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INTEGRAL EQUATIONS

Regularization Method for Nonlinear Integro-Differential Systems of Fredholm Type with Rapidly Varying Kernels

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Abstract—We consider a nonlinear singularly perturbed integro-differential system with an integral operator of Fredholm type. We develop and justify an algorithm of the regularization method both in the nonresonance and resonance cases. We show that if the kernel of the integral operator contains a rapidly decaying factor, then the original integro-differential system "is not on the spectrum;" i.e., it is uniquely solvable for any right-hand side (provided that the nonlinear orthogonality conditions are globally solvable). We solve the initialization problem, that is, the problem of describing the original data of the problem for which the convergence holds on the entire time interval considered (including the boundary-layer zone).

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The linear singularly perturbed problem

$$\varepsilon\frac{dy}{dt} = A(t)y + \int\limits_0^1 \exp\left(\frac{1}{\varepsilon}\int\limits_s^1 \mu(\theta)\,d\theta\right)K(t,s)y(s,\varepsilon)\,ds + h(t), \quad y(0,\varepsilon) = y^0, \quad t \in [0,1], \qquad (1)$$

with a Fredholm integral operator was considered in [1]. The regularized asymptotics (see [2, p. 35]) of the solution of this problem was constructed under the assumption that the spectrum $\sigma(A(t)) = \{\lambda_j(t)\}$ of the matrix $A(t)$ is stable and $\operatorname{Re}\mu(t) < 0$, $t \in [0,1]$. The passage from the linear problem (1) to the nonlinear problem

$$\varepsilon\frac{dy}{dt} = A(t)y + \int\limits_0^1 \exp\left(\frac{1}{\varepsilon}\int\limits_s^1 \mu(\theta)\,d\theta\right)K(t,s)y(s,\varepsilon)\,ds + \varepsilon f(y,t) + h(t), \quad y(0,\varepsilon) = y^0, \quad t \in [0,1], \qquad (2)$$

can hardly be trivial, at least because there arise resonance-related effects in the nonlinear case. Even in the absence of the integral operator in (2), nonlinear resonances substantially change the theory of solvability of iteration problems used in the algorithm of the regularization method (e.g., see [2, Chap. 7]). Obviously, these difficulties become even more complicated in the presence of the integral operator. This is justified by papers dealing with integro-differential equations of Volterra type (see [3]). For Eqs. (1) of Fredholm type, it was shown that the spectral value $\mu(t)$ of the integral operator does not occur in the regularization but affects the solvability of the original problem. For $\mu(t) \equiv 0$, problem (1) can be "on the spectrum." (That is, the equivalent integral system may have characteristic values of nonzero rank.) In this case, system (1) is not solvable for some $h(t)$ and $A(t)$. If $\operatorname{Re}\mu(t) < 0$, $t \in [0,1]$, then system (1) is always solvable. One can ask whether the same is true for the nonlinear problem (2). Apparently, the algorithm for constructing regularized asymptotic solutions for problem (2), which we develop below, can help us to answer this question as well as questions related to resonances.

NOTATION

Throughout the paper, we use the following notation. We use parentheses for a row vector, $b = (b_1, \dots, b_r)$, and braces for a column vector, $a = \{a_1, \dots, a_r\}$. [Thus, $a^T = (a_1, \dots, a_r)$.] The asterisk stands for transposition and complex conjugation: $b^* = \overline{(b^T)}$. We introduce two types of multi-indices, multi-indices $k = (k_1, \dots, k_n)$ of dimension $|k| = k_1 + \dots + k_n$ and multi-indices $m = (m_1, \dots, m_n, 0)$ of dimension $|m| = m_1 + \dots + m_n$. By $\lambda(t)$ we denote the row vector $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t), \lambda_{n+1}(t))$; moreover, $\lambda_{n+1}(t) \equiv \mu(t)$, and by e_j we denote the jth unit vector in the space $\mathbb{C}^{n+1}$ of complex-valued $(n+1)$-dimensional rows; i.e., $e_j = (0, \dots, 0, \underset{(j)}{1}, 0, \dots, 0)$. The expression $(m, \lambda(t))$ stands for $m_1\lambda_1(t) + \dots + m_n\lambda_n(t)$ (the last component of the vector m is zero), and the expression $(m + m_{n+1}e_{n+1}, \lambda(t))$ stands for $\sum_{j=1}^{n+1} m_j\lambda_j(t)$. We also use the following notation: if $y = (y_1, \dots, y_n)$ and $k = (k_1, \dots, k_n)$ is a multi-index, then $y^k = y_1^{k_1} \cdots y_n^{k_n}$ (in contrast to $y^{(k)} = \{y_1^{(k_1,\dots,k_n)}, \dots, y_n^{(k_1,\dots,k_n)}\}$). Here the number n of components of the row vector $y = (y_1, \dots, y_n)$ can be arbitrary. For example, $z^r \equiv (z_1, \dots, z_q)^{(r_1,\dots,r_q)} = z_1^{r_1} \cdots z_q^{r_q}$. By σ we denote the row vector

$$\sigma = (\sigma_1, \dots, \sigma_{n+1}) = \left(\exp\left\{\varepsilon^{-1} \int_0^1 \lambda_1(\theta)\, d\theta\right\}, \dots, \exp\left\{\varepsilon^{-1} \int_0^1 \lambda_{n+1}(\theta)\, d\theta\right\}\right).$$

The inner product of the complex space $\mathbb{C}^n$ of n-dimensional column vectors (or row vectors) is introduced in the usual way: by definition, we set $(y, z)_{\mathbb{C}^n} = \sum_{j=1}^n y_j\bar{z}_j$ for arbitrary vectors $y = \{y_1, \dots, y_n\}$ and $z = \{z_1, \dots, z_n\}$ in the space $\mathbb{C}^n$. Sometimes (if no misunderstanding is likely) the subscript $\mathbb{C}^n$ on the inner product is omitted. Finally, by $\varphi_j(t)$ we denote the "$\lambda_j(t)$"-eigenvector of the matrix $A(t)$ $[A(t)\varphi_j(t) \equiv \lambda_j(t)\varphi_j(t)]$, and by $\chi_i(t)$ we denote the ith column of the matrix $[\Phi^*(t)]^{-1}$, where $\Phi(t) \equiv (\varphi_1(t), \dots, \varphi_n(t))$. Then $\chi_i(t)$ is the "$\bar{\lambda}_i(t)$"-eigenvector of the matrix $A^*(t)$; moreover, $(\varphi_j(t), \chi_i(t))_{\mathbb{C}^n} = \delta_{ji}$, where δ_{ji} is the Kronecker delta, $i, j = 1, \dots, n$. Note that, to simplify the calculations, we consider problem (2) on the interval $[0, 1]$. Nevertheless, all results obtained below remain valid on any finite interval $[0, T]$.

Now let us proceed to the development of an algorithm for constructing regularized solutions of problem (2).

1. REGULARIZATION OF PROBLEM (2)

We assume that the following conditions are satisfied.

1. $h(t) \in \mathrm{C}^\infty([0, 1], C^n)$ and $A(t) \in \mathrm{C}^\infty([0, 1], C^{n^2})$.

2. The spectrum $\sigma(A(t))$ of the matrix $A(t)$ and the spectral value $\mu(t) \equiv \lambda_{n+1}(t)$ of the integral operator satisfy the following requirements:

(a) $\lambda_i(t) \neq \lambda_j(t)$ for $i \neq j$ and $\lambda_j(t) \neq 0$, $i, j = 1, \dots, n+1$, $t \in [0, 1]$;

(b) $\operatorname{Re}\lambda_j(t) \leq 0$, $j = 1, \dots, n$, and $\operatorname{Re}\lambda_{n+1}(t) < 0$, $t \in [0, 1]$.

3. The function $f(y, t) = \{f_1(y, t), \dots, f_n(y, t)\}$ is a polynomial[1] in $y = (y_1, \dots, y_n)$; i.e.,

$$f(y, t) = \sum_{0 \leq |k| \leq N_0} f^{(k)}(t)y^k, \qquad f^{(k)}(t) \in C^\infty([0, 1], \mathbb{C}^n), \qquad 0 \leq |k| \leq N_0 < \infty.$$

4. The relations $(m, \lambda(t)) \equiv \sum_{i=1}^n m_i\lambda_i(t) = \lambda_j(t)$ (for $|m| \geq 2$ and $j \in \{1, 2, \dots, n+1\}$) are either satisfied for no $t \in [0, 1]$ or hold identically for all $t \in [0, 1]$.

[1] Here the function $f(y, t)$ is chosen in the form of a polynomial to simplify the calculations. All results can readily be generalized to the case of a function $f(y, t)$ analytic in y (see [2, Chap. 7]).

It was noted in [1] that the spectral value $\lambda_{n+1}(t) \equiv \mu(t)$ is not used in the regularization of problem (1); therefore, we perform the regularization of problem (2) with the use of the functions

$$\tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)\,d\theta \equiv \frac{\psi(t)}{\varepsilon}, \qquad j = 1,\ldots,n. \tag{3}$$

For the "extension" $\tilde{y}(t,\tau,\varepsilon)$, we obtain the problem

$$\begin{gathered}\varepsilon\frac{\partial\tilde{y}}{\partial t} + \sum_{j=1}^{n}\lambda_j(t)\frac{\partial\tilde{y}}{\partial\tau_j} - A(t)\tilde{y} \\ - \int_0^1 \exp\left(\frac{1}{\varepsilon}\int_s^1 \lambda_{n+1}(\theta)\,d\theta\right)K(t,s)\tilde{y}\left(s,\frac{\psi(s)}{\varepsilon},\varepsilon\right)ds - \varepsilon f(\tilde{y},t) = h(t), \\ \tilde{y}(t,\tau,\varepsilon)|_{t=0,\ \tau=0} = y^0,\end{gathered} \tag{4}$$

where[2] $\psi(t) = (\psi_1(t),\ldots,\psi_n(t),0)$ and $\tau = (\tau_1,\ldots,\tau_n,0)$. Although the function $\tilde{y} = \tilde{y}(t,\tau,\varepsilon)$ satisfies the necessary regularization condition $\tilde{y}(t,\psi(t)/\varepsilon,\varepsilon) \equiv y(t,\varepsilon)$ [where $y(t,\varepsilon)$ is the exact solution of system (2)], problem (3) cannot be viewed as completely regularized, because the regularization of the integral term

$$J\tilde{y}(t,\tau,\varepsilon) \equiv \int_0^1 \exp\left\{\frac{1}{\varepsilon}\int_s^1 \lambda_{n+1}(\theta)\,d\theta\right\}K(t,s)\tilde{y}\left(s,\frac{\psi(s)}{\varepsilon},\varepsilon\right)ds \tag{5}$$

has not been carried out.

It is well known that, for its regularization, one should introduce the class M_ε asymptotically invariant with respect to the operator J (see [2, Chap. 2]).

Definition 1. We say that a *vector function* $y(t,\tau) = \{y_1,\ldots,y_n\}$ *belongs to the space* U if it is a sum of the form

$$y(t,\tau) \equiv y(t,\tau,\sigma) = \sum_{0\le|m|\le N}^{*} y^{(m)}(t,\sigma)e^{(m,\tau)}, \qquad N = N_y < \infty, \tag{6}$$

with coefficients

$$\begin{gathered}y^{(m)}(t,\sigma) = \sum_{0\le|p|\le N_m} y_p^{(m)}(t)\sigma^p, \qquad y_p^{(m)}(t) \equiv y_{(p_1,\ldots,p_{n+1})}^{(m_1,\ldots,m_n,0)}(t) \in C^\infty([0,1],\mathbb{C}^n), \\ N_m \equiv N_{(m_1,\ldots,m_n,0)} < \infty.\end{gathered}$$

The asterisk over the sign of the sum in (6) is used to indicate that the sum does not contain resonance exponentials (see [2, p. 234]), that is, exponentials $e^{(m,\tau)}$ of dimension $|m| \ge 2$ such that the identity $(m,\lambda(t)) \equiv \lambda_j(t)$ holds for some $j \in \{1,2,\ldots,n+1\}$ and $t \in [0,1]$.

By substituting the function (6) for $\tilde{y}$ into relation (5) and by setting $k^{(m)}(t,s) \equiv K(t,s)y^{(m)}(s)$ (we omit the dependence on σ), we obtain

$$\begin{aligned}Jy(t,\tau,\sigma) &= \sum_{0\le|m|\le N}^{*}\int_0^1 k^{(m)}(t,s)\exp\left\{\frac{1}{\varepsilon}\int_s^1 \lambda_{n+1}(\theta)\,d\theta + \frac{1}{\varepsilon}\int_0^s (m,\lambda(\theta))\,d\theta\right\}ds \\ &= \sum_{0\le|m|\le N}^{*}\sigma_{n+1}\int_0^1 k^{(m)}(t,s)\exp\left\{\frac{1}{\varepsilon}\int_0^s (m-e_{n+1},\lambda(\theta))\,d\theta\right\}ds.\end{aligned}$$

[2] One could use the notation $\psi = (\psi_1,\ldots,\psi_n)$ and $\tau = (\tau_1,\ldots,\tau_n)$, but for forthcoming calculations, it is convenient to introduce the $(n+1)$-dimensional vectors ψ and τ.

By setting $J^{(m)}(t,\varepsilon) \equiv \sigma_{n+1}\int_0^1 k^{(m)}(t,s)\exp\{\varepsilon^{-1}\int_0^s (m-e_{n+1},\lambda(\theta))\,d\theta\}\,ds$ and by integrating by parts, we obtain the chain of relations

$$
\begin{aligned}
J^{(m)}(t,\varepsilon) &\equiv \varepsilon\sigma_{n+1}\int_0^1 \frac{k^{(m)}(t,s)}{(m-e_{n+1},\lambda(s))}\,d\exp\left\{\frac{1}{\varepsilon}\int_0^s (m-e_{n+1},\lambda(\theta))\,d\theta\right\}ds \\
&= \varepsilon\sigma_{n+1}\left[\frac{k^{(m)}(t,s)}{(m-e_{n+1},\lambda(s))}\exp\left\{\frac{1}{\varepsilon}\int_0^s (m-e_{n+1},\lambda(\theta))\,d\theta\right\}\Bigg|_{s=0}^{s=1}\right. \\
&\quad \left. -\varepsilon\int_0^1 \frac{\partial}{\partial s}\left(\frac{k^{(m)}(t,s)}{(m-e_{n+1},\lambda(s))}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^s (m-e_{n+1},\lambda(\theta))\,d\theta\right\}\right. \\
&= \varepsilon\sigma_{n+1}\left[\frac{k^{(m)}(t,1)}{(m-e_{n+1},\lambda(1))}\exp\left\{\frac{1}{\varepsilon}\int_0^1 (m-e_{n+1},\lambda(\theta))\,d\theta\right\}-\frac{k^{(m)}(t,0)}{(m-e_{n+1},\lambda(0))}\right] \\
&\quad -\varepsilon\sigma_{n+1}\int_0^1 \frac{\partial}{\partial s}\left(\frac{k^{(m)}(t,s)}{(m-e_{n+1},\lambda(s))}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^s (m-e_{n+1},\lambda(\theta))\,d\theta\right\}ds \\
&= \varepsilon\left[\frac{k^{(m)}(t,1)}{(m-e_{n+1},\lambda(1))}\exp\left\{\frac{1}{\varepsilon}\int_0^1 (m,\lambda(\theta))\,d\theta\right\}-\frac{k^{(m)}(t,0)}{(m-e_{n+1},\lambda(0))}\sigma_{n+1}\right] \\
&\quad -\varepsilon\sigma_{n+1}\int_0^1 \frac{\partial}{\partial s}\left(\frac{k^{(m)}(t,s)}{(m-e_{n+1},\lambda(s))}\right)\exp\left\{\frac{1}{\varepsilon}\int_0^s (m-e_{n+1},\lambda(\theta))\,d\theta\right\}ds.
\end{aligned}
$$

By continuing this process, we obtain the asymptotic expansion

$$
J^{(m)}(t,\varepsilon) = \sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_m^{\nu}(k(t,s)))_{s=1}\sigma^m - (I_m^{\nu}(k(t,s)))_{s=0}\sigma_{n+1}], \tag{7}
$$

where we have introduced the operators

$$
I_m^0 = \frac{1}{(m-e_{n+1},\lambda(s))}, \qquad I_m^{\nu} = \frac{1}{(m-e_{n+1},\lambda(s))}\frac{\partial}{\partial s}I_m^{\nu-1}, \qquad \nu \ge 1. \tag{8}
$$

Consequently,

$$
Jy(t,\tau,\sigma) = \sum_{0\le|m|\le N}^{*}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}[(I_m^{\nu}(k(t,s)))_{s=1}\sigma^m - (I_m^{\nu}(k(t,s)))_{s=0}\sigma_{n+1}]; \tag{9}
$$

moreover, this series converges asymptotically (uniformly with respect to $t \in [0,1]$) to $Jy(t,\tau,\sigma)$ as $\varepsilon \to +0$ (see [3]). It follows that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is invariant with respect to the integral operator J.

Now assume that the series

$$
\tilde{y}(t,\tau,\sigma,\varepsilon) = \sum_{q=0}^{\infty}\varepsilon^q y_q(t,\tau,\sigma) \equiv \sum_{q=0}^{\infty}\varepsilon^q \sum_{0\le|m|\le N_q}^{*} y_q^{(m)}(t,\sigma)e^{(m,\tau)} \tag{10}
$$

converges asymptotically as $\varepsilon \to +0$ and uniformly with respect to $(t,\tau) \in [0,1] \times \{\operatorname{Re}\tau_j \le 0,\ j = 1,\dots,n\}$. Then, obviously, $J\tilde{y}(t,\tau,\sigma,\varepsilon)$ is represented by an asymptotic series uniformly convergent as $\varepsilon \to +0$ as well. This permits obtaining the definitive extension of the integral operator J as follows.

For an arbitrary element (6) of the space U, one can write out the relation

$$Jy(t,\tau,\sigma) = R_0 y(t,\tau,\sigma) + \sum_{\nu=1}^{\infty} R_\nu y(t,\tau,\sigma),$$

where the operators $R_\nu : U \to U$ (operators of order with respect to ε) are given by the formulas

$$\begin{aligned} &R_0 y(t,\tau,\sigma) \equiv 0,\\ &R_{\nu+1} y(t,\tau,\sigma) = (-1)^\nu \sum_{0\le |m|\le N}^{*} [(I_m^\nu(k(t,s)))_{s=1}\sigma^m - (I_m^\nu(k(t,s)))_{s=0}\sigma_{n+1}], \\ &\nu \ge 0, \qquad \tau = \varepsilon^{-1}\psi(t). \end{aligned} \tag{11}$$

In view of these formulas, the result of substitution of the series (10) into the integral $J\tilde{y}$ can be represented in the form

$$J\tilde{y} = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} y_s(t,\tau,\sigma)|_{\tau=\psi(t)/\varepsilon}.$$

The extension $\tilde{J}$ of the integral operator J has the following natural definition.

Definition 2. The *formal extension of the operator* J is defined as the operator $\tilde{J}$ acting on any function $\tilde{y}(t,\tau,\sigma,\varepsilon)$ of the form (10) continuous in $(t,\tau) \in [0,1] \times \{\operatorname{Re}\tau_j \le 0,\ j=1,\dots,n\}$ by the rule

$$\tilde{J}\tilde{y} \equiv \tilde{J}\left(\sum_{q=0}^{\infty} \varepsilon^k y_q(t,\tau,\sigma)\right) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} y_s(t,\tau,\sigma).$$

Now one can readily write out the problem completely regularized with respect to the original problem (2),

$$\varepsilon \frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{n} \lambda_j(t) \frac{\partial \tilde{y}}{\partial \tau_j} - A(t)\tilde{y} - \tilde{J}\tilde{y} - \varepsilon f(\tilde{y},t) = h(t), \qquad \tilde{y}(0,0,\varepsilon) = y^0. \tag{12}$$

2. SOLVABILITY OF THE ITERATION PROBLEMS

By substituting the function (10) into Eq. (12) and by matching the coefficients of like powers of ε, we obtain the iteration problems

$$L_0 y_0 \equiv \sum_{j=1}^{n} \lambda_j(t) \frac{\partial y_0}{\partial \tau_j} - A(t) y_0 = h(t), \qquad y_0(0,0) = y^0, \tag{12_0}$$

$$L_0 y_1 = -\frac{\partial y_0}{\partial t} + R_1 y_0 + \hat{f}(y_0,t), \qquad y_1(0,0) = 0, \tag{12_1}$$

$$L_0 y_2 = -\frac{\partial y_1}{\partial t} + \left(\frac{\partial f(y_0,t)}{\partial y} y_1\right)^{\wedge} + R_1 y_1 + R_2 y_0, \qquad y_2(0,0) = 0, \ \dots, \tag{12_2}$$

$$L_0 y_k = -\frac{\partial y_{k-1}}{\partial t} + R_1 y_{k-1} + R_2 y_{k-2} + \dots + R_k y_0 + \hat{P}_k(y_0,\dots,y_{k-1}), \qquad y_k(0,0) = 0. \tag{12_k}$$

Here $P_k(y_0,\dots,y_{k-1})$ is some polynomial in $y_1,\dots,y_{k-1}$ with coefficients depending on the partial derivatives of the function $f(y,t)$ at the point $y = y_0(t,\tau,\sigma)$; moreover, $P_k(y_0,\dots,y_{k-1})$ is linear with respect to y_{k-1}; the symbol $\wedge$ over $f, \dots, P_k$ indicates the embedding of the corresponding vector function in the space U in which resonance exponentials are absent. (This operation acts as follows: if a resonance exponential $e^{(m,\tau)}$ $[|m| \ge 2,\ (m, \lambda(t)) \equiv \lambda_j(t)]$ occurs in the polynomial $g(t, e^{\tau_1}, \dots, e^{\tau_n}, \sigma)$ in exponentials, then the operation $\wedge$ replaces it by the corresponding exponential e^{τ_j} of the first dimension; for details, see [2, p. 234].)

Each of the iteration problems (12_k) has the form of the system

$$L_0 y(t,\tau) \equiv \sum_{j=1}^{n} \lambda_j(t) \frac{\partial y}{\partial \tau_j} - A(t) y = H(t,\tau,\sigma), \tag{13}$$

where $H(t,\tau,\sigma) = \sum^{*}_{0 \le |m| \le N_H} H^{(m)}(t,\sigma) e^{(m,\tau)} \in U$ is the corresponding right-hand side. The space U can be represented as the direct sum of the subspaces

$$U^{(s)} = \left\{ y(t,\tau,\sigma) :\ y(t,\tau,\sigma) = \sum^{*}_{|m|=s} y^{(m)}(t,\sigma) e^{(m,\tau)} \right\}, \qquad s = 0, \dots, N;$$

i.e., $U = \sum_{s=0}^{N} \oplus U^{(s)}$. We introduce the following notation: if $y(t,\tau,\sigma)$ is an element (6) of the space U, then by $y^{(s)}(t,\tau,\sigma)$ we denote the sum $\sum^{*}_{|m|=s} y^{(m)}(t,\sigma) e^{(m,\tau)} \in U^{(s)}$. In particular,

$$y^{(1)}(t,\tau,\sigma) = \sum_{|m|=1} y^{(m)}(t,\sigma) e^{(m,\tau)} \equiv \sum_{j=1}^{n} y^{e_j}(t,\sigma) e^{\tau_j} \in U^{(1)},$$
$$H^{(1)}(t,\tau,\sigma) = \sum_{|m|=1} H^{(m)}(t,\sigma) e^{(m,\tau)} \equiv \sum_{j=1}^{n} H^{e_j}(t,\sigma) e^{\tau_j} \in U^{(1)}.$$

We need an inner product (for each $t \in [0,1]$ and each σ) in the space $U^{(1)}$. It is introduced as follows:

$$\langle y(t,\tau,\sigma), z(t,\tau,\sigma) \rangle \equiv \left\langle \sum_{j=1}^{n} y^{e_j}(t,\sigma) e^{\tau_j}, \sum_{j=1}^{n} z^{e_1}(t,\sigma) e^{\tau_j} \right\rangle \overset{\text{def}}{=} \sum_{j=1}^{n} (y^{e_j}(t,\sigma), z^{e_j}(t,\sigma)),$$

where $(\,,)$ is the ordinary inner product in C^n. One can readily see that the vector functions $\nu_j(t,\tau,\sigma) \equiv \chi_j(t) e^{\tau_j}$ [where $\chi_j(t)$ is the eigenvector of the matrix $A^*(t)$ corresponding to the eigenvalue $\bar{\lambda}_j(t)$ $(j = 1, \dots, n)$] form a basis in the kernel of the operator $L_0^* = \sum_{j=1}^{n} \bar{\lambda}_j(t) \dfrac{\partial}{\partial \tau_j}$, which is the adjoint in $U^{(1)}$ of the operator L_0. Let us prove the following assertion.

Theorem 1. *Let $H(t,\tau,\sigma) \in U$, and let conditions* 1 *and* 2 (a) *be satisfied. Equation* (13) *is solvable in the space U if and only if*

$$\langle H^{(1)}(t,\tau,\sigma), \nu_j(t,\tau) \rangle \equiv 0 \qquad (j = 1, \dots, n, \quad t \in [0,1]). \tag{14}$$

Proof. We seek a solution of system (13) in the form of the element

$$y(t,\tau,\sigma) = \sum^{*}_{0 \le |m| \le N_y} y^{(m)}(t,\sigma) e^{(m,\tau)} \tag{15}$$

of the space U, where $N_y \ge N_H$. By substituting this element into Eq. (13), we obtain (we omit the dependence on σ)

$$\sum^{*}_{0 \le |m| \le N_y} [(m, \lambda(t)) I - A(t)] y^{(m)}(t) e^{(m,\tau)} = \sum^{*}_{0 \le |m| \le N_H} H^{(m)}(t) e^{(m,\tau)}.$$

By matching the free terms and the coefficients of like exponentials, we obtain the systems of equations

$$\begin{aligned} &[\lambda_j(t)I - A(t)]y^{e_j} = H^{e_j}, \qquad j = 1,\dots,n, \qquad -A(t)y^{(0)}(t) = H^{(0)}(t),\\ &[(m,\lambda(t))I - A(t)]y^{(m)}(t) = H^{(m)}(t), \qquad 2 \le |m| \le N_H,\\ &[(m,\lambda(t))I - A(t)]y^{(m)}(t) = 0, \qquad N_H < |m| \le N_y. \end{aligned} \tag{16}$$

Since $\det A(t) \ne 0$, $t \in [0,1]$, and U does not contain resonant exponents $[(m,\lambda(t)) \notin \sigma(A(t))]$, it follows that all systems (16) except for the first one are uniquely solvable in the space $C^n([0,1],\mathbb{C}^n)$. Their solutions have the form

$$\begin{aligned} &y^{(0)}(t) = -A^{-1}(t)H^{(0)}(t),\\ &y^{(m)}(t) = [(m,\lambda(t))I - A(t)]^{-1}H^{(m)}(t) \qquad (0 \le |m| \le N_H),\\ &y^{(m)}(t) \equiv 0 \qquad (|m| > N_H). \end{aligned} \tag{17}$$

The first system in (16) is solvable if and only if conditions (14) are satisfied (see [2, p. 237]). If these conditions are satisfied, then the solutions of this system can be represented in the form $y^{e_j}(t) = \alpha^{e_j}(t)\varphi_j(t)$, where $\varphi_j(t)$ is the "$\lambda_j(t)$"-eigenvector of the matrix $A(t)$ and the $\alpha^{e_j}(t) \in C^\infty([0,1],\mathbb{C}^1)$ are arbitrary scalar functions. The proof of the theorem is complete.

We do not state the theorem on the unique solvability of system (13) (under some additional constraints). We only note that if Theorem 1 is applied to two successive iteration problems (12_l) and (12_{l+1}), then we obtain conditions for the unique solvability of system (12_l) in the space U (e.g., see [2, p. 239]).

3. CONSTRUCTION OF SOLUTIONS OF ITERATION PROBLEMS

Consider system (12_0). Since the inhomogeneity $H(t,\tau,\sigma) \equiv h(t)$ is independent of the exponentials e^{τ_j}, it follows that the orthogonality conditions (14) are necessarily satisfied for it automatically; therefore, system (12_0) has a solution in U, which can be represented in the form

$$y_0(t,\tau) = \sum_{j=1}^{n} \alpha_j^{(0)}(t)\varphi_j(t)e^{\tau_j} + y_0^{(0)}(t), \tag{18}$$

where $y_0^{(0)}(t) = -A^{-1}(t)h(t)$ and the $\alpha_j^{(0)}(t) \in C^\infty([0,1],\mathbb{C}^1)$ are arbitrary scalar functions. By subjecting the function (18) to the initial condition $y_0(0,0) = y^0$, we obtain

$$\sum_{j=1}^{n} \alpha_j^{(0)}(0)\varphi_j(0) + y_0^{(0)}(0) = y^0,$$

which is equivalent to the relations

$$\alpha_j^{(0)}(0) = (y^0 + A^{-1}(0)h(0), \chi_j(0)), \qquad j = 1,\dots,n.$$

Now let us proceed to the iteration problem (12_1). By substituting the function (18) into Eq. (12_1), we obtain the system

$$\begin{aligned} L_0 y_1 = &-\sum_{j=1}^{n}(\alpha_j^{(0)}(t)\varphi_j(t))^{\cdot} e^{\tau_j} - \dot{y}_0^{(0)}(t) - \frac{K(t,1)y_0^{(0)}(1)}{\lambda_{n+1}(1)} + \frac{K(t,0)y_0^{(0)}(0)}{\lambda_{n+1}(0)}\,\sigma_{n+1}\\ &-\sum_{j=1}^{n}\left[\frac{K(t,1)\varphi_j(1)}{\lambda_j(1)-\lambda_{n+1}(1)}\,\alpha_j^{(0)}(1)\sigma_j - \frac{K(t,0)\varphi_j(0)}{\lambda_j(0)-\lambda_{n+1}(0)}\,\alpha_j^{(0)}(0)\sigma_{n+1}\right]\\ &+\hat{f}\left(\sum_{j=1}^{n}\alpha_j^{(0)}(t)\varphi_j(t)e^{\tau_j} + y_0^{(0)}(t), t\right). \end{aligned} \tag{19}$$

By computing

$$\hat{f}\left(\sum_{j=1}^{n}\alpha_j^{(0)}\varphi_j(t)e^{\tau_j}+y_0^{(0)}(t),t\right)$$
$$=f_0(t)+\sum_{j=1}^{n}f^{e_j}(\alpha_1^{(0)},\dots,\alpha_n^{(0)},t)e^{\tau_j}+\sum_{2\le|m|\le N_1}^{*}f^{(m)}(\alpha_1^{(0)},\dots,\alpha_n^{(0)},t)e^{(m,\tau)}$$

and by subjecting the right-hand side of system (19) to the orthogonality conditions (14), we obtain the system of ordinary differential equations

$$\begin{aligned}\dot{\alpha}_j^{(0)}&=-(\dot{\varphi}_j(t),\chi_j(t))\alpha_j^{(0)}+(f^{e_j}(\alpha_1^{(0)},\dots,\alpha_n^{(0)},t),\chi_j(t)),\\ \alpha_j^{(0)}(0)&=(y^0+A^{-1}(0)h(0),\chi_j(0)),\qquad j=1,\dots,n.\end{aligned}\tag{20}$$

System (20) in the resonance case is a nonlinear system of differential equations for $\alpha_j^{(0)}(t)$; therefore, its solvability on the interval $[0,1]$ is not guaranteed. If, say, the spectrum $\{\lambda_j(t)\}$ of the matrix $A(t)$ at a given point $t\in[0,1]$ lies on one side of some line π passing through the origin of the complex λ-plane and there are no points $\lambda_j(t)$ on π, then the system is triangular. In this case, Eqs. (20) can be integrated successively, and hence their solvability on the interval $[0,1]$ becomes obvious. System (20) fails to be triangular in other cases of arrangement of the spectrum $\lambda_j(t)$ with respect to the imaginary axis. However, in any case, there exists some birational change of variables that permits one to diminish the order of the system and to reduce the study of the global solvability to the case of a simpler system of differential equations.

We do not discuss this problem here. We require that system (20) has a solution in the class $C^\infty([0,1],\mathbb{C}^n)$. Then the functions $\alpha_j^{(0)}(t)$ occurring in the solution (18) of system (12_0) can be completely computed, and system (12_0) itself has the unique solution in the space U. In this case, we find a solution of system (12_1) (to within elements of the kernel of the operator L_0 in $U^{(1)}$). The construction of functions $\alpha_j^{(1)}(t)$ occurring in the above-mentioned kernel is performed by the same scheme as for the functions $\alpha_j^{(0)}(t)$. For $\alpha_j^{(1)}(t)$, we obtain a linear system of differential equations, whose solvability on the interval $[0,1]$ is guaranteed by the smoothness of its coefficients. Let us state the corresponding result.

Theorem 2. *Let conditions* 1–4 *be satisfied, and let problem* (20) *be solvable on the interval* $[0,1]$. *Then all iteration problems* (12_k) $(k=0,1,2,\dots)$ *are uniquely solvable in the class* U (*if one solves them successively*).

4. ASYMPTOTIC CONVERGENCE OF FORMAL SOLUTIONS

Having constructed solutions $y_0(t,\tau,\sigma),\dots,y_l(t,\tau,\sigma)$ of problems $(12_0),\dots,(12_l)$ in the space U, we write out the partial sum

$$S_l(t,\tau,\sigma,\varepsilon)=\sum_{k=0}^{l}\varepsilon^k y_k(t,\tau,\sigma).$$

We denote the restriction of this sum to $\tau=\psi(t)/\varepsilon$ by $y_{\varepsilon l}(t)$. The proof of the following assertion can be carried out by analogy with [3].

Lemma 1. *Let the assumptions of Theorem* 2 *be satisfied. Then the function* $y_{\varepsilon l}(t)$ *satisfies system* (2) *modulo terms containing* ε^{l+1}; *i.e.*,

$$\varepsilon\frac{dy_{\varepsilon l}(t)}{dt}=A(t)y_{\varepsilon l}(t)+\int_0^1\exp\left(\frac{1}{\varepsilon}\int_s^1\mu(\theta)\,d\theta\right)K(t,s)y_{\varepsilon l}(s)\,ds$$
$$+\varepsilon f(y_{\varepsilon l}(t),t)+h(t)+\varepsilon^{l+1}F(t,\varepsilon),\qquad y_{\varepsilon l}(0)=y^0,\tag{21}$$

where $\|F(t,\varepsilon)\|_{C[0,1]} \le \bar{F}$, *and* $\bar{F} > 0$ *is a constant independent of* $\varepsilon \in (0, \varepsilon_0]$. (*Here* $\varepsilon_0 > 0$ *is sufficiently small.*)

To prove the asymptotic convergence of the formal solution $y_{\varepsilon l}(t)$ to the exact solution $y(t,\varepsilon)$, we use the following assertion (see [4]) on the solvability of the operator equation

$$P_\varepsilon(u) = 0. \tag{22}$$

Theorem 3. *Let an operator* P_ε *act from a Banach space* B_1 *to a Banach space* B_2 *and have two continuous derivatives in some ball* $\{\|u - u_0\| \le r\} \subset B_1$. *In addition, assume that the operator* $\Gamma_\varepsilon \equiv [P'_\varepsilon(u_0)]^{-1}$ *exists and the following conditions are satisfied*: (1′) $\|\Gamma_\varepsilon\| \le c_1\varepsilon^{-k}$; (2′) $\|P_\varepsilon(u_0)\| \le c_2\varepsilon^m$ $(m > 2k)$; (3′) $\|P''_\varepsilon(u)\| \le c_3$. *Then, for sufficiently small* $\varepsilon \in (0, \varepsilon_0]$, *Eq.* (22) *has a solution* $u_* \in B_1$ *satisfying the inequality* $\|u_* - u_0\|_{B_1} \le c\varepsilon^{m-k}$.

To use this theorem, we need some auxiliary assertions.

Lemma 2. *Let conditions* 1 *and* 2 *be satisfied. Then the normalized fundamental solution matrix* $Y(t,s,\varepsilon)$ *of the homogeneous system*

$$\varepsilon \frac{dY(t,s,\varepsilon)}{dt} = (A(t) + \varepsilon D(t,\varepsilon))Y(t,s,\varepsilon), \qquad Y(s,s,\varepsilon) = I, \qquad 0 \le s \le t \le T, \tag{23}$$

where $D(t,\varepsilon) \in C([0,T], \mathbb{C}^{n^2})$ *is a matrix such that* $\|D(t,\varepsilon)\|_{C[0,T]} \le \bar{D}$ ($\bar{D} > 0$ *is a constant independent of* $\varepsilon \in (0, \varepsilon_0]$, *where* $\varepsilon_0 > 0$ *is sufficiently small*), *is uniformly bounded*; *i.e.*,

$$\|Y(t,s,\varepsilon)\| \le c_0 = \text{const}, \qquad 0 \le s \le t \le T, \qquad \varepsilon > 0.$$

Proof. In system (23), we make the change of variables $Y(t,s,\varepsilon) = \Phi(t)Z(t,s,\varepsilon)$. Then for the matrix function $Z(t,s,\varepsilon)$, we obtain the problem

$$\varepsilon \frac{dZ(t,s,\varepsilon)}{dt} = \Lambda(t)Z(t,s,\varepsilon) + \varepsilon\Phi^{-1}(t)(D(t,\varepsilon)\Phi(t) - \Phi'(t))Z(t,s,\varepsilon), \qquad Z(s,s,\varepsilon) = \Phi^{-1}(s),$$

where $\Lambda(t) = \operatorname{diag}(\lambda_1(t), \dots, \lambda_n(t))$. This problem is equivalent to the integral system

$$\begin{aligned} Z(t,s,\varepsilon) &= \exp\left\{\frac{1}{\varepsilon}\int_s^t \Lambda(\theta)\,d\theta\right\}\Phi^{-1}(s) \\ &\quad + \int_s^t \exp\left\{\frac{1}{\varepsilon}\int_\varsigma^t \Lambda(\theta)\,d\theta\right\}\Phi^{-1}(\varsigma)(D(\varsigma,\varepsilon)\Phi(\varsigma) - \Phi'(\varsigma))Z(\varsigma,s,\varepsilon)\,d\varsigma; \end{aligned}$$

by passing in it to the norms and by taking into account the boundedness of the matrices

$$\exp\left\{\frac{1}{\varepsilon}\int_\varsigma^t \Lambda(\theta)\,d\theta\right\}, \quad \Phi(t), \quad \Phi^{-1}(t), \quad \Phi'(t), \quad A_1(t),$$

we obtain

$$\|Z(t,s,\varepsilon)\| \le c_1 + c_2\int_s^t \|Z(\varsigma,s,\varepsilon)\|\,d\varsigma \qquad (0 \le s \le t \le T, \;\; \varepsilon > 0).$$

This, together with the Gronwall–Bellman inequality (e.g., see [5, Chap. 3]), implies that

$$\|Z(t,s,\varepsilon)\| \le c_1 e^{c_2(t-s)} \le \text{const}, \qquad 0 \le s \le t \le T, \qquad \varepsilon > 0,$$

and hence the matrix $Y(t,s,\varepsilon) = \Phi(t)Z(t,s,\varepsilon)$ is uniformly bounded for $0 \le s \le t \le T$ and $\varepsilon > 0$. The proof of the lemma is complete.

Lemma 3. *Let conditions* 1 *and* 2 *be satisfied. Then the integro-differential system*

$$\varepsilon\frac{dv}{dt}-(A(t)+D(t,\varepsilon))v-\varepsilon\int_0^1\exp\left(\frac{1}{\varepsilon}\int_s^1\mu(\theta)\,d\theta\right)K(t,s)v(s,\varepsilon)\,ds=g(t,\varepsilon),\quad v(0,\varepsilon)=0,\tag{24}$$

where the matrix $D(t,\varepsilon)$ *satisfies the assumptions of Lemma* 2, *is uniquely solvable in the space* $C^1([0,1],\mathbb{C}^n)$ *for any right-hand side* $g(t,\varepsilon)\in C([0,1],\mathbb{C}^n)$, *and its solution* $v(t,\varepsilon)$ *satisfies the estimate*

$$\|v(t,\varepsilon)\|_{C[0,1]}\le\frac{\bar{c}_0}{\varepsilon}\|g(t,\varepsilon)\|_{C[0,1]}.\tag{25}$$

Proof. By using the normalized fundamental solution matrix $Y(t,s,\varepsilon)$, we construct the integral system

$$v(t,\varepsilon)=\frac{1}{\varepsilon}\int_0^t Y(t,x,\varepsilon)\left(\int_0^1\exp\left(\frac{1}{\varepsilon}\int_s^1\mu(\theta)\,d\theta\right)K(x,s)v(s,\varepsilon)\,ds\right)dx+\frac{1}{\varepsilon}\int_0^t Y(t,x,\varepsilon)g(x,\varepsilon)\,dx,$$

which is equivalent to system (24). By changing the order of integration in the repeated integral, we obtain

$$v(t,\varepsilon)=\int_0^1\exp\left(\frac{1}{\varepsilon}\int_s^1\mu(\theta)\,d\theta\right)G(t,s,\varepsilon)v(s,\varepsilon)\,ds+\frac{1}{\varepsilon}\int_0^t Y(t,x,\varepsilon)g(x,\varepsilon)\,dx\tag{26}$$

with the kernel $G(t,s,\varepsilon)=\varepsilon^{-1}\int_0^t Y(t,x,\varepsilon)K(x,s)\,dx$. Following [3], one can show that the matrix $G(t,s,\varepsilon)$ is uniformly bounded for $0\le s,t\le 1$ and $\varepsilon\in(0,\varepsilon_0]$, and then (since $\operatorname{Re}\mu(t)<0$, $t\in[0,1]$) one finds that the resolvent $R(t,s,\varepsilon)$ of the kernel

$$\tilde{K}(t,s,\varepsilon)=\int_0^1\exp\left(\frac{1}{\varepsilon}\int_s^1\mu(\theta)\,d\theta\right)G(t,s,\varepsilon)z(s,\varepsilon)\,ds$$

of the integral system (26) is uniformly bounded; i.e., $\|R(t,s,\varepsilon)\|\le M_0$ for $0\le s,t\le 1$ and $0<\varepsilon\le\varepsilon_0$, where $\varepsilon_0>0$ is sufficiently small (see [3]). Hence it follows that the integral system (26) has the unique solution

$$v(t,\varepsilon)=\frac{1}{\varepsilon}\int_0^t Y(t,x,\varepsilon)g(x,\varepsilon)\,dx+\frac{1}{\varepsilon}\int_0^1 R(t,s,\varepsilon)\left(\int_0^s Y(s,x,\varepsilon)g(x,\varepsilon)\,dx\right)ds,$$

which belongs to the space $C([0,1],\mathbb{C}^n)$ for any right-hand side $\varepsilon^{-1}\int_0^t Y(t,x,\varepsilon)g(x,\varepsilon)\,dx$ ($g(t,\varepsilon)\in C([0,1],\mathbb{C}^n)$). Therefore, the integro-differential system (24) is uniquely solvable in the space $C^1([0,1],\mathbb{C}^n)$, and the estimate (25) holds. The proof of the lemma is complete.

Now let us proceed to the proof of the main assertion.

Theorem 4. *Let conditions* 1–4 *be satisfied, and let problem* (20) *be solvable on the interval* $[0,1]$. *Then system* (2) *with* $\varepsilon\in(0,\varepsilon_0]$ (*where* $\varepsilon_0>0$ *is sufficiently small*) *has a unique solution* $y(t,\varepsilon)\in C^1([0,T],\mathbb{C}^n)$, *and the following estimate holds*:

$$\|y(t,\varepsilon)-y_{\varepsilon l}(t)\|_{C[0,T]}\le C_l\varepsilon^{l+1}\qquad(l=0,1,\ldots),\tag{27}$$

where $C_l>0$ *is a constant independent of* $\varepsilon\in(0,\varepsilon_0]$.

Proof. In Theorem 3, for the operator $P_\varepsilon(u)$ we take the operator

$$P_\varepsilon(u) \equiv \varepsilon \frac{du}{dt} - A(t)u - \varepsilon \int_0^1 \exp\left\{\frac{1}{\varepsilon}\int_s^1 \mu(\theta)\,d\theta\right\} K(t,s)u(s,\varepsilon)\,ds - \varepsilon f(u+y^0,t)$$
$$- A(t)y^0 - \int_0^1 \exp\left\{\frac{1}{\varepsilon}\int_s^1 \mu(\theta)\,d\theta\right\} K(t,s)y^0\,ds.$$

This operator acts from the Banach space

$$B_1 = \dot{C}^1([0,T],\mathbb{C}^n) = \{v(t) \in C^1([0,T],\mathbb{C}^n),\ v(0)=0\}$$

to the Banach space $B_2 = C[0,1]$ with norms

$$\|v(t)\|_{B_2} = \max_{0\le t\le 1} |v(t)|, \qquad \|v(t)\|_{B_1} = \|v(t)\|_{B_2} + \|\dot{v}(t)\|_{B_2}.$$

For the initial approximation, we take the function $u_0 = y_{\varepsilon l}(t) - y^0$. By (21), we have $P_\varepsilon(u_0) = \varepsilon^{l+1}F_l(t,\varepsilon)$, $\|F_l(t,\varepsilon)\|_{C[0,1]} \le \bar{F}_l$. Therefore, assumption $(2')$ in Theorem 3 holds for $m = N+1$. Since $f(y,t)$ is a polynomial in y with smooth coefficients on $[0,1]$, it follows that assumption $(3')$ in Theorem 3 is satisfied as well. It remains to verify assumption $(1')$. The operator $P'_\varepsilon(u_0)$ has the form

$$P'_\varepsilon(u_0)v \equiv \varepsilon \frac{dv}{dt} - A(t)v - \varepsilon \int_0^1 \exp\left\{\frac{1}{\varepsilon}\int_s^1 \mu(\theta)\,d\theta\right\} K(t,s)v(s,\varepsilon)\,ds - \varepsilon \frac{\partial f(y_{\varepsilon l}(t),t)}{\partial y} v;$$

therefore, to estimate the norm $\|\Gamma_\varepsilon\| \equiv \|(P'_\varepsilon(u_0)^{-1})\|$, one should estimate the norm of the solution of the equation

$$P'_\varepsilon(u_0)v = g(t,\varepsilon), \qquad v(0,\varepsilon) = 0, \tag{28}$$

for an arbitrary element $g(t,\varepsilon) \in C[0,1]$ for each $\varepsilon > 0$. By Lemma 3, where $D(t,\varepsilon) \equiv \dfrac{\partial f(y_{\varepsilon l}(t),t)}{\partial y}$, system (28) is uniquely solvable in the space B_1, and the following estimate holds:

$$\|v(t,\varepsilon)\|_{C[0,1]} \le \frac{\bar{c}_0}{\varepsilon}\|g(t,\varepsilon)\|_{C[0,1]}. \tag{29}$$

By using Eq. (28), we obtain

$$\varepsilon \left\|\frac{dv}{dt}\right\|_{C[0,1]} \le \|A(t)\|_{C[0,1]}\|v\|_{C[0,1]} + \varepsilon k_0\|v\|_{C[0,1]} + \varepsilon \left\|\frac{\partial f(y_{\varepsilon l}(t),t)}{\partial y}\right\|_{C[0,1]} \|v\|_{C[0,1]},$$

whence it follows that the inequality

$$\|\dot{v}(t,\varepsilon)\|_{C[0,T]} \le \frac{k_1}{\varepsilon}\|v(t,\varepsilon)\|_{C[0,T]} \le \frac{\widehat{v}}{\varepsilon^2} k_1 \|g(t,\varepsilon)\|_{C[0,T]}$$

holds for sufficiently small $\varepsilon > 0$. Then

$$\|\Gamma_\varepsilon g\|_{\dot{C}[0,1]} \equiv \|v(t,\varepsilon)\|_{\dot{C}[0,1]} = \|v(t,\varepsilon)\|_{C[0,1]} + \|\dot{v}(t,\varepsilon)\|_{C[0,1]} \le \frac{c_2}{\varepsilon^2}\|g(t,\varepsilon)\|_{C[0,1]},$$

and it follows that $\|\Gamma_\varepsilon\| \equiv \|(P'_\varepsilon(u_0)^{-1})\| \le c_2/\varepsilon^2$; therefore, assumption $(1')$ in Theorem 3 is satisfied for $k = 2$. Consequently, Eq. (22) has a unique solution $u_*(t,\varepsilon) \in B_1$, and it satisfies the estimate $\|u_* - u_0\|_{B_1} \le c_{N-1}\varepsilon^{N-1}$ $(N > 3)$; therefore, the original problem (2) has a unique solution $y(t,\varepsilon) = u_*(t,\varepsilon) + y^0 \in C^1([0,T],\mathbb{C}^n)$ such that

$$\|y(t,\varepsilon) - y_{\varepsilon l}(t)\|_{C[0,1]} \le C_{l-1}\varepsilon^{l-1} \qquad (l = 4, 5, \ldots).$$

By writing out this inequality for the partial sum $y_{\varepsilon,l+2}(t)$ and by using the inequality $\|a - b\| \ge \|a\| - \|b\|$ and the uniform boundedness of the coefficients of this partial sum, we obtain the estimate (27). The proof of the theorem is complete.

5. PASSAGE TO THE LIMIT IN PROBLEM (2). SOLUTION OF THE INITIALIZATION PROBLEM

If the assumptions of Theorem 4 are satisfied, then the exact solution of problem (2) can be represented in the form

$$y(t,\varepsilon) = y_{\varepsilon 0}(t) + \varepsilon F_0(t,\varepsilon), \tag{30}$$

where $\|F_0(t,\varepsilon)\|_{C[0,1]} \le \bar{F}_0$, $\bar{F}_0$ is a constant independent of ε for $\varepsilon \in (0,\varepsilon_0]$, $\varepsilon_0 > 0$ is sufficiently small, $y_{\varepsilon 0}(t)$ has the form [see relations (18) and (20)]

$$y_0(t,\tau) = \sum_{j=1}^{n} \alpha_j^{(0)}(t)\varphi_j(t)\exp\left\{\frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)\,d\theta\right\} + y_0^{(0)}(t), \qquad y_0^{(0)}(t) = -A^{-1}(t)h(t), \tag{31}$$

and the functions $\alpha_j^{(0)}(t)$ satisfy problem (20). It follows from the representation (30) that if the spectrum $\sigma(A(t))$ of the matrix $A(t)$ lies to the left of the imaginary axis ($\operatorname{Re}\lambda_j(t) < 0$, $t \in [0,1]$, $j = 1,\ldots,n$), then one has the uniform convergence

$$\|y(t,\varepsilon) - y_0^{(0)}(t)\|_{C[\delta,1]} \to 0 \qquad (\varepsilon \to +0). \tag{32}$$

[Here δ is an arbitrary constant in the interval $(0,1)$.] But if the spectrum $\sigma(A(t))$ contains pure imaginary numbers, then the passage to the limit (32) in the strong sense cannot be carried out in the general case. Therefore, in this case, one usually considers the following *initialization problem*: single out a class $\Sigma = \{y^0, h(t), K(t,s)\}$ of original data of problem (2) for which the uniform convergence (as $\varepsilon \to +0$) of the exact solution $y(t,\varepsilon)$ of the considered problem to some limit function $\bar{\bar{y}}(t)$ on the entire interval $[0,1]$ is guaranteed. Let us study this problem.

It follows from relation (31) that the uniform convergence $y(t,\varepsilon) \to y_0^{(0)}(t)$ $(\varepsilon \to +0)$ on the entire interval $[0,1]$ is guaranteed if the functions $\alpha_j^{(0)}(t)$ satisfying problem (20) are identically zero. Since the vector function $f^{e_j}(\alpha_1^{(0)},\ldots,\alpha_n^{(0)},t)$ has the form (see [2, pp. 242–243])

$$f^{e_j}(\alpha_1^{(0)},\ldots,\alpha_n^{(0)},t) = \tilde{f}^{e_j}(t)\alpha_j + \sum_{\substack{|m^j|\ge 2\\ (m^j,\lambda(t))\equiv\lambda_j(t)}} \tilde{f}^{m^j}(t)(\alpha_1^{(0)})^{m_1^j}\cdots(\alpha_n^{(0)})^{m_n^j},$$

where $m^j = (m_1^j,\ldots,m_n^j,0)$ is a multi-index, it follows that problem (20) has the zero solution if and only if

$$(y^0 + A^{-1}(0)h(0), \chi_j(0)) = 0, \qquad j = 1,\ldots,n. \tag{33}$$

We have thereby proved the following assertion.

Theorem 5. *Let conditions* 1–4 *be satisfied. Then the convergence*

$$\|y(t,\varepsilon) - \bar{\bar{y}}(t)\|_{C[0,1]} \to 0 \qquad (\varepsilon \to +0) \tag{34}$$

takes place if and only if relations (33) *are satisfied. In this case,* $\bar{\bar{y}}(t) \equiv y_0^{(0)}(t)$.

Remark. Relations (33) imply that the initialization class $\Sigma = \{y^0, h(t), K(t,s)\}$ is independent of the kernel $K(t,s)$, and the initial vector y^0 coincides with the limit solution $\bar{\bar{y}}(t) = -A^{-1}(t)h(t)$ at the initial time $t = 0$. Finally, note that a survey of the main results on integro-differential equations can be found in [6, pp. 402–410].

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