ISSN 0012-2661, Differential Equations, 2014, Vol. 50, No. 13, pp. 1695–1717. © Pleiades Publishing, Ltd., 2014.

DYNAMICAL SYSTEMS

Analytical-Numerical Methods of Finding Hidden Oscillations in Multidimensional Dynamical Systems

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Abstract— In nonlinear dynamical systems, attractors can be regarded as self-excited and hidden attractors. Self-excited attractors can be localized numerically by a standard computational procedure, in which after a transient process a trajectory starting from a point of unstable manifold in a neighborhood of equilibrium reaches a state of oscillation, and therefore one can readily identify it. In contrast, for a hidden attractor, the basin of attraction does not intersect with small neighborhoods of equilibria. While classical attractors are self-excited, attractors can therefore be obtained numerically by the standard computational procedure. For localization of hidden attractors, it is necessary to develop special procedures, since there are no similar transient processes leading to such attractors.

In this paper, we propose a new efficient analytical–numerical method for the study of hidden oscillations in multidimensional dynamical systems.

DOI: 10.1134/S0012266114130023

1. INTRODUCTION

The theory of nonlinear oscillations of dynamical systems created in the 1930s was originally clear enough to permit generations of researchers to use it for the solution of problems in various fields of science. In these applications, most of the problems to be studied had a structure such that the existence of oscillatory modes was unquestionable; thus the main effort of researchers was focused on the analysis of properties and the form of such oscillations. In the 1970s the situation changed fundamentally. It became clear that, in addition to orbitally stable cycles and tori, which have a common nature, strange attractors of complicated topological structure can exist in dynamical systems. In the next years, numerous mathematicians were concentrated on the study of the structure of strange attractors, their dimension, conditions for their appearance as a result of a cascade of bifurcations [43–46, 33].

Note that most particular mathematical models of dynamical systems do not admit "qualitative integration" with the use of a pure mathematical analysis. Therefore, numerous results dealing with mechanisms of the generation of attractors, their localization in the phase space, and the evaluation of their characteristics were obtained with the use of computer modeling [46]. The matter is that the attractors of classical Lorenz [24], Rössler [25], and Chua [23] systems as well as the attractors of models of classical automated control systems contain arbitrarily small neighborhoods of unstable equilibria in their attraction domains. Such attractors are *self-excited* in the sense that a computational procedure "issuing" from an arbitrary point of an unstable manifold in a neighborhood of an equilibrium "achieves" an attractor and computes it. Unlike self-excited attractors, *hidden* attractors do not contain equilibria in their attraction domains. The existence of such attractors (embedded orbitally asymptotically stable cycles) is a well-known fact in the case of two-dimensional systems in which they can easily be detected. Other well-known examples of the existence of hidden attractors in multidimensional models of automated control systems are given by counterexamples to the Aizerman and Kalman conjectures [35, 31], where the unique stable-in-small equilibrium co-exists with an orbitally stable cycle [37, 38]. Effectively verified conditions for the existence of hidden orbitally stable cycles in some class of multidimensional systems were obtained in [26, 28].

In 2010, Leonov suggested a new method for finding hidden oscillations in multidimensional dynamical systems, which is based on the use of the method of harmonic linearization, the method of a small parameter, and the method of describing functions in combination with the applied theory of bifurcations [8]. The further development of that method [9–22] permitted one to detect a chaotic hidden attractor in the Chua contour for the first time. The above-mentioned papers excited interest in the study of multidimensional dynamical systems, which either have no equilibrium or have stable-in-small equilibria and simultaneously have orbitally stable cycles, or strange attractors [1–7].

The main idea of finding hidden oscillations of a dynamical system $\dot{x} = f(x)$ used in [9–22] is based on homotopy and implies the following. One considers the one-parameter family of systems

$$\dot{x} = \varphi(x, \varepsilon), \qquad \varepsilon \in [0, 1], \tag{1}$$

such that $\varphi(x, 1) = f(x)$, and, for small values of $\varepsilon > 0$, system (1) has a ready-to-detect self-excited orbitally asymptotically stable cycle. The evolution of that cycle as ε grows up to 1 is traced. The following alternative is possible: either a bifurcation of the attractor destruction takes place for some $\varepsilon \in (0, 1)$, or a hidden attractor of the considered dynamical system is detected for $\varepsilon = 1$.

Obviously, the construction of the function $\varphi = (x, \varepsilon)$ with the above-mentioned properties is a key point in the represented algorithm. A class of systems for which the desired function can be constructed was studied in [9–22], and an algorithm for its construction was suggested.

In the present paper, for the same class of systems, we suggest a similar procedure of finding hidden attractors. In numerous cases, the below-suggested procedure proves to be “less expensive” at the stage of the preparation of the numerical implementation of the algorithm for finding a hidden attractor and, at the same time, permits one, for example, to detect hidden attractors in the classical and generalized Chua systems [23–30], construct a counterexample to the well-known Kalman conjecture [31], and detect hidden oscillations in control systems for aircraft [21, 22, 41, 42].

2. EXISTING CONDITIONS FOR SELF-EXCITING CYCLES IN MULTIDIMENSIONAL SYSTEMS

Consider a system of the form

$$\frac{dx}{dt} = Ax + B\xi, \qquad \xi = \varphi(\sigma), \qquad \sigma = C^*x, \tag{2}$$

where A, B, and C are real constant matrices of the sizes $n \times n$, $n \times m$, and $n \times m$, respectively, $m \le n$, and $x \in R^n$. The sign $(*)$ stands for transposition and below, in the complex case, for Hermitian conjugation. Throughout the following, we assume that $\xi_j = \varphi_j(\sigma_j)$, $j = 1, 2, \ldots, m$, where the $\varphi_j(\sigma_j)$ are continuous functions differentiable for $\sigma_j = 0$.

Below, for the statement of assertions, it is convenient to use the $m \times m$ transfer matrix $W(p) = C^*(A - pI_n)^{-1}B$ of system (2), where p is a complex variable. Throughout the following, we assume that the ranks of the matrices $\|B, AB, \ldots, A^{n-1}B\|$ and $\|C, A^*B, \ldots, (A^*)^{n-1}B\|$ are equal to n. In this case, in accordance with [32], we say that system (2) is controllable and observable. By Theorem 1.2.4 in [32], the controllability and observability of system (2) are equivalent to the nonsingularity of the matrix $W(p)$. The latter implies that, for any root p_0 of the polynomial $\delta(p) = \det(pI_n - A)$, there exists a minor $\nu(p)$ of the matrix $W(p)$ such that $\lim_{p \to p_0} \delta(p)\nu(p) = 0$.

Let the functions $\varphi_j(\sigma_j)$ satisfy the conditions

$$0 \le \frac{\varphi_j(\sigma_{j2}) - \varphi_j(\sigma_{j1})}{\sigma_{j2} - \sigma_{j1}} \le \mu_j \quad \text{for all} \quad \sigma_j \in (-\infty, \infty), \quad \sigma_{j1} \ne \sigma_{j2}, \quad \varphi_j(0) = 0, \quad j = 1, 2, \ldots, m. \tag{3}$$

Obviously, assumptions (3) imply that system (2) has a solution (a point of equilibrium) $x = 0$. If x_0 is some point of equilibrium of system (2), then it satisfies the relation

$$C^*x_0 + C^*A^{-1}B\varphi(C^*x_0) = 0,$$

which can be represented in the form

$$\sigma_0 + W(0)\varphi(\sigma_0) = 0, \quad \text{where} \quad \sigma_0 = \mathrm{col}(\sigma_1^0, \dots, \sigma_m^0), \quad \varphi(\sigma_0) = \mathrm{col}(\varphi_1(\sigma_1^0), \dots, \varphi_m(\sigma_m^0)). \tag{4}$$

All criteria stated below for the existence of cycles of system (2) are based on the assumption that $x = 0$ is the unique point of equilibrium of the system. For this, it is necessary and sufficient that system (4) has only the trivial solution $\sigma_0 = 0$. We set $W(0) = (w_{ij})_{m\times m}$.

Lemma 1. *Suppose that all entries in some row with index j in the matrix $W(0)$ except for the element w_{jj} are zero. Let the inequality $w_{ij} > -\mu_j^{-1}$ be true. Then $\sigma_j = 0$.*

Lemma 2. *Let all entries w_{kl} in rows with indices i and j for which at least one of the indices k and l does not coincide with i or j, respectively, be zero. If the conditions*

$$w_{ii} > \mu_i^{-1}, \qquad w_{jj} > -\mu_j^{-1}, \qquad w_{ij}w_{ji} \le 0 \tag{5}$$

are satisfied, then system (4) has only the trivial solution.

Let us present the proof of Lemma 2. Lemma 1 can be proved in a similar way.

Proof. To be definite, suppose that $i = 1$ and $j = 2$. Then the first two equations in system (4) have the form

$$\begin{aligned} \sigma_1 + w_{11}\varphi_1(\sigma_1) + w_{12}\varphi_2(\sigma_2) = 0,\\ \sigma_2 + w_{21}\varphi_1(\sigma_1) + w_{22}\varphi_2(\sigma_2) = 0. \end{aligned} \tag{6}$$

If $w > -\mu^{-1}$, then the line $\sigma + w\xi = 0$ does not meet the sector $0 < \xi/\sigma < \mu$ on the plane (σ, ξ). It follows from conditions (3) that $0 \le \varphi_k(\sigma_k)/\sigma_k \le \mu_k$, $k = 1, 2$. Therefore, by assuming that system (6) has a solution $\sigma \ne 0$, $\sigma_2 = 0$ and by using the condition $w_{11} > -\mu_1^{-1}$, we readily obtain a contradiction with the first equation in the system. In a similar way, by assuming that $\sigma_1 = 0$, from the second equation of the system and the condition $w_{22} > -\mu_2^{-1}$, we obtain $\sigma_2 = 0$. Now, by assuming that system (6) has a solution $\sigma_1 \ne 0$, $\sigma_2 = 0$, we obtain

$$\left(\frac{\sigma_1}{\varphi_1(\sigma_1)} + w_{11}\right)\left(\frac{\sigma_2}{\varphi_2(\sigma_2)} + w_{22}\right) = w_{12}w_{21}.$$

By (5), both factors on the left-hand side in the last relation are strictly positive, and its right-hand side is nonpositive. This contradiction completes the proof of the lemma.

We say that the matrix $W(0)$ "admits reduction by version 1" if all entries in the ith row of this matrix except for the entry w_{ii} are zero. We say that the matrix $W(0)$ "admits reduction by version 2" if all entries in some rows of this matrix with indices i and j except for the entries w_{ii}, w_{jj}, w_{ij}, and w_{ji} are zero. The reduction of a matrix by the version 1 is defined as the matrix in which all entries in the row and column with index i are replaced by zeros. The reduction of a matrix by version 2 is defined as the matrix in which all entries in the columns with the indices i and j are replaced by zeros.

Lemma 3. *Let the matrix $W(0)$ admit successive reduction by versions 1 and 2 until it becomes the zero $m \times m$ matrix. If the condition $w_{ii} > -\mu_i^{-1}$ is always satisfied for the reduction by version 1 and conditions (5) are satisfied for reduction by version 2, then system (4) has only the trivial solution $\sigma_1^0 = \sigma_2^0 = \cdots = \sigma_m^0 = 0$ [system (2) has the unique equilibrium $x = 0$].*

The validity of the assertion of Lemma 3 follows from Lemmas 1 and 2.

Remark 1. If the functions $\varphi_j(\sigma_j)$ satisfy the relations

$$\mu_j^1 \le \frac{\varphi_j(\sigma_{j2}) - \varphi_j(\sigma_{j1})}{\sigma_{j2} - \sigma_{j1}} \le \mu_j^2 \quad \text{for all} \quad \sigma_j \in (-\infty, \infty), \quad \sigma_{j1} \ne \sigma_{j2}, \quad \varphi_j(0) = 0, \quad j = 1, 2, \dots, m, \tag{3$'$}$$

then system (2) has the unique equilibrium $x = 0$ provided that the assumptions of Lemma 3 are satisfied for the matrix $W(0)[I_m + \mu W(0)]^{-1}$, where $\mu = \operatorname{diag}(\mu_1^2 - \mu_1^1, \mu_2^2 - \mu_2^1, \ldots, \mu_m^2 - \mu_m^1)$. It is well known that, in the scalar case ($m = 1$), the condition for the existence of only the trivial equilibrium $x = 0$ of system (2) can be reduced to the requirement of the existence of the unique point $\sigma = 0$ of the intersection of the graph of the nonlinearity $\varphi(\sigma)$ with the "characteristic line" $\sigma + W(0)\varphi = 0$.

Theorem 1. *Let the nonlinearities $\varphi_j(\sigma_j)$ in system (2) satisfy relations (3′). Suppose that there exists a number $\lambda > 0$ such that the following conditions are satisfied.*

1. *The matrix $A + B\varphi'(0)C^*$, where $\varphi'(0) = \operatorname{diag}(\varphi_1'(0), \ldots, \varphi_m'(0))$ has exactly two eigenvalues with positive real parts and does not have them in the strip $-\lambda \le \operatorname{Re} p \le 0$.*

2. *The matrix $A + BhC^*$, where $h = \operatorname{diag}(h_1, h_2, \ldots, h_m)$, is a Hurwitz matrix, and $|\varphi(\sigma) - hC^*x| < \gamma < \infty$.*

3. *The inequality*

$$\det \operatorname{Re} \left[I_m + \mu^1 W(i\omega - \lambda)^*\right]\left[I_m + \mu^2 W(i\omega - \lambda)\right] \ne 0, \quad \mu^k = \operatorname{diag}(\mu_1^k, \mu_2^k, \ldots, \mu_m^k), \quad k = 1, 2, \tag{7}$$

holds for all $\omega \in [0, \infty)$.

Then system (2) has at least one orbitally stable cycle whose attraction domain contains almost all points of a neighborhood of the equilibrium $x = 0$.

Proof. Let $C^* = \operatorname{col}(c_1^*, c_2^*, \ldots, c_m^*)$, where c_j^* is a row vector, and let $x_1(t)$ and $x_2(t)$ be two solutions of system (2). Set $z(t) = x_1(t) - x_2(t)$. Obviously, $z(t)$ is a solution of the system

$$\frac{dz}{dt} = Az + B\psi(t, \sigma), \qquad \sigma = C^* z, \tag{8}$$

where $\psi(t, \sigma) = R(t)\sigma$, $R(t) = \operatorname{diag}(r_1(t), \ldots, r_m(t))$, $r_j(t) = \dfrac{[\varphi_j(\sigma_{j2}(t)) - \varphi_j(\sigma_{j1}(t))]c_k^* z(t)}{|c_j^* z(t)|}$ for $c_j^* z(t) \ne 0$, and $r_j(t) = 0$ for $c_j^* z(t) = 0$. By taking into account (3′) for the function $\psi(t, \sigma) = R(t)\sigma$, we obtain

$$\mu_j^1 \sigma_j^2 \le \psi_j(t, \sigma_j)\sigma_j \le \mu_j^2 \sigma_j^2, \qquad j = 1, 2, \ldots, m. \tag{9}$$

We introduce the function $\nu(z) = z^* H z$. We choose the matrix $H = H^*$ so as to ensure that the inequality

$$\dot\nu(z) + 2\lambda\nu(z) \le -\varepsilon |z|^2 \tag{10}$$

holds with some $\varepsilon > 0$, where the derivative of the function $\nu(z)$ is computed according to system (8).

For the validity of relation (10), it is sufficient that the inequality

$$2z^* H[(A + \lambda I_n)z + B\psi] + \sum_{j=1}^{m} \left(\mu_j^2 \sigma_j - \psi_j\right)\left(\psi_j - \mu_j^1 \sigma_j\right) \le -\varepsilon(|z|^2 + |\psi|^2) \tag{11}$$

be valid for arbitrary $z \in R^n$ and for arbitrary $\psi \in R^m$ satisfying relation (9). By the frequency theorem 1.2.7 in [32], there exists a matrix $H = H^*$ satisfying inequality (11) if and only if $\operatorname{Re}[I_m + \mu^1 W(i\omega - \lambda)]^*[I_m + \mu^2 W(i\omega - \lambda)] < 0$ for all $\omega \in [0, \infty)$. The last inequality is equivalent to condition (7).

By setting $\psi_j = \varphi_j'(0)\sigma_j$ in (11) and by taking into account relation (9), we obtain the matrix inequality

$$H[A + B\varphi'(0)C^* + \lambda I_n] + [A + B\varphi'(0)C^* + \lambda I_n]^* H \le -\varepsilon I_n. \tag{12}$$

This inequality, together with assumption 1 of the theorem and Lemma 1.2.4 from [32], implies that H is a nonsingular matrix and has exactly 2 negative eigenvalues and $n - 2$ positive ones.

It follows from the above argument that the relation

$$\dot{V}[x_1(t) - x_2(t)] + 2\lambda V[x_1(t) - x_2(t)] \le -\varepsilon |x_1(t) - x_2(t)|^2 \tag{13}$$

holds for the function $V(x) = x^* H x$ with the found matrix H and for two arbitrary solutions $x_1(t)$ and $x_2(t)$ of system (2). By setting $x_2(t) \equiv 0$ in (13), we obtain the inequality

$$\dot{V}[x(t)] + 2\lambda V[x(t)] \le -\varepsilon |x(t)|^2,$$

which holds for an arbitrary solution $x(t)$ of system (2). This inequality, together with Theorem 2.9 in [33], implies that $\Omega = \{x : \ x^* H x \le 0\}$ is a positively invariant set for the trajectories of system (2) and its boundary $\partial\Omega = \{x : \ x^* H x = 0\}$ is intersected inwards by all trajectories of this system that meet it.

We set $P = A + B\varphi'(0)C^*$, $f(x) = B[\varphi(C^* x) - \varphi'(0)C^* x]$ and rewrite system (2) in the form

$$\dot{x} = Px + f(x). \tag{14}$$

By virtue of assumption 1 of the theorem, there exists a nonsingular transformation $x = Qy$ reducing system (14) to the form

$$\begin{aligned} \dot{y}_1 &= -P_{11} y_1 + g_1(y), \\ \dot{y}_2 &= P_{22} y_2 + g_2(y), \end{aligned} \tag{15}$$

where P_{11} and P_{12} are anti-Hurwitz matrices of dimensions $(n-2) \times (n-2)$ and 2×2, respectively; moreover, $P_{22} + P_{22}^*$ is a positive definite matrix, and $g_i(y) = o(|y|)$, $i = 1, 2$. Let $N = Q^* H Q$ and $P_1 = Q^{-1} P Q$. Let us show that $y^* N y < 0$ for $y_1 = 0$ and $y_2 \ne 0$ and $y^* N y > 0$ for $y_1 \ne 0$ and $y_2 = 0$. To this end, we represent the matrices N and P_1 in the form

$$N = \begin{pmatrix} N_{11} & L \\ L^* & N_{22} \end{pmatrix}, \qquad P = \begin{pmatrix} -P_{11} & 0 \\ 0 & P_{22} \end{pmatrix},$$

where $N_{11} = N_{11}^*$ is an $(n-2) \times (n-2)$ matrix and $N_{22} = N_{22}^*$ is a 2×2 matrix. From relation (12) for the quadratic form $y^* H y$ with $y_1 = 0$ and $y_2 \ne 0$, we have

$$2 y_2^* N_{22} (P_{22} + \lambda I_2) y_2 \le -\varepsilon |Qy|^2 \le -\frac{\varepsilon}{|Q^{-1}|^2} |y_2|^2.$$

Since $P_{22} + \lambda I_{n-2}$ is an anti-Hurwitz matrix, it follows from this inequality that $N_{22} < 0$. In a similar way, by using the fact that $-P_{11} + \lambda I_{n-2}$ is a Hurwitz matrix, one can show that $y^* N y > 0$ if $y_1 \ne 0$ and $y_2 = 0$.

Let k be the minimum eigenvalue of the positive definite matrix $P_{22} + P_{22}^*$. Then

$$\frac{d}{dt}(|y_2|^2) = y_2^* (P_{22} + P_{22}^*) y_2 + 2 y_2^* g_2(y) \ge k |y_2|^2 + 2 y_2^* g_2(y). \tag{16}$$

Note that the relation $|y_1| \le \vartheta |y_2|$ with some $\vartheta > 0$ holds for an arbitrary $y = \operatorname{col}(y_1, y_2)$ in the set $\Omega = \{y : \ y^* Q^* H Q y \le 0\}$. Therefore, $g_2(y) = o(|y_2|)$. Set $Fy = \operatorname{col}(0, y_2)$. It follows from the property of $g_2(y)$ and relation (16) that, for a sufficiently small $\vartheta > 0$, the surface $\partial G_1 = \{x : \ |F Q^{-1} x| = \theta, \ x^* H x \le 0\}$ is contact-free for the trajectories of system (2) and is intersected outwards by all trajectories of this system that meet it.

Let a matrix R be a solution of the Lyapunov equation

$$R(A + BhC^*) + (A + BhC^*)^* R = -I_n.$$

By assumption 2 of the theorem, R is a positive definite matrix. Consider the function $U(x) = x^* R x$. For the derivative of the function $U(x)$ according to system (2), we have

$$\dot{U} = x^*[(A + BhC^*)^* R + R(A + BhC^*)]x + 2x^* R(\varphi(\sigma) - hC^* x) \le -|x|^2 + \|R\| \gamma |x|. \tag{17}$$

By q we denote the maximum eigenvalue of the matrix R and take an arbitrary number $\rho > 4\|R\|^2\gamma^2 q$. Then it follows from (17) that the surface of the ellipsoid $\partial G_2 = \{x: \ x^*Rx = \rho\}$ is contact-free for the trajectories of system (2) and is intersected inwards by all trajectories of this system that meet it.

By D we denote the domain bounded by the surfaces $\partial\Omega$, ∂G_2, and ∂G_2. This domain is closed and bounded, does not contain the equilibrium $x = 0$ of system (2), and is intersected inwards by all trajectories of that system. This, together with relation (13) and the boundedness of the function $|\varphi(\sigma)' - hC^*x|$, implies that the assumptions of Theorem 8.4 in [33] are satisfied; this theorem implies that the ω-limit set of the trajectory of any solution $x(t)$ of system (2) such that $x(0) \in D$ contains at least one orbitally stable cycle of that system.

It follows from above-performed constructions and relation (13) that if $|x(0)|$ is sufficiently small and $x(0)$ does not belong to the stable manifold of the equilibrium $x = 0$ of system (2), then $x(\tau) \in D$ for some $\tau > 0$.

The proof of Theorem 1 is complete.

Theorem 2. *Let assumption* (3′) *be satisfied, and let the assumptions* 1 *and* 3 *of Theorem* 1 *be satisfied. Let the matrix* $A+B\tilde{M}C^*$ *with some matrix* $\tilde{M} = \operatorname{diag}(\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_m)$, *where* $\tilde{\mu} \in (\mu_j^1, \mu_j^2)$, *have no eigenvalue in the strip* $-\lambda \le \operatorname{Re}\rho \le 0$. *Then all solutions* $x(t)$ *of system* (2) *with* $\varphi(\sigma) = \tilde{M}C^*x$ *such that* $x(0) \in \Omega = \{x: \ x^*Hx \le 0\}$ *have the property* $|x(t)| \to \infty$ *as* $t \to \infty$.

Proof. We denote $P = A + B\tilde{M}C^*$ and set $\psi = \tilde{M}C^*x$ in (11). One can show that the matrix P satisfies the inequality $H(P + \lambda I_n) + (P + \lambda I_n)^*H \le -\varepsilon I_n$. Since the matrix H has exactly 2 negative eigenvalues and $n - 2$ positive ones, it follows from Theorem 2.3 in [33] that the matrix $P + \lambda I_n$ has exactly two eigenvalues in the half-plane $\operatorname{Re} p > 0$. By virtue of the assumptions of the theorem, the matrix P also has exactly 2 eigenvalues in the half-plane $\operatorname{Re} p > 0$ and no one on the imaginary axis. The linear system obtained from system (2) for $\varphi(\sigma) = \tilde{M}C^*x$ has the form (14) with $f(x) = 0$. After the nonsingular transformation $x = Qy$, it acquires the form $\dot{y}_1 = -P_{11}y_1$, $\dot{y}_2 = P_{22}y_2$, where $y = \operatorname{col}(y_1, y_2)$, with a Hurwitz $(n-2) \times (n-2)$ matrix $(-P_{11})$ and an anti-Hurwitz 2×2 matrix P_{22}. If $x(0) \in \Omega$, then, as was shown above, $y_2 \ne 0$. Therefore, $|y_2(t)| \to \infty$ as $t \to \infty$ and hence $|x(t)| \to \infty$.

The proof of Theorem 2 is complete.

Remark 2. The assumption 2 of Theorem 1 can be replaced by any condition of Levinson dissipativity of system (2) providing the existence of an open bounded set Λ containing the point $x = 0$ such that its closure $\overline{\Lambda}$ is positively invariant for trajectories of system (2) and its boundary $\partial\Lambda$ is intersected inwards by all trajectories of this system that meet it. For example, it was shown in [27, 29] that system (2) with a nonlinearity satisfying condition (3′) is Levinson dissipative if there exist numbers $\nu_j^1, \nu_j^2 \in (\mu_j^1, \mu_j^2)$ and $\sigma_j^0 > 0$ such that $\nu_j^1(\sigma_j)^2 \le \sigma_j\varphi_j(\sigma_j) \le \nu_j^2(\sigma_j)^2$ for $|\sigma_j| \ge \sigma_j^0$, and the relation

$$\det \operatorname{Re}\,[I_m + \nu^1 W(i\omega)]^*[I_m + \nu^2 W(i\omega)] \ne 0, \qquad \nu^k = \operatorname{diag}(\nu_1^k, \nu_2^k, \ldots, \nu_m^k), \qquad k = 1, 2,$$

is satisfied for all $\omega \in [0, \infty)$.

3. ALGORITHM FOR FINDING HIDDEN ATTRACTORS

Consider some system that can be represented in the form (2). Let us perform a “linear analysis” of that system. Suppose that there exist numbers μ_j^1 and μ_j^2 $(j = 1, 2, \ldots, m)$ such that relation (7) holds for some $\lambda > 0$ and, in addition, some matrix $A + BMC^*$, where $M = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_m)$, $\mu_j \in (\mu_j^1, \mu_j^2)$, has exactly two eigenvalues with positive real parts and have no one in the strip $-\lambda \le \operatorname{Re} p \le 0$. Let $A + BNC^*$, $N = \operatorname{diag}(\nu_1, \nu_2, \ldots, \nu_m)$, be a Hurwitz matrix for all $\nu_j \in (\mu_1^2, \mu_j^2)$. Finally, let the assumptions of Lemma 3 and the conditions stated in Remark 1 be satisfied. Then, by using Theorem 1, one can readily choose a nonlinearity $\psi(\sigma)$ so as to ensure that system (2) with such a nonlinearity has at least one orbitally asymptotically stable cycle whose attraction domain contains all points in a small neighborhood of the equilibrium $x = 0$ of the system. Let $x_0 \neq 0$ be some (arbitrary) point in a neighborhood of the equilibrium $x = 0$. We numerically find a solution $x_0(t)$ of system (2) with a nonlinearity $\psi(\sigma)$ on the interval $[0, T]$, where T is sufficiently large, and with the initial condition $x_0(0) = x_0$. The value $x_0(T)$ is sufficiently close to the cycle. Now consider a family of systems (2) with nonlinearities $\varepsilon_j\varphi(\sigma) + (1 - \varepsilon_j)\psi(\sigma)$, where $\varepsilon_i = 0.1i$, $i = 0, 1, \ldots, 10$. The solutions of these systems are denoted by $x_i(t)$. For the numerical integration of each of systems of the family, as an initial condition $x_i(0)$, we take $x_{i-1}(T)$. If, for the integration of all systems of the family, we obtain an attractor, then for $j = 10$, we find an attractor of system (2) with the nonlinearity $\varphi(\sigma)$. But if the numerical integration does not detect an attractor for some value ε_i, then this implies a bifurcation and the vanishing of the attractor.

Remark 3. If an attractor is not detected on some step of the implementation of the described algorithm, then this could imply that a next attractor has a very small attraction domain. In this case, it is reasonable to diminish the discretization increment with respect to ε and repeat the search procedure with the smaller increment.

Let us consider several examples of artificially constructed systems for which we demonstrate the operation of the algorithm of finding hidden attractors.

Example 1. Consider the system

$$\begin{aligned} \dot{x}_1 &= 1.241x_1 + 8.45x_2 - 1.4365\frac{x_1^4 + 0.2}{0.34x_1^4 + 0.2}\tanh x_1, \\ \dot{x}_2 &= x_1 - x_2 + x_3 + 0.1x_2^3, \\ \dot{x}_3 &= -12.1x_2 - 0.005x_3. \end{aligned} \tag{18}$$

System (18) has the equilibria $(0, 0, 0)$ and $(\pm 3.3862, \pm 1.3987 \times 10^{-3}, \mp 3.3862)$. In this case, the zero equilibrium is stable in the small, and two other equilibria are saddle-foci. That system can be represented in the form (2) with

$$A = \begin{pmatrix} 1.241 & 8.45 & 0 \\ 1 & -1 & 1 \\ 0 & -12.1 & -0.005 \end{pmatrix}, \qquad B = \begin{pmatrix} -8.45 & 0 \\ 0 & 0.1 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\varphi_1(\sigma_1) = 0.17\frac{\sigma_1^4 + 0.2}{0.34\sigma_1^4 + 0.2}\tanh\sigma_1, \qquad \varphi_2(\sigma_2) = \sigma_2^3.$$

By performing the "linear analysis" of system (18), we find that $A + BMC^*$, $M = \operatorname{diag}(\mu_1, \mu_2)$, is a Hurwitz matrix for $\mu_1 \in [0.148, 0.2]$ and $\mu_2 \in [0, 0.8]$. If $\mu_1 \in [0.215, 0.9]$ and $\mu_2 \in [0, 2]$, then the matrix $A + BMC^*$, $M = \operatorname{diag}(\mu_1, \mu_2)$, has exactly 2 eigenvalues with positive real parts and no one in the strip $-1 \leq \operatorname{Re} p \leq 0$.

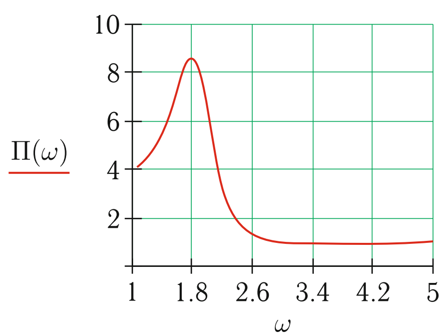


Fig. 1.

Let $\Pi(\omega) = \det \operatorname{Re}\,[I_2 + \mu^1 W(i\omega - \lambda)]^*[I_2 + \mu^2 W(i\omega - \lambda)]$, where $\lambda = 0.6$, $\mu^2 = \operatorname{diag}(0.77, 1)$, and $\mu^1 = \operatorname{diag}(0.17, 0)$. The graph of the function $\Pi(\omega)$ is shown in Fig. 1. Therefore, relations (7) hold for the above-mentioned matrices μ^1 and μ^2. We replace the functions $\varphi_1(x_1)$ and $\varphi_2(x_2)$ in system (18) by arbitrary continuous piecewise differentiable functions $\psi_1(x_1)$ and $\psi_2(x_2)$ such that $\psi_1'(0) \in [0.215, 0.77]$, $\psi_2'(0) \in [0, 1]$, the conditions $\psi_1'(x_1) \in [0.17, 0.77]$ and $\psi_2'(x_2) \in [0, 1]$ are satisfied on all differentiability intervals, and, for example, the relations

$$\lim_{|x_1|\to\infty} \frac{\psi_1(x_1)}{x_1} = 0.18, \qquad \lim_{|x_2|\to\infty} \frac{\psi_2(x_2)}{x_2} = 0.5 \tag{19}$$

hold. By using Remark 1, one can readily show that, in this case, the system has the unique equilibrium $(0, 0, 0)$. By Theorem 1, system (18) with the nonlinearities satisfying conditions (19) has at least one orbitally stable cycle whose attraction domain contains almost all points of a neighborhood of the equilibrium $(0, 0, 0)$.

As $\psi_1(x_1)$ and $\psi_2(x_2)$ we take the functions

$$\psi_1(x_1) = \begin{cases} 0.18x_1 - 0.11 & \text{for} \quad x \le -0.5, \\ 0.4x_1 & \text{for} \quad |x| \le 0.5, \\ 0.18x_1 + 0.11 & \text{for} \quad x \ge 0.5, \end{cases} \qquad \psi_2(x_2) = 0.5x_2.$$

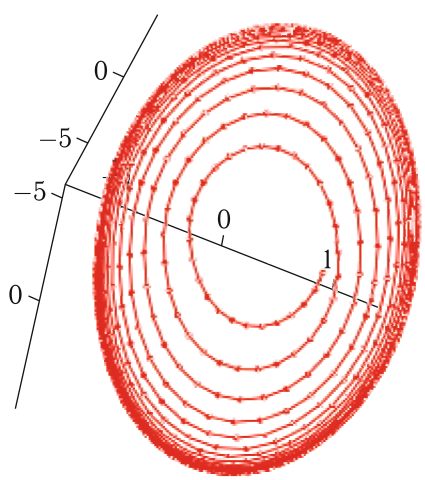
By "starting" from a system with such nonlinearities, we implement the above-described algorithm of the search of hidden oscillations of system (18). Figure 2 represents a cycle of the "starting" system with the initial conditions $(0.1, 0.2, 0.3)$. Figures 3–5 represent the evolution of an attractor of the system under changes of ε.

Note that a point with the coordinates $(3.386, 1.3987\times 10^{-3}, -3.386)$ lies in an attraction domain of the found attractor, while trajectories issuing from the close point $(3.386, 1.3987\times 10^{-2}, -3.386)$ very rapidly go to infinity.

Example 2. Consider system (2) with a single nonlinearity, where

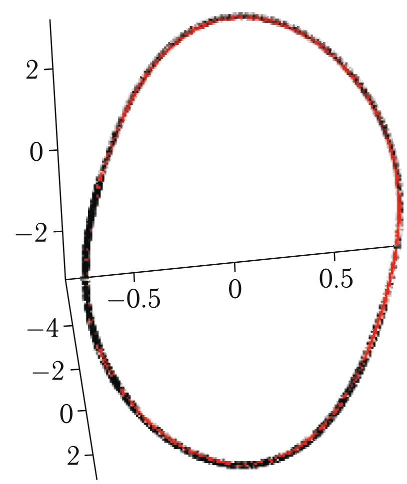
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} -18 \\ -1 \\ -1 \end{pmatrix}, \qquad \varphi(\sigma) = (7 + 0.0567\sigma^2)\operatorname{arctanh}\sigma.$$

Here the matrix $A(\mu) = A + \mu BC^*$ is a Hurwitz matrix for $\mu \in (0, 2) \cup (5, \infty)$. For $\mu = 2$ and $\mu = 5$, this matrix has exactly one negative eigenvalue and a pair of pure imaginary ones.



$\varepsilon = 0$

Fig. 2.



$\varepsilon = 0.6$

Fig. 3.

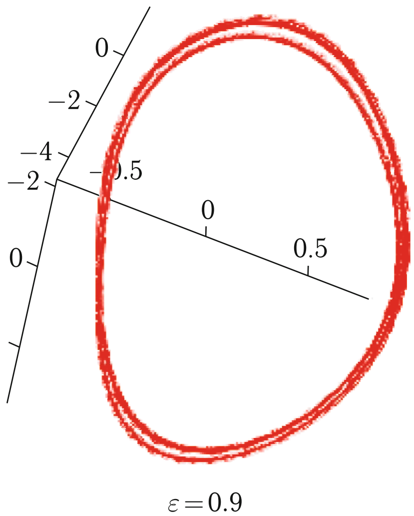


Fig. 4.

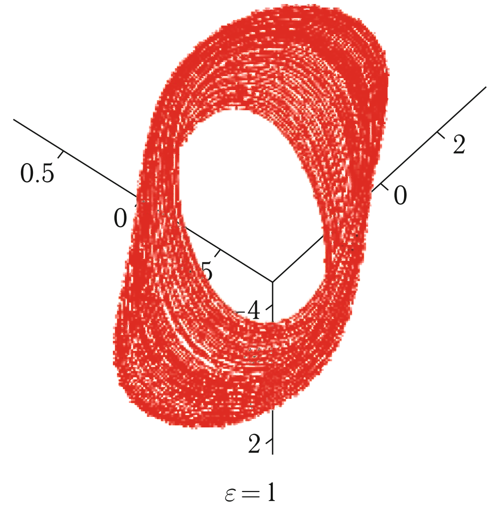


Fig. 5.

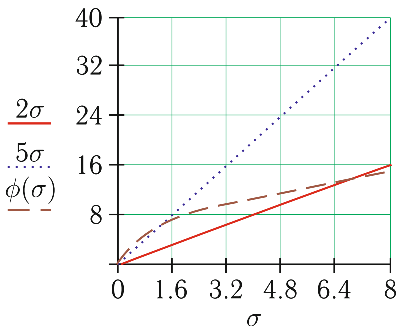


Fig. 6.

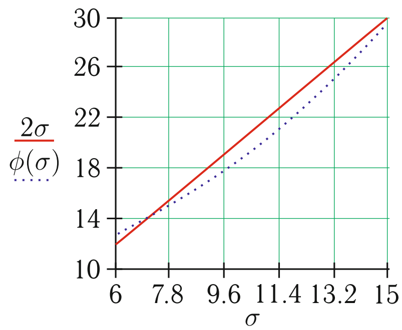


Fig. 7.

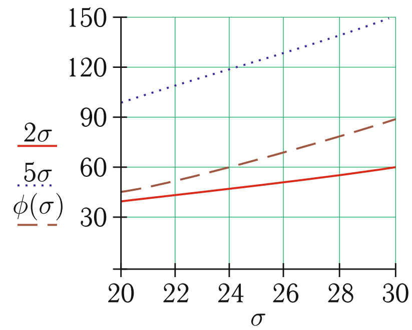


Fig. 8.

If $\mu \in (2, 5)$, then the matrix $A(\mu)$ has a pair of eigenvalues in the right open half-plane and no one in the strip $-3 \le \operatorname{Re} p \le 0$. The unique equilibrium σ of the considered system is stable in small. Figures 6–8 show that the graph of the odd nonlinearity $\varphi(\sigma)$ "alternately visit" sectors of the Hurwitz property and instability of degree 2. Such a behavior of the nonlinearity permits one to claim that the considered system has hidden attractors.

For this system, the transfer function has the form $W(p) = \dfrac{p^2 + p + 18}{p^3 + p^2 + 10p}$. One can readily see that the relation

$$\operatorname{Re}\{[1 + 0.4W(i\omega - \lambda)][1 + 10W(i\omega - \lambda)]^*\} > 0$$

holds for all $\omega \ge 0$ and $\lambda = 0.6$; this implies that assumption 3 of Theorem 1 is satisfied in the scalar case.

Consider the auxiliary system with the nonlinearity $\varphi(\sigma)$ replaced by the function

$$\psi_1(\sigma) = \begin{cases} \sigma - 1.5 & \text{for} \quad \sigma \le -0.5, \\ 4\sigma & \text{for} \quad |\sigma| \le 0.5, \\ \sigma + 1.5 & \text{for} \quad \sigma \ge 0.5. \end{cases}$$

By Theorem 1, a system with such a nonlinearity has a self-exciting cycle whose attraction domain contains points in an arbitrarily small neighborhood of the point of equilibrium $x = 0$. By using the above-represented algorithm for detecting hidden oscillations, we find a cycle of the considered system. The stages of operation of the algorithm are presented in Figs. 9–11 [projections of the cycle onto the plane (x, y)].

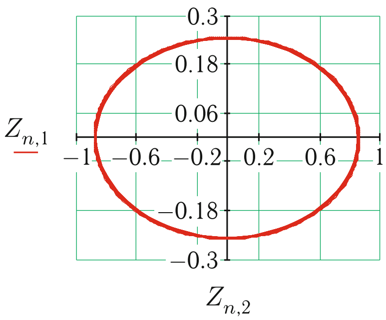


Fig. 9. $\varepsilon = 0.$

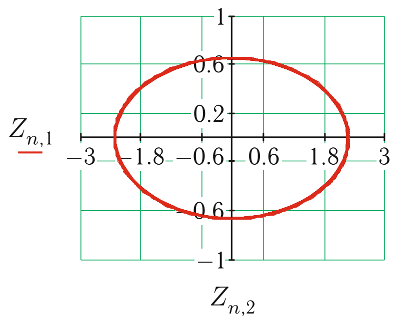


Fig. 10. $\varepsilon = 0.5.$

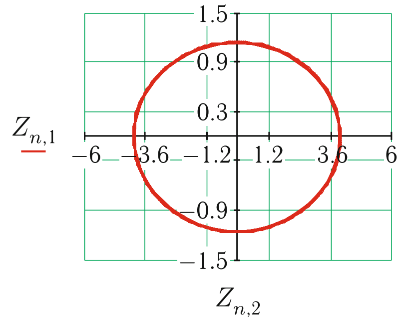


Fig. 11. $\varepsilon = 1.$

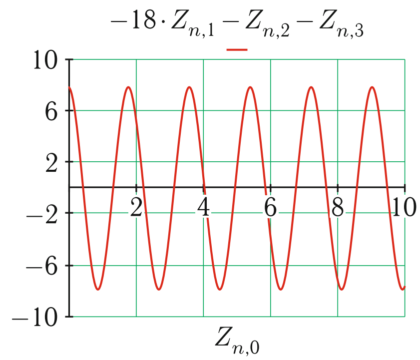


Fig. 12.

Figure 12 presents the graph of $\sigma(t)$ for the found cycle.

In Fig. 12, one can see that $|\sigma(t)| < 8$. By using Theorem 2, we choose a nonlinearity $\psi_2(\sigma)$ in the considered system so as to ensure that the system has a self-exciting cycle, which necessarily differs from the above-found cycle. We set

$$\psi_2(\sigma) = \begin{cases} 6\sigma + 24 & \text{for} \quad \sigma \le -8, \\ 4\sigma & \text{for} \quad |\sigma| \le 8, \\ 6\sigma - 24 & \text{for} \quad \sigma \ge 8. \end{cases}$$

By using again the above-described algorithm of detecting hidden oscillations, we find one more orbitally stable cycle of the considered system. The stages of the operation of the algorithm are presented in Figs. 13–15.

Note that, in addition to the found orbitally stable cycles, the considered system has two more unstable cycles, which can be found by some well-known method of finding unstable cycles, for example, "the shooting method" described in the monograph [34]. Figure 16 presents the projection to the plane (y, z) of the minimum global attractor of the considered system found by numerical integration; this attractor consists of the stable equilibrium $(0, 0, 0)$, two orbitally stable cycles with the initial conditions $(-0.28697, -3.74819, 3.68689)$ and $(4.39634, -40.59566, -66.18277)$, and two unstable cycles with the initial conditions $(-0.13393, -1.7377, 2.01380)$ and $(2.02008, 5.13178, -24.27500)$.

Example 3. Let us show that, in some cases, the above-suggested method for detecting hidden attractors of systems of the form (2) can be used for finding attractors of systems of a more general form. Consider the third-order system

$$\begin{aligned} \dot{x} &= -z, \\ \dot{y} &= -x - z, \\ \dot{z} &= 2x - 1.3y - 2z + x^2 + z^2 - xz. \end{aligned} \tag{20}$$

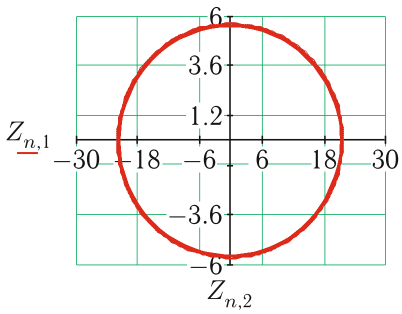


Fig. 13. $\varepsilon = 0$.

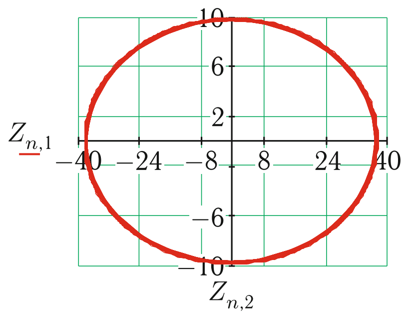


Fig. 14. $\varepsilon = 0.3$.

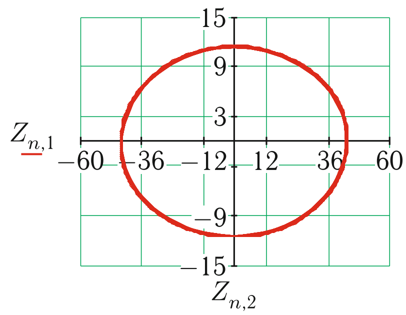


Fig. 15. $\varepsilon = 1$.

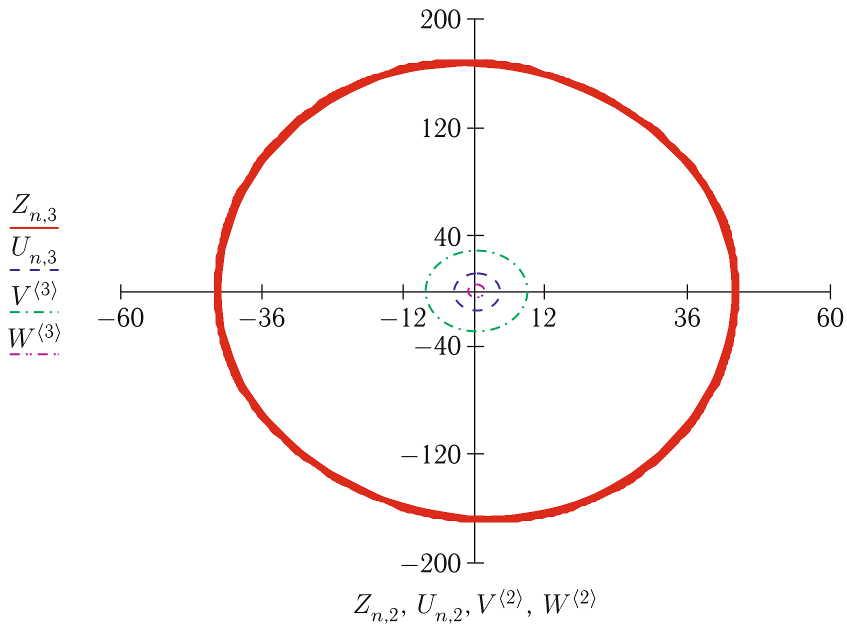


Fig. 16.

This system was considered in detail in [7], where it was shown that this system has the unique stable, in small, equilibrium $O(0,0,0)$ and a hidden chaotic attractor. Let us try to find this attractor numerically.

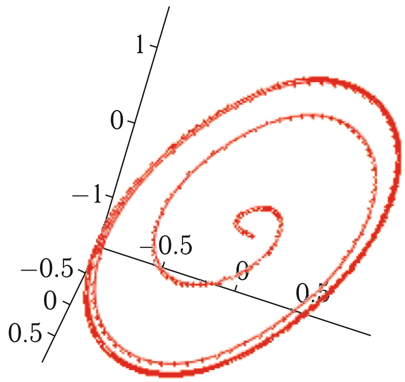
We rewrite system (20) in the form

$$\begin{aligned}\dot{x} &= -z,\\ \dot{y} &= -x - z,\\ \dot{z} &= 2x - 1.3y - 2z + (1-\varepsilon)\varphi(x-z) + \varepsilon(x^2 + z^2 - xz)\end{aligned} \tag{21}$$

and choose a function $\varphi(x-z)$ such that system (21) with $\varepsilon = 0$ has a self-exciting attractor. If $\varepsilon = 0$, then system (21) has the form (2) with

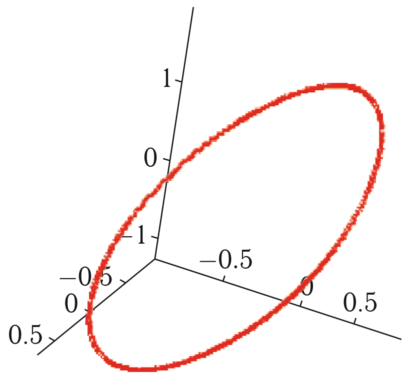
$$A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 2 & -1.3 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad W(p) = \frac{p^2 + p}{p^3 + 2p^2 + 0.7p + 1.3}.$$

The linear analysis provides the following results. For $\mu \in (-\infty, -0.037559)$, the matrix $A(\mu) = A + \mu BC^*$ has two eigenvalues in the open right half-plane and has no one in the strip $-1 \le \operatorname{Re} p \le 0$.



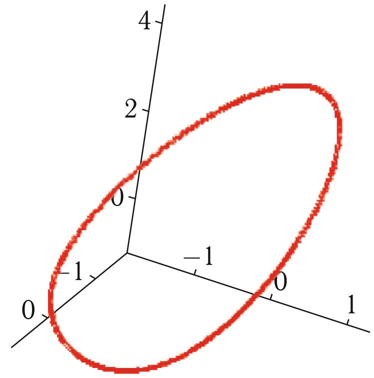
$(Z^{\langle 3\rangle}, Z^{\langle 1\rangle}, Z^{\langle 2\rangle})$

Fig. 17. $\varepsilon = 0$.



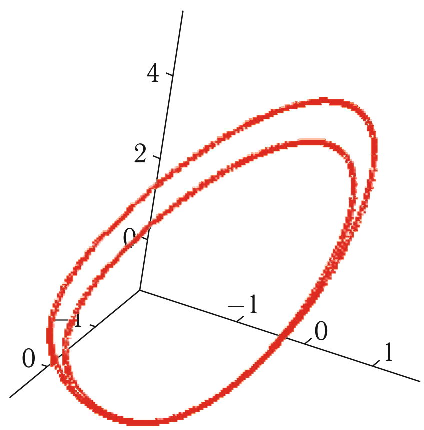
$(Z^{\langle 3\rangle}, Z^{\langle 1\rangle}, Z^{\langle 2\rangle})$

Fig. 18. $\varepsilon = 0.6$.



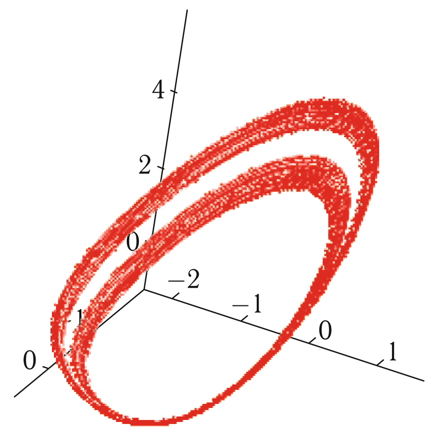
$(Z^{\langle 3\rangle}, Z^{\langle 1\rangle}, Z^{\langle 2\rangle})$

Fig. 19. $\varepsilon = 0.95$.



$(Z^{\langle 3\rangle}, Z^{\langle 1\rangle}, Z^{\langle 2\rangle})$

Fig. 20. $\varepsilon = 0.99$.



$(Z^{\langle 3\rangle}, Z^{\langle 1\rangle}, Z^{\langle 2\rangle})$

Fig. 21. $\varepsilon = 1$.

For $\mu \approx -0.037559$, the matrix $A(\mu)$ has a pair of pure imaginary eigenvalues and one negative eigenvalue. If $\mu > -0.037559$, then $A(\mu)$ is a Hurwitz matrix. If $\mu_1 = -1.2$, $\mu_2 = 2.5$, and $\lambda = 1$, then one can readily see that $\operatorname{Re}\{[1 + \mu_1 W(i\omega - \lambda)][1 + \mu_2 W(i\omega - \lambda)]^*\} > 0$ for all $\omega \geq 0$. By using Theorem 1, we construct a function $\varphi(\sigma) = \varphi(x - z)$ such that system (21) with $\varepsilon = 0$ has a cycle self-exciting in an arbitrarily small neighborhood of the point of equilibrium $O(0, 0, 0)$. For example, we set

$$\varphi(\sigma) = \begin{cases} 2\sigma + 2 & \text{for} \quad \sigma \leq -1, \\ \sigma^3 - \sigma & \text{for} \quad |\sigma| \leq 1, \\ 2\sigma - 2 & \text{for} \quad \sigma \geq 1. \end{cases}$$

By issuing from that cycle and by varying ε in system (21) from 0 to 1 with a small increment, we try to detect a hidden attractor of system (20). Figures 17–21 present the results of operation of the algorithm.

4. KALMAN PROBLEM

The Kalman and Aizerman problems [35, 31] well known in the theory of nonlinear regulated systems were posed in the middle of the XX century and readily attracted the attention of specialists in control theory and the theory of differential equations. The matter of these problems is the following. Consider system (2), where A is an $n \times n$ matrix, B and C are n vectors, and $\varphi(\sigma)$ is a scalar function with $\varphi(0) = 0$. Let all systems (2) with $\varphi(\sigma) = \mu\sigma$ be asymptotically stable for $\mu \in (\mu_1, \mu_2)$.

Aizerman conjecture. System (2) with an arbitrary piecewise continuous nonlinearity $\varphi(\sigma)$ satisfying the condition $\mu_1 < \dfrac{\varphi(\sigma)}{\sigma} < \mu_2$ for $\sigma \neq 0$ is globally stable.

Kalman conjecture. System (2) with any piecewise continuous nonlinearity $\varphi(\sigma)$ satisfying the condition $\mu_1 < \varphi'(\sigma) < \mu_2$ at the points of differentiability is globally stable.

For $n = 2$, the first counterexample to the Aizerman conjecture was constructed by Krasovskii in 1952 [36]. In 1958, Pliss [37] suggested a method for constructing three-dimensional nonlinear systems that satisfy the Aizerman condition and have periodic solutions. Later, this method was generalized by Leonov to systems of arbitrary dimension [38]. The proof of the fact that the Kalman problem has the positive solution for $n = 2$ and $n = 3$ was obtained in [39]. Finally, in 1988 Barabanov [40] proved the existence of fourth-order systems for which the Kalman problem has the negative solution. As was noted in [14], the Barabanov result is an "existence theorem" and should be verified carefully. In other words, one should find examples of particular fourth-order systems that satisfy the Kalman condition and have, for example, periodic solutions.

Let us present the procedure of constructing a counterexample to the Kalman conjecture based on Theorem 1.

Consider system (2) with

$$A = \begin{pmatrix} 3.5 & 1 & -4.5 & -2 \\ -4 & -2 & 3.5 & 2.5 \\ 2.5 & 1 & -2.5 & -1 \\ -0.5 & -1 & -0.5 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -25.55 \\ 26.05 \\ -15.55 \\ -4.55 \end{pmatrix}, \qquad C = \begin{pmatrix} -9.2 \\ -20.2 \\ -21 \\ 10 \end{pmatrix},$$

$$\varphi(\sigma) = \tanh\sigma = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}}.$$

The function $\varphi(\sigma)$ satisfies the conditions $0 < \varphi'(\sigma) < 2$, while $A + \mu BC^*$ is a Hurwitz matrix if $0 < \mu < 9.9$. Therefore, the considered system satisfies all assumptions of the Kalman conjecture. By using the above-described procedure, we show that the considered system has a cycle.

One can readily see that, in this case, assumption 3 of Theorem 1 is satisfied for $\mu_1 = 0.1$, $\mu_2 = 20$, and $\lambda = 0.3$. As $\psi(\sigma)$ we take the function $15\tanh\sigma + 0.1\sigma$. Theorem 1 guarantees that the considered system with such a nonlinearity has an orbitally stable cycle. The numerical integration of the system with such a function and with the initial conditions $(0.1; 0.3; 0.3; 0.2)$ by the Runge–Kutta method leads to that cycle. The projection of the cycle onto the plane (x_3, x_4) is shown in Fig. 22. Now consider the family of systems (2) with the nonlinearities $\varepsilon_j\varphi(\sigma) + (1 - \varepsilon_j)\psi(\sigma)$, where $\varepsilon_j = 0.1j$, $j = 0, \ldots, 10$. To this family, we apply the above-described procedure of successive construction of the solutions $x_j(t)$. The cycles obtained for $j = 5$ and $j = 10$ are shown in Figs. 23 and 24.

5. HIDDEN ATTRACTORS OF THE CHUA SYSTEM

Chua chains are analogs of self-generators, that is, generators of oscillations with a feedback. Systems of differential equations governing the behavior of Chua chains [23] are three-dimensional dynamical systems with one scalar nonlinearity. In dimensionless coordinates, such a system can be represented in the form [14]

$$\begin{aligned} \dot{x} &= \alpha(y - x) - \alpha\varphi(x), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y - \gamma z, \end{aligned} \tag{22}$$

where the function $\varphi(x)$ specifies a nonlinear element (a "Chua diode"). Depending on the form of the function $\varphi(x)$, one can distinguish a classical and generalized Chua systems. A generalized

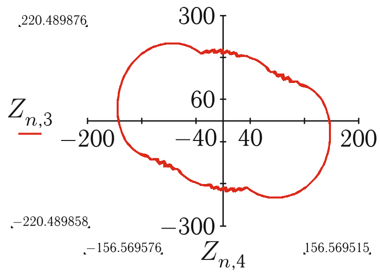


Fig. 22.

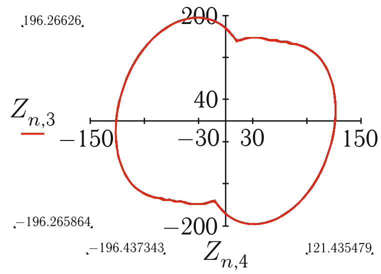


Fig. 23.

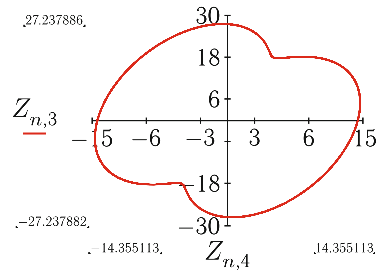


Fig. 24.

Chua system is defined as system (22) with a nonlinearity of the form [30]

$$\varphi(x) = m_1 x + 0.5(m_0 - m_1)(|x+1| - |x-1|) + 0.5(s - m_0)(|x+\delta_0| - |x-\delta_0|). \qquad (23)$$

In relations (22)–(23) α, β, m_0, and m_1 are parameters of the classical Chua system, δ_0 and s are parameters of the generalized Chua system responsible for the stability of the zero equilibrium.

First, we consider the classical Chua system, and, following [14], we choose its parameters as follows: $\alpha = 8.4562$, $\beta = 12.0732$, $\gamma = 0.0052$, $m_0 = -0.1768$, and $m_1 = -1.1468$. We rewrite system (22)–(23) in the form

$$\dot{X} = AX + B\varphi_1(\sigma), \qquad \sigma = C^*X; \qquad X = \mathrm{col}(x, y, z), \qquad (24)$$

$$A = \begin{pmatrix} -\alpha(m_1+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \qquad B = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \varphi_1(\sigma) = \varphi(\sigma) - m_1\sigma.$$

Note that $A + 0.97BC^*$ is a Hurwitz matrix; therefore, the solution $x = 0$, $y = 0$, $z = 0$ of the considered system is stable in the small. This fact does not permit one to detect numerically an attractor of the system after the "start" from a neighborhood of the above-mentioned equilibrium.

By setting $\varphi_1(\sigma) = \mu\sigma$ in (24), we perform the linear analysis of the system, i.e., we extract the stability and instability sectors of the linear system $\dot{X} = (A + \mu BC^*)X$ for various values of $\mu \in (-\infty, \infty)$. If $\mu \in (-\infty, 0.14723)$, then the matrix $A_\mu = A + \mu BC^*$ has one positive eigenvalue and two complex conjugate eigenvalues in the open left half-plane. For some $\tilde{\mu}_1 \in (0.14723, 0.147231)$, the positive eigenvalue passes into the left half-plane, and A_μ becomes a Hurwitz matrix. For $\mu \in (0.147231, 0.20986)$ the matrix A_μ remains a Hurwitz matrix (the Hurwitz sector). For some $\tilde{\mu}_2 \in (0.20986, 0.20987)$, two eigenvalues of A_μ become pure imaginary, and one remains in the open left half-plane. For $\mu \in (0.20987, 0.9596)$, the matrix A_μ has two complex conjugate eigenvalues with positive real parts and one negative eigenvalue (the sector of instability of degree 2). For some $\tilde{\mu}_3 \in (0.9596, 0.9597)$, the matrix A_μ has a pair of pure imaginary eigenvalues and one negative eigenvalue again. Finally, for $\mu \in (0.9597, \infty)$, the matrix A_μ is a Hurwitz matrix.

We take $\mu_1 = 0.17 \in (\tilde{\mu}_1, \tilde{\mu}_2)$, $\mu_2 = 4 > \tilde{\mu}_3$, and $\lambda = 0.5$. One can readily see that relation (7) holds for these values. Let us construct an auxiliary nonlinearity $\psi(\sigma)$ so as to ensure that system (24) with such a nonlinearity satisfies all assumptions of Theorem 1. One can readily see that as $\psi(\sigma)$ one can take, for example, the function

$$\psi(\sigma) = \begin{cases} 0.18\sigma - 0.62 & \text{for} \quad \sigma \le -1, \\ 0.4(|\sigma+1| - |\sigma-1|) & \text{for} \quad |\sigma| \le 1, \\ 0.18\sigma + 0.62 & \text{for} \quad \sigma \ge 1. \end{cases}$$

The result of the above-described algorithm of detecting a hidden attractor in the Chua classical system is shown in Figs. 25–27 [the projection to the plane (x, y)].

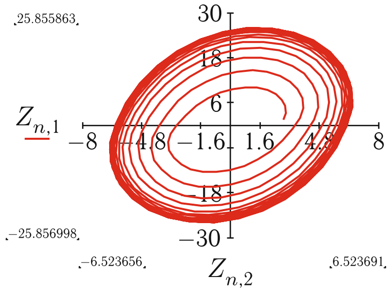


Fig. 25. $\varepsilon = 0$.

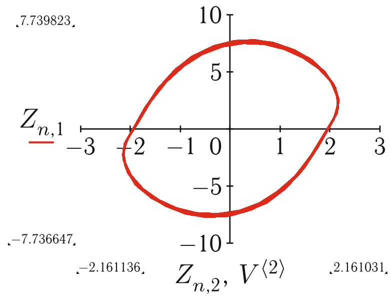


Fig. 26. $\varepsilon = 0.5$.

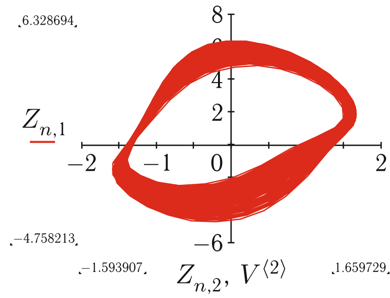


Fig. 27. $\varepsilon = 1$.

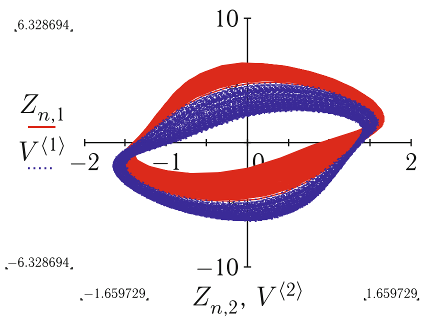


Fig. 28. $\varepsilon = 1$.

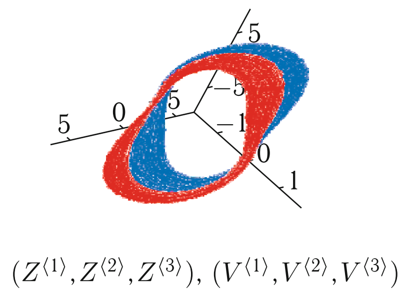


Fig. 29. $\varepsilon = 1$.

The hidden strange attractor of the system shown in Fig. 27 is obtained by numerical integration with the initial conditions $(3.414309, 1.41477, -3.666077)$. Note that this attractor is not symmetric around the origin, while the original system is preserved under the replacement of (x, y, z) by $(-x, -y, -z)$. The latter permits one to assume that the system has one more hidden attractor, which is obtained by the numerical integration with the initial conditions $(-3.414309, -1.41477, 3.666077)$. This assumption proves to be true. Two hidden attractors-twins are shown in Figs. 28 and 29.

Now consider the generalized Chua system (22) with a nonlinear $\psi(\sigma)$ of the form (23) and with the parameter values $\alpha = 8.4562$, $\beta = 12.0732$, $\gamma = 0.0052$, $m_0 = 0.14$, $m_1 = -1.1468$, $s = -0.9668$, and $\delta_0 = 0.2$. This system is stable in small.

We begin the process of detection of hidden oscillations with the construction of an auxiliary nonlinearity $g(\sigma)$ such that system (24) with such a nonlinearity satisfies all assumptions of Theorem 1. For example, let

$$g(\sigma) = \begin{cases} 2\sigma + 0.3 & \text{for} \quad \sigma \le -0.2, \\ 0.5\sigma & \text{for} \quad -0.2 \le \sigma \le 0.2, \\ 2\sigma - 0.3 & \text{for} \quad \sigma \ge 0.2. \end{cases}$$

The result of operation of the algorithm for the detection of a hidden oscillation in the Chua generalized system is shown in Figs. 30–32 [the projection onto the plane (x, y)].

Figure 33 represents the graph of $x(t)$ for the found cycle Γ.

In Fig. 33, one can see that the relation $|x(t)| = |\sigma(t)| < 0.76$ holds for the cycle Γ. Now we construct one more auxiliary system, which has a cycle necessarily different from the above-found cycle Γ of the Chua generalized system. As such a system we take system (24) with the nonlinearity

$$g_1(x) = \begin{cases} 0.18x - 0.64 & \text{for} \quad x \le 2, \\ 0.5x & \text{for} \quad -2 \le x \le 2, \\ 0.18x + 0.64 & \text{for} \quad x \ge 2. \end{cases}$$

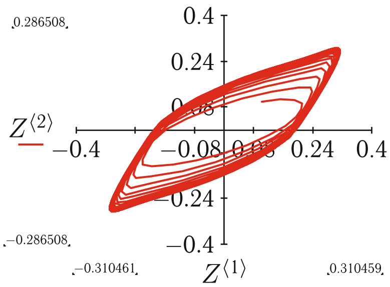


Fig. 30. $\varepsilon = 0$.

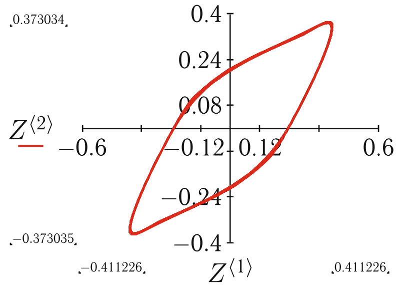


Fig. 31. $\varepsilon = 0.5$.

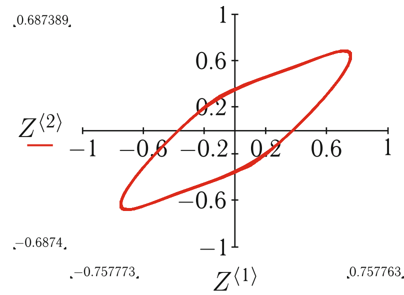


Fig. 32. $\varepsilon = 1$.

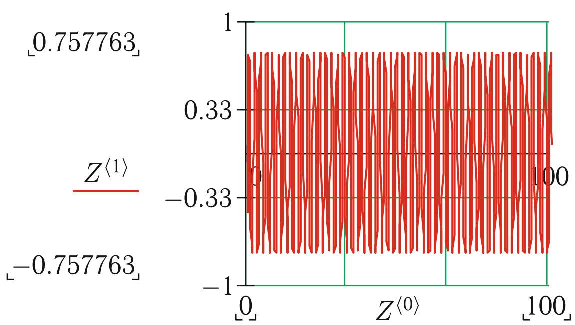


Fig. 33.

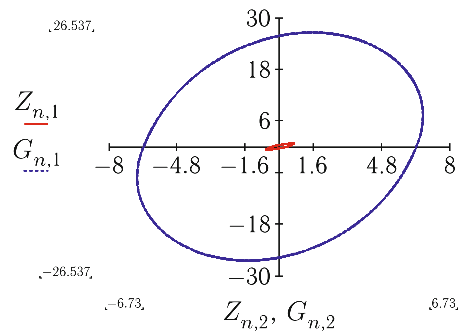


Fig. 34.

The system with such a nonlinearity satisfies all assumptions of Theorem 1; therefore, it has an orbitally stable cycle Γ_1, which is detected by the computational procedure after the start from any point of a neighborhood of the zero equilibrium. That cycle necessarily differs from the above-found cycle Γ of the Chua system. Indeed, it lies in the cone $\Omega = \{x : \; x^*Hx \le 0\}$. Should it coincide with the cycle Γ, it would satisfy the condition $|x(t)| = |\sigma(t)| < 0.76$ for all $t \in (-\infty, \infty)$. But then Γ_1 would be a trajectory of the linear system (24) with $\varphi_1(\sigma) = 0.5\sigma$. The eigenvalues of the matrix $A + 0.5BC^*$ of that system are $\lambda_1 = -4.3652$ and $\lambda_{2,3} = 0.1666 \pm 2.8669i$; i.e., it has no eigenvalue in the strip $-0.5 \le \operatorname{Re} p \le 0$. Therefore, by Theorem 2, all solutions of such a linear system with initial conditions in Ω are unbounded as $t \to \infty$. Therefore, $\Gamma_1 \ne \Gamma$. The "large" cycle Γ_1 and the "small" cycle Γ are shown in Fig. 34.

Now we repeat the above-described procedure for finding a hidden attractor issuing from some point of the cycle Γ_1. Namely, consider the family of systems (24) with the nonlinearities $\zeta_j(x) = \varepsilon_j\varphi_1(x) + (1 - \varepsilon_1)g_1(x)$. When approaching the attractor, it is reasonable to diminish the discretization step in ε. The result of operation of the algorithm for finding an attractor other than the attractor Γ is shown in Figs. 35–37.

The found attractor of the Chua generalized system is not symmetric around the origin. By repeating the arguments used above in the study of the Chua classical system, for the considered system, we find the symmetric attractor-twin. Figures 38–40 present the projections of hidden attractors of the Chua generalized system onto the planes (x, y), (x, z), and (y, z), respectively.

Figure 41 represents the attractor of the Chua generalized system in R^3.

In addition to the stable equilibrium $(0, 0, 0)$, the considered system has two more saddle equilibria $(\pm 7.2365, 3.1154 \times 10^{-3}, \mp 7.2365)$. The numerical integration shows that, among trajectories issuing from a small neighborhood of saddle equilibria, there are unbounded increasing ones [for example, trajectories with the initial conditions $(\pm 7.2, 0.03, \mp 7.2)$]. Note that the attraction domain of the cycle is "sufficiently large." Thus, for example, the trajectories with the initial conditions $(\pm 7.2, 0, \mp 7.2)$ in a small neighborhood of unstable equilibria are attracted to the cycle. Therefore, by issuing from an arbitrary point of a sufficiently small neighborhood of any of three equilibria, one cannot numerically detect hidden attractors of the studied system.

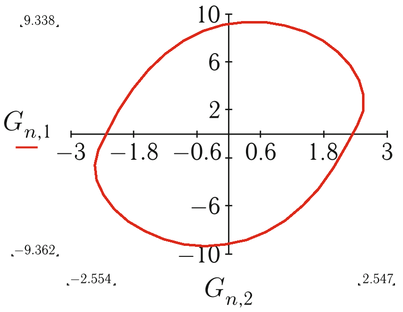


Fig. 35. $\varepsilon = 0.4$.

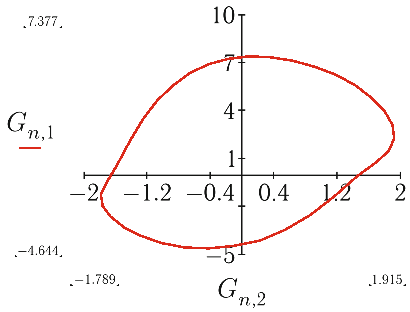


Fig. 36. $\varepsilon = 0.8$.

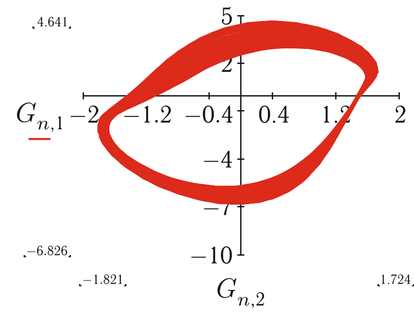


Fig. 37. $\varepsilon = 1$.

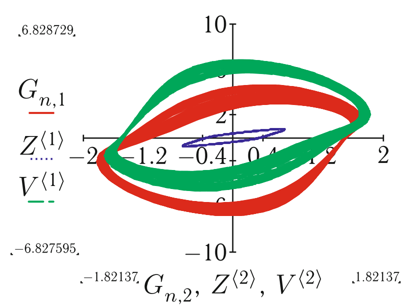


Fig. 38.

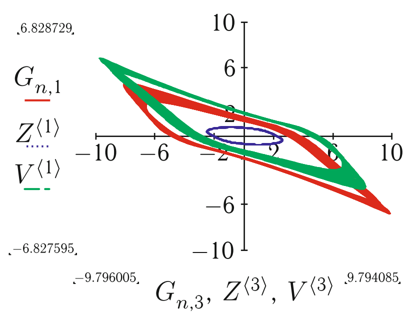


Fig. 39.

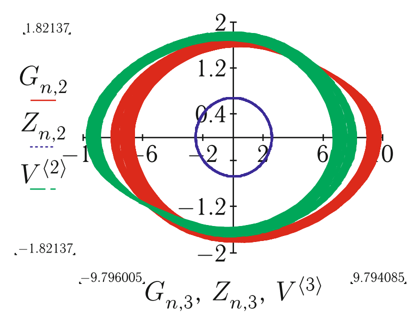
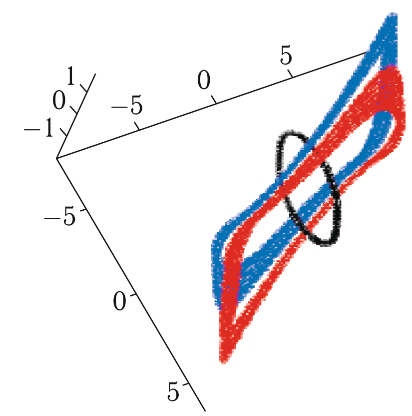


Fig. 40.



$(Z^{\langle 3\rangle}, Z^{\langle 1\rangle}, Z^{\langle 2\rangle})$, $(G^{\langle 1\rangle}, G^{\langle 2\rangle}, G^{\langle 3\rangle})$, $(V^{\langle 1\rangle}, V^{\langle 2\rangle}, V^{\langle 3\rangle})$

Fig. 41.

6. HIDDEN ATTRACTORS OF AIRCRAFT CONTROL SYSTEM

In the presence of external perturbations in aircraft control system, one can face unstable modes, which can lead to catastrophic results. Such modes (hidden oscillations) are usually not detected by an approximate linear analysis of control systems used in engineering. As a rule, systems constructed on the basis of such an analysis operate normally in the case in which defining and perturbing influences in the system are sufficiently small. But if perturbations become large, then the control system cannot necessarily parry related discoordinations caused by a nonlinearity like a "saturation" in a control contour. Oscillational processes generated in the system and corresponding to maximum possible amplitudes of the input influence on the control object are referred to as "wind up." The paper [41] dealt with the survey of methods of the elimination of wind-up used in engineering and to the development of new methods of anti-windup correction for aircraft control systems. Below we illustrate the possibility of the generation of hidden oscillations in a rocket control system in the case of a nonlinearity of "saturation" type in the control contour (similar results were obtained in [22]).

The linearized model of the control system for the flexible space launch vehicle is described by the system of equations [42]

$$\begin{aligned} \dddot{\psi}(t) + a_y^{\dot{\psi}(t)}\dot{\psi}(t) + a_y^{\psi}\psi(t) &= a_y^{\delta_r}\delta_r(t) + f_y(t), \\ \ddot{\tilde{\psi}}(t) + 2\xi_1\omega_1\dot{\tilde{\psi}}(t) + \omega_1^2\tilde{\psi}(t) &= l_1\omega_1^2\delta_r(t) + \tilde{f}_y(t), \\ \psi_g(t) &= \psi(t) + \tilde{\psi}(t). \end{aligned} \tag{25}$$

Here the first equation describes the dynamics of the vehicle as an absolutely rigid body, where ψ is the yaw angle, and δ_r is the angle of the rudder deviation. The second equation describes the dynamics of the control system, and the third one describes the system output. The external perturbations $f_t(t)$ and $\tilde{f}_y(t)$ are assumed to be absent.

By taking into account some constraints for the dynamics and the rudder servomotor, the operation of the rudder servomotor can be described as follows:

$$\delta_r = M\,\mathrm{sat}\left(\frac{u}{M}\right),$$

where $u(t)$ is the control signal generated by the angular stabilization system. Here

$$\mathrm{sat}(\sigma) = \mathrm{sgn}(\sigma)\min(1, |\sigma|), \qquad u(t) = -k_p\psi_g(t) - k_D\dot{\psi}_g(t),$$

where k_p and k_D are the proportional and differential gain factors, respectively.

By setting $X = \left(\psi, \dot{\psi}, \tilde{\psi}, \dot{\tilde{\psi}}\right)$, we rewrite system (26) in the form (2) with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_y^{\psi} & 0 & -a_y^{\dot{\psi}} & 0 \\ 0 & -\omega_1^2 & 0 & -2\xi_1\omega_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ a_\gamma^{\delta_r} \\ l_1\omega_1^2 \end{pmatrix}, \qquad C = \begin{pmatrix} -k_p \\ -k_p \\ -k_D \\ -k_D \end{pmatrix}.$$

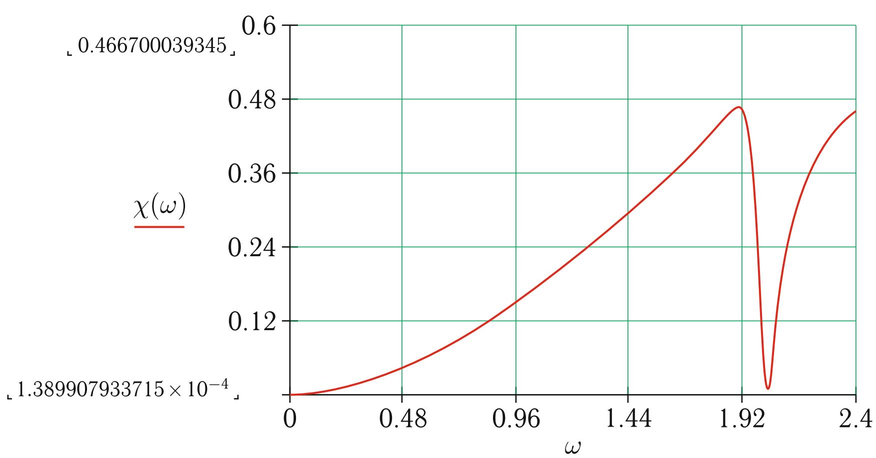


Fig. 42.

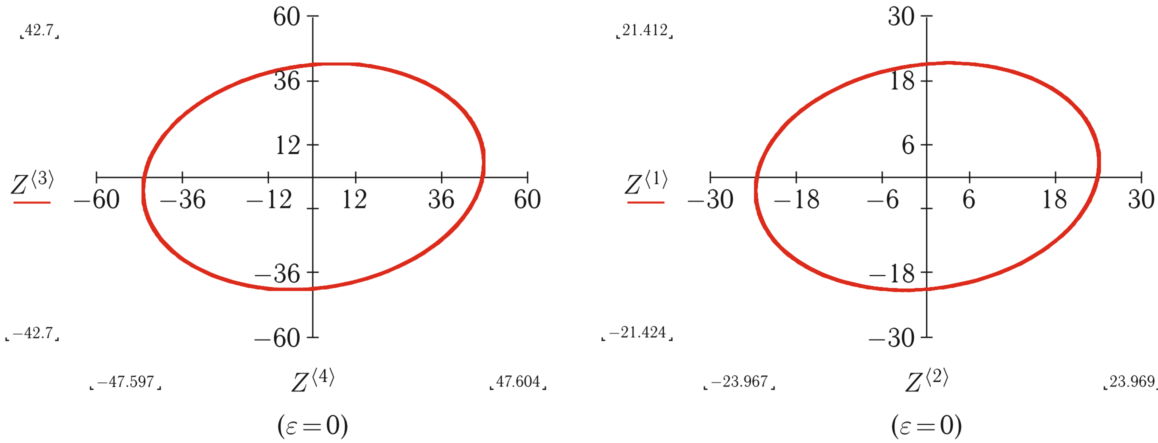


Fig. 43.

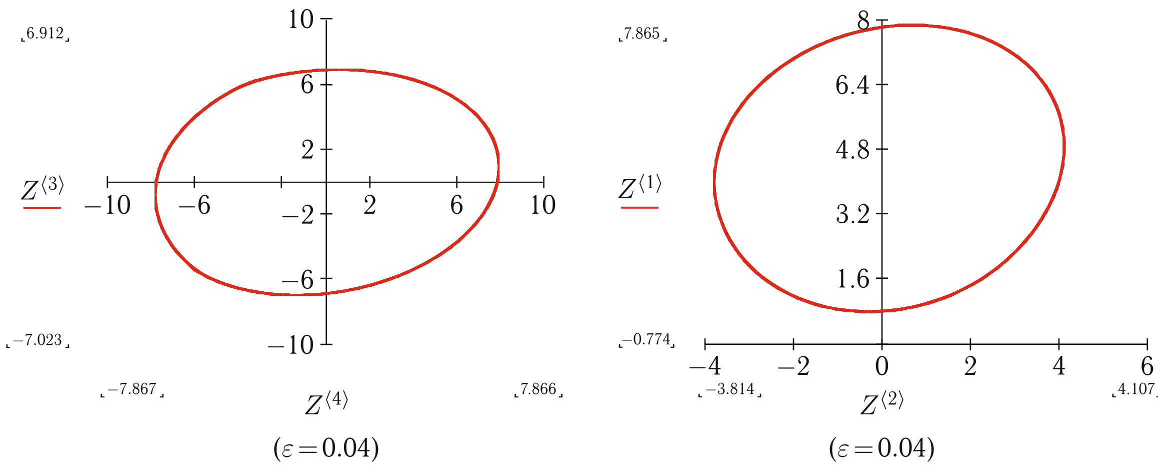


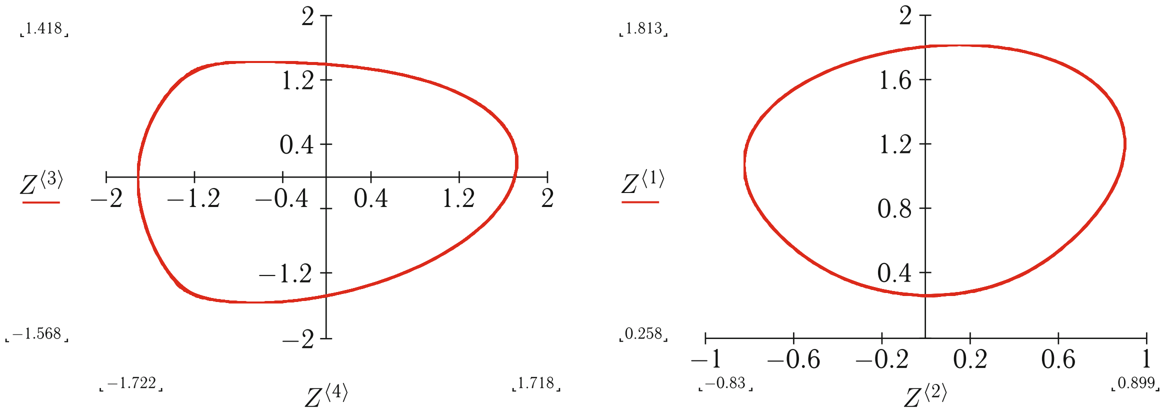
Fig. 44.

We define the following values of coefficients of the studied model [42] : $a_\gamma^\psi = -4c^{-2}$, $a_\gamma^{\dot\psi} = 0.4c^{-1}$, $a_\gamma^{\delta_r} = 14c^{-2}$, $\omega_1 = 2c^{-1}$, $\xi_1 = 0.03$, $l_1 = -0.12c^{-2}$, $k_p = 5$, $k_D = 2$, $M = 0.174$. Then

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0.4 & 0 \\ 0 & -4 & 0 & -12 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 14 \\ -0.48 \end{pmatrix}, \qquad C = \begin{pmatrix} -5 \\ -5 \\ -2 \\ -2 \end{pmatrix},$$

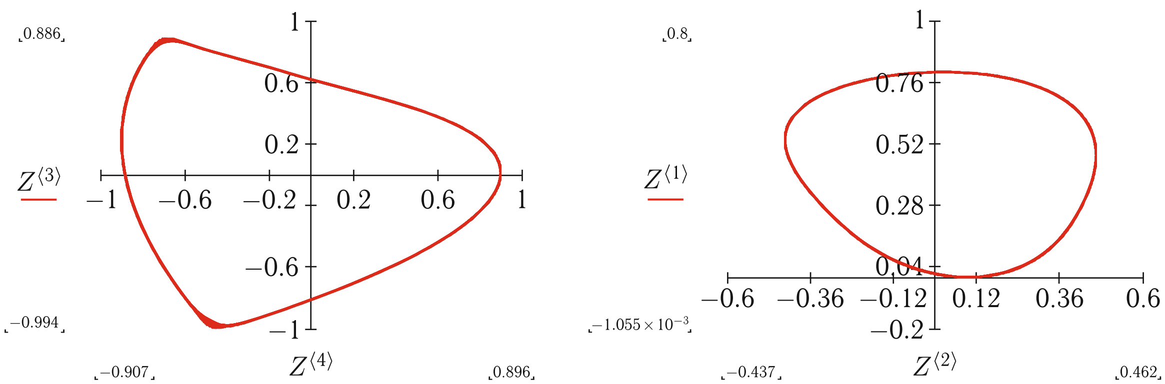
$$W(p) = \frac{2(1690p^3 + 4411p^2 + 7705p + 18100)}{(5p^2 + 2p - 20)(25p^2 + 3p + 100)}.$$

Let us perform the linear analysis of the system: we extract sectors of stability and instability of the linear system $\dot X = (A + \mu BC^*)X$ for various values of $\mu \in (0, \infty)$. For $\mu \in (0, 0.05524)$, the matrix $A_\mu = A + \mu BC^*$ has one positive eigenvalue and three eigenvalues in the open left half-plane. For some $\tilde\mu_1 \in (0.05524, 0.05525)$, the positive eigenvalue passes into the left half-plane, and A_μ becomes a Hurwitz matrix. For $\mu \in (0.05525, 0.056)$, A_μ remains a Hurwitz matrix (the Hurwitz sector). For some $\tilde\mu_2 \in (0.056, 0.057)$, two eigenvalues of A_μ become pure imaginary, and two ones remain in the open left half-plane. For $\mu \in (0.057, 0.1633)$ the matrix A_μ has two complex conjugate eigenvalues with positive real parts and two eigenvalues with negative real parts (the sector of the instability of degree 2). For some $\tilde\mu_3 \in (0.1633, 0.1634)$, the matrix A_μ has a pair of pure imaginary



($\varepsilon = 0.2$) ($\varepsilon = 0.2$)

Fig. 45.



($\varepsilon = 0.5$) ($\varepsilon = 0.5$)

Fig. 46.

eigenvalues again and two eigenvalues with negative real parts. Finally, A_μ is a Hurwitz matrix for $\mu \in (0.1634, \infty)$.

If we set $\varphi(\sigma) = \sigma$ in the considered system, then we obtain a globally stable linear system.

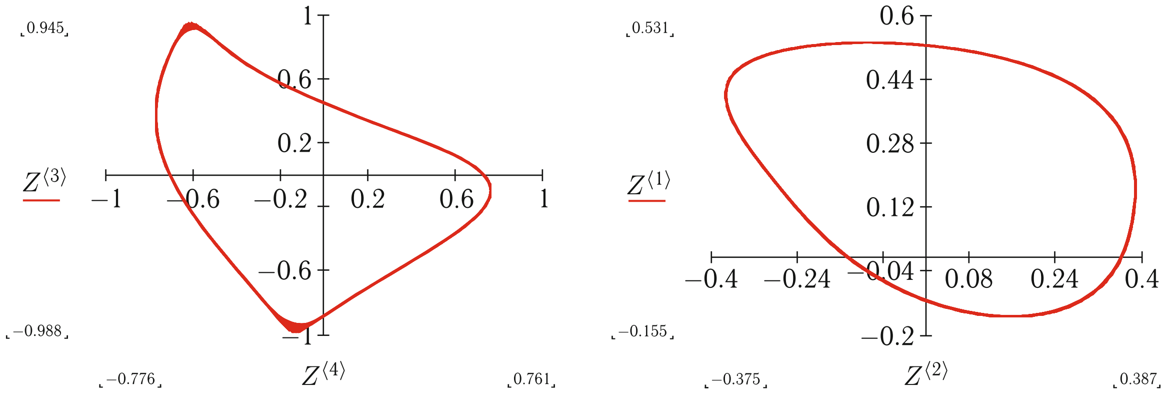
Set $\mu_1 = 0.0555$, $\mu_2 = 0.14$, and $\lambda = 0.009$. The graph of the function

$$\chi(\omega) = \mu_1 \mu_2 |W(i\omega - \lambda)|^2 + (\mu_1 + \mu_2) \operatorname{Re} W(i\omega - \lambda) + 1$$

is shown in Fig. 42. Therefore, assumptions 3 of Theorem 1 are satisfied in this case. In addition, for all $\mu \in [\mu_1, \mu_2]$ the matrix $A_\mu = A + \mu B C^*$ has two eigenvalues in the open right half-plane and has no one in the strip $-\lambda \le \operatorname{Re} p \le 0$.

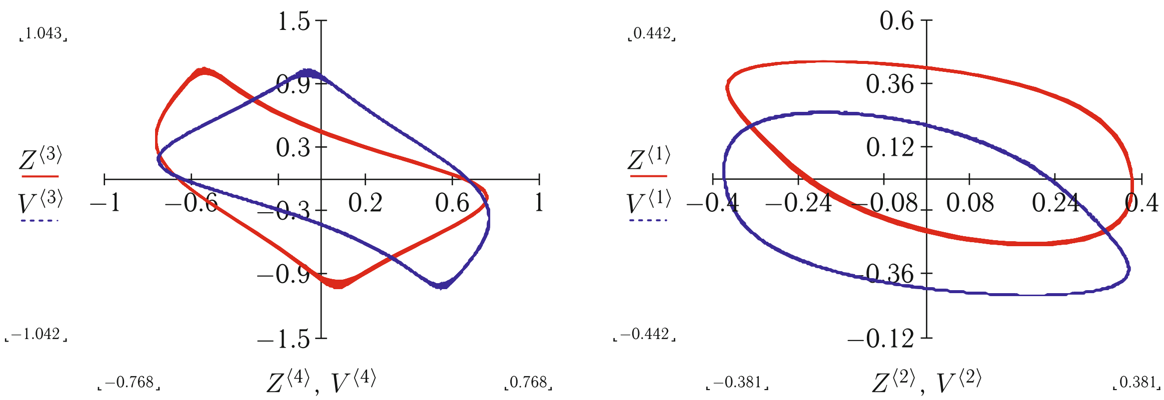
In accordance with chosen parameters of the system, here the nonlinearity of the "saturation" type has the form $\varphi(\sigma) = 0.5(|\sigma + 0.174| - |\sigma - 0.174|)$. By using the performed linear analysis, as the "starting" inequality providing the existence of a cycle at a starting point from an arbitrarily small neighborhood of the equilibrium, we take

$$\psi(\sigma) = \begin{cases} 0.0555\sigma - 0.174\left(1 - \dfrac{0.0555}{0.13}\right) & \text{for} \quad \sigma < -\dfrac{0.174}{0.13}, \\ 0.13\sigma & \text{for} \quad |\sigma| \le \dfrac{0.174}{0.13}, \\ 0.0555\sigma + 0.174\left(1 - \dfrac{0.0555}{0.13}\right) & \text{for} \quad \sigma > \dfrac{0.174}{0.13}. \end{cases}$$



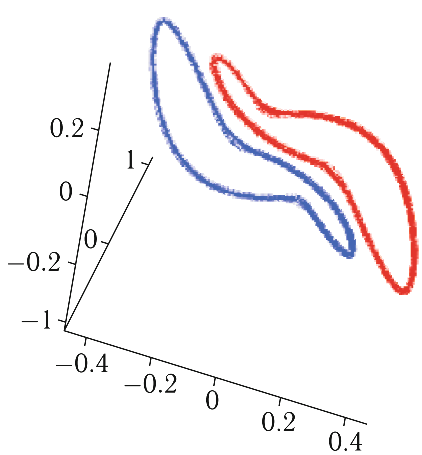
$(\varepsilon = 0.8)$ $(\varepsilon = 0.8)$

Fig. 47.



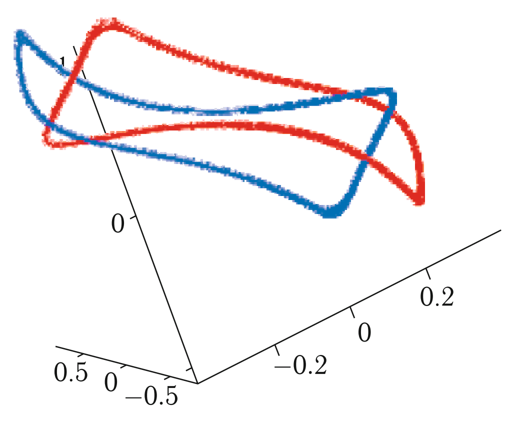
$(\varepsilon = 1)$ $(\varepsilon = 1)$

Fig. 48.



$(Z^{\langle 1\rangle}, Z^{\langle 2\rangle}, Z^{\langle 3\rangle}), (V^{\langle 1\rangle}, V^{\langle 2\rangle}, V^{\langle 3\rangle})$

Fig. 49.



$(Z^{\langle 2\rangle}, Z^{\langle 3\rangle}, Z^{\langle 4\rangle}), (V^{\langle 2\rangle}, V^{\langle 3\rangle}, V^{\langle 4\rangle})$

Fig. 50.

The results of the step operation of the algorithm for the localization of the hidden attractor of system (25) are represented in Figs. 43–48 (projections onto the coordinate subspaces of dimension 2). In Fig. 48, one can see that the system has two hidden attractors-twins. Figures 49 and 50 present the projections of the hidden attractors-twins onto the subspace of dimension 3.

We have clearly demonstrated that the use of only linear analysis for the design of aircraft control systems does not guarantee the absence of complicated oscillatory modes in the system, and they can be detected only by its special nonlinear analysis.

ACKNOWLEDGMENTS

The research was supported by the Russian Scientific Foundation (project no. 14-21-00041) and Saint Petersburg State University.

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