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ORDINARY DIFFERENTIAL EQUATIONS

Equiconvergence Theorems for Singular Sturm–Liouville Operators with Various Boundary Conditions

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Abstract—We consider the Sturm–Liouville operator $L(y) = -d^2y/dx^2 + q(x)y$ in the space $L_2[0,\pi]$, where the potential $q(x)$ is a complex-valued distribution of the first order of singularity; namely, $q(x) = u'(x)$, where $u \in L_2[0,\pi]$. (The derivative is understood in the sense of distributions.) We study the uniform equiconvergence on the entire interval $[0,\pi]$ of the expansions of a function $f \in L_2$ in the system of eigenfunctions and associated functions of the operator L with the Fourier trigonometric series expansion. We also estimate the equiconvergence rate.

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In the present paper, we study the Sturm–Liouville operator generated in the space $L_2[0,\pi]$ by the differential expression

$$l(y) = -y'' + q(x)y \tag{1}$$

and the boundary conditions presented below. We assume that the potential $q(x)$ is a complex-valued distribution of the first order of singularity; namely, $q(x) = u'(x)$, where $u \in L_2[0,\pi]$, where the derivative is understood in the sense of distributions. (The definition and properties of such operators are described in detail in [1–3].) We study the uniform equiconvergence on $[0,\pi]$ of the expansions of a function $f \in L_2$ in the system of eigenfunctions and associated functions of L with the Fourier trigonometric series expansion.

The uniform equiconvergence was proved in [4] for the case of the Dirichlet boundary conditions $y(0) = 0$ and $y(\pi) = 0$. In the present paper, we study the remaining types of separated boundary conditions. We not only solve the equiconvergence problem but also estimate the equiconvergence rate. A detailed history of this problem can be found in [5].

Consider all types of separated boundary conditions starting from the case of the Dirichlet–Neumann boundary conditions $y(0) = 0$, $y^{[1]}(\pi) = 0$, where

$$y^{[1]}(x) = y'(x) - u(x)y(x)$$

is the first quasiderivative (see [2, Sec. 1]). Here we need the asymptotics of the eigenfunctions of L, which was obtained in Theorem 1 in [6]. We rearrange the terms in this asymptotics as follows:

$$y_n(x) = \sqrt{2/\pi}\sin(mx)+\phi_n(x), \quad w_n(x) = \sqrt{2/\pi}\sin(mx)+\psi_n(x), \quad m = n-1/2, \quad n \ge N_u; \tag{2}$$

here $\phi_n(x)$ and $\psi_n(x)$ are functions such that

$$\sup_{0\le x\le\pi} \|\{\phi_n(x)\}\|_{l_2} \le C_u, \qquad \sup_{0\le x\le\pi} \|\{\psi_n(x)\}\|_{l_2} \le C_u. \tag{3}$$

In addition, the sequences $\{\|\phi_n(x)\|_{L_2}\} = \{\Phi_n\}$ and $\{\|\psi_n(x)\|_{L_2}\} = \{\Psi_n\}$ belong to the space l_2, and one has the inequalities

$$\|\Phi_n\|_{l_2} \le C_u, \qquad \|\Psi_n\|_{l_2} \le C_u. \tag{4}$$

By α_n we denote the following sum of two integrals:

$$\alpha_n = \sqrt{\frac{2}{\pi}}\frac{1}{\pi}\int_0^\pi (\pi - t)(u_R(t) + 2iu_I(t))\cos(2mt)\,dt$$
$$+ \sqrt{\frac{2}{\pi}}\frac{1}{2m\pi}\int_0^\pi (\pi - t)(u_R^2(t) - u_I^2(t) + 4iu_R(t)u_I(t))\sin(2mt)\,dt.$$

Then one can represent the function $\psi_n(x)$ in the form

$$\psi_n(x) = \psi_{n,0} + \psi_{n,1} + \psi_{n,2},$$

where

$$\psi_{n,0} = \alpha_n \sin(mx) - \sqrt{\frac{2}{\pi}}\int_0^x \overline{u}(t)\sin(m(x-2t))\,dt,$$

$$\psi_{n,1}(x) = \sqrt{\frac{2}{\pi}}\sin(mx)\left(-\frac{1}{2m}\int_0^x \overline{u}^2(t)\sin(2mt)\,dt\right) + \sqrt{\frac{2}{\pi}}\cos(mx)\left(-\frac{x}{\pi}\int_0^\pi \overline{u}(t)\sin(2mt)\,dt\right.$$
$$+ \frac{x}{2m\pi}\int_0^\pi \overline{u}^2(t)(\cos(2mt) - 1)\,dt - \frac{2x}{\pi}\int_0^\pi\int_0^t \overline{u}(t)\overline{u}(s)\cos(2mt)\sin(2ms)\,ds\,dt$$
$$\left. + \frac{1}{2m}\int_0^x \overline{u}^2(t)(1 - \cos(2mt))\,dt + 2\int_0^x\int_0^t \overline{u}(t)\overline{u}(s)\cos(2mt)\sin(2ms)\,ds\,dt\right), \qquad (5)$$
$$\sup_{0\le x\le\pi}\|\{\psi_{n,2}(x)\}\|_{l_2} \le C_u, \qquad \{\|\psi_{n,2}(x)\|_{L_2}\} = \{\Psi_{n,2}\} \in l_2, \qquad \|\Psi_{n,2}\|_{l_2} \le C_u.$$

Theorem 1. *Let the operator L be generated by the differential expression $-y'' + q(x)y$, where $q(x) = u'(x)$ in the sense of distributions and the complex-valued function $u(x)$ belongs to $L_2[0,\pi]$, and by the Dirichlet–Neumann boundary conditions $y(0) = 0$, $y^{[1]}(\pi) = 0$. Next, let $\{y_n(x)\}_{n=1}^\infty$ be a system of eigenfunctions and associated functions of L with $\|y_n(x)\|_{L_2} = 1$, and let $\{w_n(x)\}_{n=1}^\infty$ be the biorthogonal system. Then the uniform equiconvergence of the expansion of a function f in the system $\{y_n(x)\}_{n=1}^\infty$ and in the sine system holds on the entire interval $[0,\pi]$, and the equiconvergence rate can be characterized as follows:*

$$\left\|\sum_{n=1}^l c_n y_n(x) - \sum_{n=1}^l \sqrt{\frac{2}{\pi}}c_{n,0}\sin((n-1/2)x)\right\|_C$$
$$\le C_u\left(\sum_{n\ge l^{1/2-\varepsilon}} |c_{n,0}|^2\right)^{1/2} + \|f\|_{L_2}(v_u([l^{1/2-\varepsilon}]) + C_u l^{-\varepsilon}), \qquad (6)$$

where $c_n = (f(x), w_n(x))$, $c_{n,0} = \sqrt{2/\pi}(f(x), \sin((n-1/2)x))$, $\varepsilon \in (0, 1/2)$ is an arbitrary small positive number, $l^{1/2-\varepsilon} > N_u$, and

$$v_u(k) = C_u\left(\left(\sum_{n\ge k}\|\psi_n(x)\|_{L_2}^2\right)^{1/2} + \left(\sum_{n\ge k}\|\phi_n(x)\|_{L_2}^2\right)^{1/2}\right), \qquad k \ge N_u. \qquad (7)$$

Proof. Consider the operators $B_{l,N}: L_2[0,\pi]\to C[0,\pi]$ acting on functions $f\in L_2[0,\pi]$ by the following rule:

$$(B_{l,N})f(x)=\sum_{n=N}^{l}c_n y_n(x)-\sum_{n=N}^{l}\sqrt{\frac{2}{\pi}}\,c_{n,0}\sin((n-1/2)x),\qquad N=N_u,\qquad l\ge N. \tag{8}$$

Set $m=n-1/2$, $n=1,2,\ldots$ Then, by Theorem 2 in [6], every function $f\in L_2[0,\pi]$ admits the representation

$$\begin{aligned}(B_{l,N})f(x)&=\sum_{n=N}^{l}\sqrt{\frac{2}{\pi}}(f(t),\psi_{n,0}(t))\sin(mx)+\sum_{n=N}^{l}\sqrt{\frac{2}{\pi}}(f(t),\psi_{n,1}(t))\sin(mx)\\&\quad+\sum_{n=N}^{l}\sqrt{\frac{2}{\pi}}(f(t),\psi_{n,2}(t))\sin(mx)+\sum_{n=N}^{l}\sqrt{\frac{2}{\pi}}(f(t),\sin(mx))\phi_n(x)\\&\quad+\sum_{n=N}^{l}\sqrt{\frac{2}{\pi}}(f(t),\psi_n(t))\phi_n(x).\end{aligned} \tag{9}$$

Let us estimate the norm of the operator $B_{l,N}$. To simplify the notation, for $N=1$, we set $B_{l,1}=B_l$.

Let us successively consider the terms on the right-hand side in relation (9). The most complicated thing is to estimate the first term. We have the inequality

$$\left\|\sum_{n=N}^{l}\sqrt{\frac{2}{\pi}}\,\overline{\alpha}_n(f(t),\sin(mx))\sin(mx)\right\|_C\le C_u\|f\|_{L_2}.$$

Now one should estimate the expression

$$\sum_{n=N}^{l}\int_0^{\pi}f(t)\int_0^{t}u(s)\sin(m(t-2s))\,ds\,dt\sin(mx). \tag{10}$$

To this end, we transform the product of sines,

$$\sin(m(t-2s))\sin(mx)=(1/2)(\cos(m(t-2s-x))-\cos(m(t-2s+x))),$$

and substitute the result into the expression (10),

$$\begin{aligned}&\sum_{n=N}^{l}\int_0^{\pi}f(t)\int_0^{t}u(s)\sin(m(t-2s))\,ds\,dt\sin(mx)\\&\qquad=\frac12\sum_{n=N}^{l}\int_0^{\pi}f(t)\int_0^{t}u(s)\cos(m(t-2s-x))\,ds\,dt\\&\qquad\quad-\frac12\sum_{n=N}^{l}\int_0^{\pi}f(t)\int_0^{t}u(s)\cos(m(t-2s+x))\,ds\,dt\equiv S_1+S_2.\end{aligned} \tag{11}$$

Consider the first term on the right-hand side in relation (11) in detail (the second one can be considered in a similar way):

$$\begin{aligned}S_1&=\frac12\sum_{n=N}^{l}\int_0^{\pi}f(t)\int_0^{t}u(s)\cos(n(t-2s-x))\cos\left(\frac{t}{2}-s-\frac{x}{2}\right)ds\,dt\\&\quad+\frac12\sum_{n=N}^{l}\int_0^{\pi}f(t)\int_0^{t}u(s)\sin(n(t-2s-x))\sin\left(\frac{t}{2}-s-\frac{x}{2}\right)ds\,dt\equiv S_{11}+S_{12}.\end{aligned} \tag{12}$$

Then

$$S_{11} = \int\limits_0^\pi f(t) \int\limits_0^t u(s) D_l(t - x - 2s) \cos\left(\frac{t}{2} - s - \frac{x}{2}\right) ds\, dt$$
$$- \int\limits_0^\pi f(t) \int\limits_0^t u(s) D_{N-1}(t - x - 2s) \cos\left(\frac{t}{2} - s - \frac{x}{2}\right) ds\, dt \equiv S_{11}^{(1)} + S_{11}^{(2)}. \qquad (13)$$

Here $D_l(\xi) = 1/2 + \sum_{n=1}^{l} \cos(n\xi)$ is the Dirichlet kernel. Since t and s belong to the interval $[0, \pi]$, it follows that the second term on the right-hand side in relation (13) can be estimated as

$$S_{11}^{(2)} \le C_u \|f\|_{L_2}.$$

Now let us estimate the term $S_{11}^{(1)}$ in (13). To this end, we define an operator $A_{l,-x}$ on the space $L_2[0, \pi]$ by the rule

$$(A_{l,-x} u)(t) = \int\limits_0^t u(s) D_l(t - 2s - x)\, ds.$$

It was shown in [4] that $\|A_{l,-x}\|_{L_2} \le C_u$, which implies an estimate for the term S_{11} in (12). Since the term S_{12} is bounded by $C_u \|f\|_{L_2}$ as well, we have the estimate

$$\left\| \sum_{n=N}^{l} \sqrt{2/\pi}\, (f(t), \psi_{n,0}(t)) \sin(mx) \right\| \le C_u \|f\|_{L_2}.$$

We return to the main relation (9). Let us verify that

$$\left\| \sqrt{\frac{2}{\pi}} \sum_{n=N}^{l} (f(t), \psi_{n,1}(t)) \sin(mx) \right\|_{L_2} \le C_u \|f\|_{L_2}.$$

Let us use the asymptotic formulas obtained above for the function $\psi_{n,1}(x)$. We have

$$(f(t), \psi_{n,1}(t)) = -\frac{1}{2m} \int\limits_0^\pi f(t) \sin(mt) \int\limits_0^t u^2(s) \sin(2ms)\, ds\, dt$$
$$- \int\limits_0^\pi f(t) \cos(mt) \frac{t}{\pi} \int\limits_0^\pi u(s) \sin(2ms)\, ds\, dt + \int\limits_0^\pi f(t) \cos(mt) \frac{t}{2m\pi} \int\limits_0^\pi u^2(s)(\cos(2ms) - 1)\, ds\, dt$$
$$- \int\limits_0^\pi f(t) \cos(mt) \frac{2t}{\pi} \int\limits_0^\pi \int\limits_0^s u(s) u(\tau) \cos(2ms) \sin(2m\tau)\, d\tau\, ds\, dt$$
$$+ \frac{1}{2m} \int\limits_0^\pi f(t) \cos(mt) \int\limits_0^t u^2(s)(1 - \cos(2ms))\, ds\, dt$$
$$- \int\limits_0^\pi \frac{2t f(t)}{\pi} \cos(mt) \int\limits_0^\pi \int\limits_0^s u(s) u(\tau) \cos(2ms) \sin(2m\tau)\, d\tau\, ds\, dt$$
$$+ 2 \int\limits_0^\pi f(t) \cos(mt) \int\limits_0^t \int\limits_0^s u(s) u(\tau) \cos(2ms) \sin(2m\tau)\, d\tau\, ds\, dt \equiv \sum_{i=1}^{7} J_i. \qquad (14)$$

Let us successively estimate all terms in relation (14). For the first term, we have

$$\left|\sum_{n=N}^{l}\sin(mx)J_1\right| \le \int_0^\pi |u(s)|^2 \left|\sum_{n=N}^{l}\frac{\sin(mx)}{2m}\sin(mx)\int_0^\pi f(t)\sin(mt)\,dt\right| ds$$
$$\le \int_0^\pi |u(s)|^2 \sum_{n=N}^{l}\frac{1}{2m}\left|\int_0^\pi f(t)\sin(mt)\,dt\right| ds \le C_u\|f\|_{L_2}, \qquad (15)$$

where the last inequality holds because $\{\int_0^\pi f(t)\sin(mt)\,dt\}_{n=N}^\infty$ belongs to l_2 and $\|H_s f\|_{L_2} \le \|f\|_{L_2}$, where H_s is the cutoff operator in the space $L_2[0,\pi]$.

For the second term in (14), we have

$$\left|\sum_{n=N}^{l} J_2\sin(mx)\right|$$
$$\le \frac{1}{\pi}\left(\sum_{n=N}^{l}\left|\int_0^\pi tf(t)\cos(mt)\,dt\right|^2\right)^{1/2}\left(\sum_{n=N}^{l}\left|\int_0^\pi u(s)\sin(2ms)\,ds\right|^2\right)^{1/2} \le C_u\|f\|_{L_2}. \qquad (16)$$

The third term in (14) can be estimated as

$$\left|\sum_{n=N}^{l} J_3\sin(mx)\right| \le C_u\left(\sum_{n=N}^{l}\frac{1}{m^2}\right)^{1/2}\left(\sum_{n=N}^{l}\left|\int_0^\pi tf(t)\cos(mt)\,dt\right|^2\right)^{1/2} \le C_u\|f\|_{L_2}. \qquad (17)$$

The fourth term can be estimated by analogy with the first one.

To estimate the fifth term, we set $U(t) = \int_0^t u^2(s)\,dt$ (this function is absolutely continuous) and integrate $u^2(s)$. We have

$$\left|\sum_{n=N}^{l}\frac{\sin(mx)}{2m}\int_0^\pi f(t)\cos(mt)\int_0^t u^2(s)\,ds\,dt\right| \le \sum_{n=N}^{l}\frac{1}{2m}\left|\int_0^\pi f(t)U(t)\cos(mt)\,dt\right|$$
$$\le C\left(\sum_{n=N}^{l}\left|\int_0^\pi f(t)U(t)\cos(mt)\,dt\right|^2\right)^{1/2} \le C\|f(t)U(t)\|_{L_2} \le C_u\|f\|_{L_2}. \qquad (18)$$

To estimate the sixth term, we introduce a function $\xi(s,\tau)$ such that $\xi(s,\tau) = 1$ if $\tau \le s$ and $\xi(s,\tau) = 0$ otherwise. Then $u(s)u(\tau)\xi(s,\tau) \in L_2[0,\pi][0,\pi]$. We find that the following inequality holds for $x \in [0,\pi]$:

$$\left|\sum_{n=N}^{l} J_6\sin(mx)\right|$$
$$\le \frac{2}{\pi}\left(\sum_{n=N}^{l}\left|\int_0^\pi tf(t)\cos(mt)\,dt\right|^2\right)^{1/2}\left(\sum_{n=N}^{l}\left|\int_0^\pi\int_0^s u(s)u(\tau)\cos(2ms)\sin(2m\tau)\,d\tau\,ds\right|^2\right)^{1/2}$$
$$\le C\|f\|_{L_2}\left(\sum_{n=N}^{l}\left|\int_0^\pi\int_0^\pi u(s)u(\tau)\xi(s,\tau)\cos(2ms)\sin(2m\tau)\,d\tau\,ds\right|^2\right)^{1/2} \le C_u\|f\|_{L_2}. \qquad (19)$$

Let us proceed to the last term in relation (14). By $\widetilde{H}_s$ we denote the cutoff operator $\widetilde{H}_s f(t) = \xi_{[s,\pi]} f(t)$. Then for each $x \in [0,\pi]$, we have

$$\left|\sum_{n=N}^{l} \frac{1}{2} J_7 \sin(mx)\right| = \left|\int_0^\pi u(s) \sum_{n=N}^{l} \cos(2ms)\sin(mx)(\widetilde{H}_s f(t), \cos(mt))(H_s u(\tau), \sin(2m\tau))\,ds\right|$$
$$\le \int_0^\pi |u(s)| \left|\sum_{n=N}^{l} (\widetilde{H}_s f(t), \cos(mt))(H_s u(\tau), \sin(2m\tau))\right| ds$$
$$\le C \int_0^\pi |u(s)|\, \|\widetilde{H}_s f(t)\|_{L_2} \, \|H_s u\|_{L_2} \le C_u \|f\|_{L_2}. \tag{20}$$

By taking into account the estimates (15)–(20), we obtain the inequality

$$\left\|\sqrt{\frac{2}{\pi}} \sum_{n=N}^{l} (f(t), \psi_{n,1}(t)) \sin(mx)\right\|_{L_2} \le C_u \|f\|_{L_2}. \tag{21}$$

Let us continue to estimate terms in the main relation (9).

An estimate for the third term can be obtained with regard to relation (5) in a rather simple way,

$$\sup_{0\le x\le\pi} \left|\sum_{n=N}^{l} \sqrt{\frac{2}{\pi}} (f(t), \psi_{n,2}(t)) \sin(mx)\right| \le \sqrt{\frac{2}{\pi}} \sum_{n=N}^{l} \left|\int_0^\pi f(t)\overline{\psi_{n,2}(t)}\,dt\right|$$
$$\le \sqrt{\frac{2}{\pi}} \sum_{n=1}^{l} \|f\|_{L_2} \int_0^\pi |\psi_{n,2}(t)|^2\,dt \le C_u \|f\|_{L_2}. \tag{22}$$

By virtue of the estimate (4), the fourth term in (9) satisfies the inequality

$$\left\|\sum_{n=N}^{l} \sqrt{\frac{2}{\pi}} (f(t), \sin(mt)) \phi_n(x)\right\|_C \le \sqrt{\frac{2}{\pi}} \sup_{0\le x\le\pi} \sum_{n=N}^{l} (|(f(t), \sin(mt))|\, |\phi_n(x)|) \le C_u \|f\|_{L_2}. \tag{23}$$

By virtue of the estimate (3), for the last term in (9), we have

$$\left\|\sum_{n=N}^{l} \sqrt{\frac{2}{\pi}} (f(t), \psi_n(t)) \phi_n(x)\right\|_C \le \sup_{0\le x\le\pi} \left|\sum_{n=N}^{l} \sqrt{\frac{2}{\pi}} \|f(t)\|_{L_2} \, \|\psi_n(t)\|_{L_2} \, \phi_n(x)\right|$$
$$\le \sqrt{\frac{2}{\pi}} \|f\|_{L_2} \sup_{0\le x\le\pi} \left|\sum_{n=N}^{l} \|\psi_n(t)\|_{L_2} \phi_n(x)\right| \le C_u \|f\|_{L_2}. \tag{24}$$

As a result, we have obtained an estimate for the left-hand side of the main relation (9). The operator B_l can be represented as the sum $B_l = B_{N-1} + B_{l,N}$, and since $\|B_{N-1}\|_{L_2\to C} \le C_u$ (a sum of finitely many terms), we have $\|B_l\|_{L_2\to C} \le C_u$.

We have thereby proved the uniform boundedness of the operators B_l for each function $u \in L_2[0,\pi]$.

The operator B_l acts on the eigenfunctions and associated functions of the operator L as follows:

$$(B_l y_k)(x) = \sum_{n=1}^{l} (y_k(x), w_n(x)) y_n(x) - \frac{2}{\pi} \sum_{n=1}^{l} (y_k(x), \sin(mx)) \sin(mx). \tag{25}$$

The first term in (25) is zero for $m < k$ and is equal to $y_k(x)$ for $m \geq k$. The second term is a partial sum of the Fourier series of the function y_k. Since $y_k \in W_2^1[0,\pi]$, it follows that the Fourier series of y_k uniformly converges to y_k on the interval $[0,\pi]$, and we obtain

$$\lim_{l\to\infty} \|B_l y_k\|_C = 0. \tag{26}$$

In what follows, we use the auxiliary Theorem 2.7 in [1].

Let $u \in L_2[0,\pi]$. Then the system $\{y_n(x)\}_{n=1}^{\infty}$ of eigenfunctions and associated functions of L is a Riesz basis in $L_2[0,\pi]$.

Hence it follows that each function $f(x)$ can be approximated by linear combinations of functions of the system $\{y_k(x)\}$ and

$$\lim_{l\to\infty} \|B_l f\|_C = 0. \tag{27}$$

We have thereby proved the equiconvergence of expansions in the system of eigenfunctions and associated functions of L and in the sine system. Let us estimate the equiconvergence rate.

Set $g_k(x) = \sum_{n=1}^{k} c_n y_n(x)$ for each $k \geq N_u$ [where $c_n = (f(x), w_n(x))$]. Then the inequality

$$\|B_l f\|_C \leq \|B_l(f - g_k)\|_C + \|B_l g_k\|_C \tag{28}$$

holds for any function $f \in L_2[0,\pi]$ and any positive integer l.

Let us estimate the norm of the first term on the right-hand side in this inequality in the space $C[0,\pi]$:

$$\|B_l(f - g_k)\|_C \leq C_u \left(\sum_{n=k+1}^{\infty} |c_{n,0}|^2\right)^{1/2} + \|f\|_{L_2} v_u(k+1), \tag{29}$$

where $c_{n,0} = \sqrt{2/\pi}(f(x), \sin(mx))$ and the $v_u(k)$ are the numbers defined in (7).

By taking into account the asymptotic formulas (2), we obtain

$$\begin{aligned}
\|f(x) - g_k(x)\|_{L_2} &\leq \left\| \sum_{n=k+1}^{\infty} \frac{2}{\pi}(f(x), \sin(mx))\sin(mx) \right\|_{L_2} + \left\| \sum_{n=k+1}^{\infty} \frac{2}{\pi}(f(x), \psi_n(x))\sin(mx) \right\|_{L_2} \\
&\quad + \left\| \sum_{n=k+1}^{\infty} \frac{2}{\pi}(f(x), \sin(mx))\phi_n(x) \right\|_{L_2} + \left\| \sum_{n=k+1}^{\infty} \frac{2}{\pi}(f(x), \psi_n(x))\phi_n(x) \right\|_{L_2} \\
&\leq C_u \left(\sum_{n=k+1}^{\infty} |c_{n,0}|^2\right)^{1/2} \\
&\quad + \|f\|_{L_2} \left(\left(\sum_{n=k+1}^{\infty} \|\psi_n\|_{L_2}^2\right)^{1/2} + \left(\sum_{n=k+1}^{\infty} \|\phi_n\|_{L_2}^2\right)^{1/2} + \sum_{n=k+1}^{\infty} \|\psi_n\|_{L_2} \|\phi_n\|_{L_2} \right) \\
&= C_u \left(\sum_{n=k+1}^{\infty} |c_{n,0}|^2\right)^{1/2} + v_u(k+1)\|f\|_{L_2}.
\end{aligned}$$

Since the norm of the operator B_l does not exceed the constant C_u, it follows that inequality (29) holds.

Let us now proceed to estimating the second term in (28). Let $l > k$. By S_l we denote the operator mapping the space $W_2^1[0,\pi]$ into the space $C[0,\pi]$ by the rule

$$S_l h(x) = (2/\pi) \sum_{n=l+1}^{\infty} (h(t), \sin(mt))\sin(mx).$$

All eigenfunctions and associated functions of the operator L belong to the space $W_2^1[0,\pi]$; therefore, the action of the operator S_l on them is well defined, and consequently,

$$B_l g_k(x) = g_k(x) - \frac{2}{\pi}\sum_{n=1}^{l}(g_k(t),\sin(mt))\sin(mx) = S_l g_k(x).$$

Hence it follows that

$$\|B_l g_k\|_C \le \|S_l(g_k - g_N)\|_C + \|S_l g_N\|_C. \tag{30}$$

Let $m > k \ge N_u$. Then

$$\|S_l(g_k - g_N)\|_C = \left\|\sum_{n=N+1}^{k} c_n S_l y_n(x)\right\|_C = \left\|\sum_{n=N+1}^{k} c_n S_l \phi_n(x)\right\|_C,$$

because $y_n(x) = \sqrt{2/\pi}\sin(mx) + \phi_n(x)$ [see (2)]. In what follows, we use the relation

$$\sum_{n=N+1}^{k}|c_n|^2 \le \sum_{n=N+1}^{\infty}|c_n|^2 \le 2\left(\frac{2}{\pi}\sum_{n=N+1}^{\infty}|(f(x),\sin(mx))|^2 + \sum_{n=N+1}^{\infty}|(f(x),\psi_n(x))|^2\right) \le C_u\|f\|_{L_2}^2.$$

Therefore,

$$\begin{aligned}\left\|\sum_{n=N+1}^{k} c_n S_l\phi_n(x)\right\|_C &\le \left(\sum_{n=N+1}^{k}|c_n|^2\right)^{1/2}\left(\sum_{n=N+1}^{k}\|S_l\phi_n(x)\|_C^2\right)^{1/2} \\ &\le C_u\|f\|_{L_2}\left(\sum_{n=N+1}^{k}\|S_l\phi_n(x)\|_C^2\right)^{1/2}.\end{aligned}$$

Let us estimate the norm $\|S_l\phi_n(x)\|_C$ with regard of the relation $\phi_n(0) = 0$,

$$\begin{aligned}\|S_l\phi_n(x)\|_C &\le \frac{2}{\pi}\sum_{j=l+1}^{\infty}|(\phi_n(x),\sin((j+1/2)x))| = \frac{2}{\pi}\sum_{j=l+1}^{\infty}\frac{|(\phi_n(x)',\cos((j+1/2)x))|}{j+1/2} \\ &\le \frac{2}{\pi}\left(\sum_{j=l+1}^{\infty}\frac{1}{(j+1/2)^2}\right)^{1/2}\left(\sum_{j=l+1}^{\infty}|(\phi_n(x)',\cos((j+1/2)x))|^2\right)^{1/2} \\ &\le C_u l^{-1/2}\|\phi_n(x)\|_{W_2^1}.\end{aligned} \tag{31}$$

It follows from [2, Sec. 2] that $\phi_n(x)' = n\eta_n(x) + u(x)y_n(x)$ and $\|\phi_n(x)\|_{W_2^1} \le C_u n\eta_n$, where $\|\{\eta_n\}\|_{l_2} \le C_u$. Hence we obtain the estimate

$$\|S_l\phi_n(x)\|_C \le C_u l^{-1/2} n\eta_n.$$

As a result, the first term on the right-hand side in inequality (30) can be estimated as follows:

$$\|S_l(g_k - g_N)\|_C \le C_u\|f\|_{L_2}\left(\sum_{n=1}^{k} l^{-1}n^2\eta_n^2\right)^{1/2} \le C_u\|f\|_{L_2} l^{-1/2}k.$$

Since the action of the operator S_l on the function g_N can be estimated just as in (31), we have

$$\|S_l g_N(x)\|_C \le C_u l^{-1/2}\|g_N(x)\|_{W_2^1}.$$

The number N is fixed and depends only on an antiderivative of the potential u; therefore,

$$\|g_N(x)\|_{W_2^1} \le C_u\|f\|_{L_2}, \qquad \|S_l g_N\|_C \le C_u l^{-1/2}\|f\|_{L_2}.$$

As a result, we obtain

$$\|B_l g_k\|_C \le \|f\|_{L_2} C_u k l^{-1/2}. \tag{32}$$

Now it suffices to take $k = [l^{1/2-\varepsilon}]$, where $\varepsilon \in (0, 1/2)$ is arbitrary, and it readily follows from relations (28), (29), and (32) that the desired inequality (6) holds. The proof of the theorem is complete.

Let us proceed to the study of the remaining forms of separated boundary conditions. We need the closed-form expression for the asymptotics of eigenfunctions and associated functions obtained in Theorem 1 and Remark 2 in [7].

Theorem 2. *Let the operator L be generated by the differential expression $-y'' + q(x)y$ and the boundary conditions presented below, where $q(x) = u'(x)$ in the sense of distributions and $u(x)$ is a complex-valued function in $L_2[0,\pi]$. Next, let $\{y_n(x)\}_{n=1}^{\infty}$ be the system of eigenfunctions and associated functions of the operator L with $\|y_n(x)\|_{L_2} = 1$, and let $\{w_n(x)\}_{n=1}^{\infty}$ be the biorthogonal system. Then the uniform equiconvergence of the expansions of the function f in the system $\{y_n(x)\}_{n=1}^{\infty}$ and in the system $\{F(mx)\}$ occurs on the entire interval $[0,\pi]$, and the equiconvergence rate can be estimated as follows:*

$$\left\| \sum_{n=1}^{l} c_n y_n(x) - \sum_{n=1}^{l} \sqrt{\frac{2}{\pi}} c_{n,0} F(mx) \right\|_C \le C_u \left(\sum_{n \ge l^{1/2-\varepsilon}} |c_{n,0}|^2 \right)^{1/2} + \|f\|_{L_2} (v_u([l^{1/2-\varepsilon}]) + C_u l^{-\varepsilon}), \tag{33}$$

where $c_n = (f(x), w_n(x))$, $c_{n,0} = \sqrt{2/\pi}(f(x), F(mx))$, $\varepsilon \in (0, 1/2)$ is an arbitrary small positive number, $l^{1/2-\varepsilon} > N_u$, and

$$v_u(k) = C_u \left(\left(\sum_{n \ge k} \|\psi_n\|_{L_2}^2 \right)^{1/2} + \left(\sum_{n \ge k} \|\phi_n\|_{L_2}^2 \right)^{1/2} \right), \qquad k \ge N_u.$$

Here $F(\alpha) = \sin(\alpha)$ and $m \in \mathbb{N}$ in the case of the Dirichlet boundary conditions $[y(0) = 0, y(\pi) = 0]$; $F(\alpha) = \cos(\alpha)$ and $m \in \mathbb{N} \cup 0$ for the Neumann boundary conditions $[y^{[1]}(0) = 0, y^{[1]}(\pi) = 0]$; $F(\alpha) = \sin(\alpha)$ and $m = n - 1/2$, $n \in \mathbb{N}$, for the Dirichlet–Neumann boundary conditions $[y(0) = 0, y^{[1]}(\pi) = 0]$; and $F(\alpha) = \cos(\alpha)$ and $m = n - 1/2$, $n \in \mathbb{N}$, for the Neumann–Dirichlet boundary conditions $[y^{[1]}(0) = 0, y(\pi) = 0]$.

Proof. In the case of the Neumann–Dirichlet and Neumann boundary conditions, the proof is similar to the above-proved Theorem 1 with the replacement of sines by cosines. As a result, in view of [4], the proof of Theorem 2 is complete.

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