad k = 1, \ldots, p_i, \qquad i = 1, 2, 3,$$

are uniquely solvable for the unknowns a_{12} and a_{22}.

By using Lemmas 1 and 2, we obtain the following assertion on the unique solvability of the inverse problem.

Theorem 4. *Let* λ_1, λ_2, λ_3 *and* μ_k, ν_k *be three numbers and two sequences of real numbers, respectively, satisfying Conditions* 1, 2, *and* 4. *Under these conditions, there exist an absolutely continuous function* $q(x)$ *and numbers* a *and* a_{ij}, $i, j = 1, 2$, *such that* $\{\mu_k\}$ *is the spectrum of problem* L_1, $\{\nu_k\}$ *is the spectrum of problem* L_2, *and the numbers* λ_1, λ_2, *and* λ_3 *are the eigenvalues of the unique problem* L_0.

By using Theorems 1 and 4 and by arguing by contradiction, one can prove the following assertion.

Lemma 3. *There exist three numbers* λ_1, λ_2, *and* λ_3 *satisfying Condition* 4.

In what follows, we present a procedure for the identification of problems L_0, L_1, and L_2.

5. PROCEDURE OF IDENTIFICATION OF PROBLEMS L_0, L_1, AND L_2

On the basis of the proof of Theorem 4, one can construct an algorithm for the unique identification of problems L_0, L_1, and L_2.

1. On the basis of two sequences of real numbers μ_k and ν_k satisfying Conditions 1 and 2, we find an absolutely continuous function $q(x)$ and numbers a and a_{11}, $b = a_{21} + a_{22}$; i.e., we construct problems L_1 and L_2. They are found with the use of well-known methods of identification of a Sturm–Liouville problem with separated boundary conditions [3, pp. 38–92].

2. For the found function $q(x)$, we find linearly independent solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ of Eq. (1) with condition (4).

3. For one, two, or three numbers λ_i satisfying Condition 4, we find the unknown coefficients a_{12} and a_{22} of problem L_0.

4. For the found coefficients b and a_{22}, from the relation $b = a_{21}+a_{22}$, we find the coefficient a_{21}. We thereby completely reconstruct problem L_0.

6. EXAMPLES OF THE SOLUTION OF THE INVERSE PROBLEM

In all examples, we assume that the μ_k are the roots of the equation

$$\sqrt{\mu}\,\sin\sqrt{\mu}\,\pi - \cos\sqrt{\mu}\,\pi = 0$$

and the ν_k are the roots of the equation

$$(\sqrt{\nu} + 1/\sqrt{\nu})\sin\sqrt{\nu}\,\pi - 2\cos\sqrt{\nu}\,\pi = 0.$$

In this case, we have $q(x) \equiv 0$, $a_{11} = 0$, $a = -1$, and $a_{21} + a_{22} = b = 1$. Below, for simplicity, we assume that these values have already been found at step 1 for the identification of problems L_1 and L_2 (see Section 5). In addition, we assume that linearly independent solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ of Eq. (1) with condition (4) have been found. Then, in this case, we have

$$\Delta(\lambda) = -1 + \sqrt{\lambda}\,\sin\sqrt{\lambda}\,\pi + a_{12}\left(1 - \frac{\sin\sqrt{\lambda}\,\pi}{\sqrt{\lambda}}\right) + a_{22}(1-\cos\sqrt{\lambda}\,\pi) + a_{12}a_{22}\left(\frac{\sin\sqrt{\lambda}\,\pi}{\sqrt{\lambda}}\right).$$

Example 1. If $\lambda_1 = 4$, then $\Delta(\lambda_1) = 0$ and $\left[\frac{d}{d\lambda}\Delta(\lambda_i)\right]_{\lambda=\lambda_1} = 0$. Hence we obtain the relations $\Delta(\lambda_1) = -1 + a_{12} = 0$ and $\left[\frac{d}{d\lambda}\Delta(\lambda_i)\right]_{\lambda=\lambda_1} = \frac{\pi}{2} + \frac{\pi}{8}(a_{22}-1) = 0$. Consequently, we obtain the unique solution $a_{12} = 1$, $a_{22} = -3$ ($a_{21} = 4$). Here problem L_0 has been uniquely reconstructed on the basis of a single (multiple) eigenvalue.

Example 2. If $\lambda_1 = 4$ and $\lambda_2 = 9$, then the system of equations $\Delta(\lambda_1) = 0$ and $\Delta(\lambda_2) = 0$ has the unique solution $a_{12} = 1$, $a_{22} = 0$ ($a_{21} = 1$). Here problem L_0 is uniquely reconstructed on the basis of two distinct eigenvalues.

Note that if, as the reconstruction data, we supplement the numbers $\lambda_1 = 4$ and $\lambda_2 = 9$ with $\lambda_3 = 1/4$, then the uniqueness of the solution is preserved, and we obtain the same solution $a_{12} = 1$, $a_{22} = 0$; i.e., the use of three numbers does not violate the uniqueness of the reconstruction of problem L_0.

In the general case, let us show that if three numbers λ_1, λ_2, and λ_3 satisfy certain conditions, then they are eigenvalues of problem L_0 and permit one to determine the coefficients a_{12} and a_{22} uniquely.

We introduce the notation

$$z_{i1} := 1 - by_2(\pi,\lambda_i), \qquad z_{i2} := a_{11}y_2(\pi,\lambda_i) - y_1(\pi,\lambda_i) + 1,$$
$$z_{i3} := y_2(\pi,\lambda_i), \quad v_i := b + y_1'(\pi,\lambda_i) - a_{11}y_2'(\pi,\lambda_i), \quad i = 1,2,3;$$

$$D := \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix}, \quad D_1 := \begin{vmatrix} v_1 & z_{12} & z_{13} \\ v_2 & z_{22} & z_{23} \\ v_3 & z_{32} & z_{33} \end{vmatrix}, \quad D_2 := \begin{vmatrix} z_{11} & v_1 & z_{13} \\ z_{21} & v_2 & z_{23} \\ z_{31} & v_3 & z_{33} \end{vmatrix}, \quad D_3 := \begin{vmatrix} z_{11} & z_{12} & v_1 \\ z_{21} & z_{22} & v_2 \\ z_{31} & z_{32} & v_3 \end{vmatrix}.$$

One can readily show that if the numbers λ_1, λ_2, and λ_3 satisfy the following conditions: (i) $D \neq 0$ and (ii) $D_1 D_2 = D D_3$, then they are eigenvalues of problem L_0. For these eigenvalues, the coefficients $a_{12} = D_1/D$ and $a_{22} = D_2/D$ are the only possible for problem L_0.

Let us return to Example 2. We have $\lambda_1 = 4$, $\lambda_2 = 9$, and $\lambda_3 = 1/4$. By substituting these values into the expressions for D, D_1, and D_2, we obtain

$$D = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 2 \end{vmatrix} = 4, \qquad D_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1/2 & 1 & 2 \end{vmatrix} = 4, \qquad D_2 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 1/2 & 2 \end{vmatrix} = 0.$$

Hence it follows that $a_{12} = 1$ and $a_{22} = 0$.

Here the same problem L_0 has been reconstructed on the basis of three distinct eigenvalues.

In the next example, we consider the case in which the use of the third eigenvalue is essential for the unique reconstruction of problem L_0.

Example 3. If $\lambda_1 = 1/4$ and $\lambda_2 = 9$, then the system $\Delta(\lambda_1) = 0$, $\Delta(\lambda_2) = 0$ has two solutions: the first solution $a_{12} = 0$, $a_{22} = 1/2$ ($a_{21} = 1/2$) and the second one $a_{12} = -1/2$, $a_{22} = 3/4$ ($a_{21} = 1/4$). If, as the reconstruction data, we supplement the eigenvalues $\lambda_1 = 1/4$ and $\lambda_2 = 9$ with the eigenvalue $\lambda_3 \approx 4.6042174$, then we obtain the unique problem L_0. For it, we have $a_{12} = 0$ and $a_{22} = 1/2$. But if, as the reconstruction data, we supplement the eigenvalues $\lambda_1 = 1/4$ and $\lambda_2 = 9$ with the eigenvalue $\lambda_3 \approx 4.8573191$, then we obtain a different unique problem L_0 with the same two eigenvalues. For this problem, we obtain $a_{12} = -1/2$ and $a_{22} = 3/4$.

Indeed, let $\lambda_1 = 1/4$, $\lambda_2 = 9$, and $\lambda_3 \approx 4.6042174$. By substituting these values into the expressions for D, D_1, and D_2, we obtain

$$D \approx \begin{vmatrix} -1 & 1 & 2 \\ 1 & 2 & 0 \\ 0.7939936 & 0.1030032 & 0.2060064 \end{vmatrix} \approx -3.5879871 \neq 0,$$

$$D_1 \approx \begin{vmatrix} 1/2 & 1 & 2 \\ 1 & 2 & 0 \\ 0.05150160 & 0.1030032 & 0.2060064 \end{vmatrix} \approx 1.92 \times 10^{-8},$$

$$D_2 \approx \begin{vmatrix} -1 & 1/2 & 2 \\ 1 & 1 & 0 \\ 0.7939936 & 0.05150160 & 0.2060064 \end{vmatrix} \approx -1.7939936.$$

Hence it follows that

$$a_{12} \approx -0.53 \times 10^{-8} \approx 0, \qquad a_{22} \approx 0.50000000 = 1/2.$$

Let $\lambda_1 = 1/4$, $\lambda_2 = 9$, and $\lambda_3 \approx 4.8573191$. By substituting these values into the expressions for D, D_1, and D_2, we obtain

$$D \approx \begin{vmatrix} -1 & 1 & 2 \\ 1 & 2 & 0 \\ 0.728787 & 0.1983065 & 0.271213 \end{vmatrix} \approx -3.3321739 \neq 0,$$

$$D_1 \approx \begin{vmatrix} 1/2 & 1 & 2 \\ 1 & 2 & 0 \\ -0.317368 & 0.1983065 & 0.27121308 \end{vmatrix} \approx 1.6660870,$$

$$D_2 \approx \begin{vmatrix} -1 & 1/2 & 2 \\ 1 & 1 & 0 \\ 0.728787 & -0.317368 & 0.271213 \end{vmatrix} \approx -2.499130.$$

Hence it follows that

$$a_{12} \approx -0.50000000420 \approx -\frac{1}{2}, \qquad a_{22} \approx 0.75000000210 \approx \frac{3}{4}.$$

Thus, problem L_0 has been uniquely reconstructed on the basis of three distinct eigenvalues. Moreover, the use of the third eigenvalue is important for the unique reconstruction of problem L_0.

Counterexample

Condition 4 in Theorem 4 is important. Problem L_0 cannot in general be reconstructed uniquely on the basis of three arbitrary numbers. Indeed, if $\lambda_1 = 4$, $\lambda_2 = 16$, and $\lambda_3 = 36$, then the system of equations $\Delta(\lambda_i) = 0$, $i = 1, 2, 3$, has infinitely many solutions $a_{12} = 1$, $a_{22} = C$ ($a_{21} = 1 - C$), where C is an arbitrary real number.

7. APPROXIMATE SOLUTION OF THE INVERSE PROBLEM

By using asymptotic formulas for the eigenvalues and a representation of the characteristic determinant (5), one can obtain approximate formulas for the unknown coefficients a_{12} and a_{22}. Indeed, let

$$\sqrt{\lambda_i} = N_i + \mathcal{O}\left(\frac{1}{N_i}\right), \tag{6}$$

where $i = 1, 2$, N_1 is a sufficiently large odd positive integer, and N_2 is a sufficiently large even positive integer.

The following asymptotic formulas hold:

$$\begin{aligned} y_1(x,\lambda) &= \cos sx + \frac{1}{s}u(x)\sin sx + \mathcal{O}\left(\frac{1}{s^2}\right), \quad y_2(x,\lambda) = \frac{1}{s}\sin sx - \frac{1}{s^2}u(x)\cos sx + \mathcal{O}\left(\frac{1}{s^3}\right), \\ y_1'(x,\lambda) &= -s\sin sx + u(x)\cos sx + \mathcal{O}\left(\frac{1}{s}\right), \quad y_2'(x,\lambda) = \cos sx + \frac{1}{s}u(x)\sin sx + \mathcal{O}\left(\frac{1}{s^2}\right), \end{aligned} \tag{7}$$

where $u(x) = 2^{-1}\int_0^x q(t)\,dt$, for a sufficiently large $\lambda = s^2 \in \mathbb{R}$.

By substituting the expressions (6) and (7) into the relations $\Delta(\lambda_1) = 0$ and $\Delta(\lambda_2) = 0$, we obtain

$$a_{12} + 2a_{22} = a_{11} + b + y_1'(\pi,\lambda_1) + \mathcal{O}\left(\frac{1}{N_1}\right), \qquad a_{12} = -a_{11} + b + y_1'(\pi,\lambda_2) + \mathcal{O}\left(\frac{1}{N_2}\right).$$

Consequently, if the numbers λ_1 and λ_2 satisfy condition (6), then we obtain the approximate formulas

$$a_{12} \approx -a_{11} + b + y_1'(\pi,\lambda_{N_2}), \qquad a_{22} \approx a_{11} + \frac{1}{2}(y_1'(\pi,\lambda_{N_1}) - y_1'(\pi,\lambda_{N_2})). \tag{8}$$

Note that the larger the indices N_1 and N_2, the more accurate the approximate formulas are. Let us present numerical examples justifying this conclusion.

These examples were considered in the following cases: (i) $N_1 = 11$, $N_2 = 12$; (ii) $N_1 = 101$, $N_2 = 102$; (iii) $N_1 = 1001$, $N_2 = 1002$.

We have the following assertions.

(i) If $\lambda_{N_1} = 121$ ($N_1 = 11$) and $\lambda_{N_2} \approx 144.635553635613$ ($N_2 = 12$), then from (8), we obtain

$$a_{12} \approx 0.0017256937 \approx 0, \qquad a_{22} \approx 0.4991371532 \approx \frac{1}{2}.$$

(ii) If $\lambda_{N_1} = 10201$ $(N_1 = 101)$ and $\lambda_{N_2} \approx 10404.6366049357$ $(N_2 = 102)$, then from (8), we obtain

$$a_{12} \approx 0.0000240272 \approx 0, \qquad a_{22} \approx 0.4999879864 \approx \frac{1}{2}.$$

(iii) If $\lambda_{N_1} = 1002001$ $(N_1 = 1001)$ and $\lambda_{N_2} \approx 1004004.63661962$ $(N_2 = 1002)$, then relation (8) implies

$$a_{12} \approx 2.54612226 \times 10^{-7} \approx 0, \qquad a_{22} \approx 0.499999872 \approx \frac{1}{2}.$$

Therefore, we obtain accuracy up to two significant digits after the decimal point in the first case, up to four significant digits after the decimal point in the second case, and up to six significant digits after the decimal point in the third case. Consequently, the higher the indices N_1 and N_2, the more accurate the approximate formulas are.

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