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CONTROL THEORY

Second-Order Necessary Optimality Conditions for a Discrete Optimal Control Problem

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Abstract—We obtain second-order necessary optimality conditions for a discrete optimal control problem. These conditions are proved without the a priori assumption of the differentiability of the right-hand side of the control system with respect to the variable corresponding to the control parameter and under general constraints on control.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Discrete optimal control problems are of interest both theoretically (numerous algorithms for solving problems in continuous time involve time discretization) as well as practically (for example, some models in economics are studied in discrete time).

The present paper deals with second-order necessary optimality conditions for the discrete optimal control problem

$$\begin{aligned}&\varphi(x(N+1)) \to \min,\\ &x(t+1) = f(t, x(t), u(t)), \qquad t \in [0, N], \qquad x(0) = x_0,\\ &u(t) \in U(t), \qquad t \in [0, N].\end{aligned} \tag{1}$$

Here N is a nonnegative integer, $[0, N] := \{0, 1, \ldots, N\}$, $f:\ [0, N]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ and $\varphi:\ \mathbb{R}^n \to \mathbb{R}$ are given functions, and $U(\cdot)$ is a given mapping of $[0, N]$ into $\mathbb{R}^m$.

An *admissible process* in problem (1) is a pair (x, u), $u = (u(0), \ldots, u(N))$, $u(i) \in \mathbb{R}^m$, $x = (x(0), \ldots, x(N+1))$, $x(i) \in \mathbb{R}^n$, such that u satisfies the condition $u(t) \in U(t)$, $t \in [0, N]$, and x is a solution of the equation $x(t+1) = f(t, x(t), u(t))$, $t \in [0, N]$, with the initial condition $x(0) = x_0$. The component u of an admissible process (x, u) is called an *admissible control*, and the component x is referred to as the *corresponding trajectory*. The minimum in problem (1) is taken over the set of admissible processes (x, u).

An admissible process $(\bar{x}, \bar{u})$ is called an *optimal process* in problem (1) if it provides a local minimum of the functional φ, i.e., if $\varphi(\bar{x}(N+1)) \le \varphi(x(N+1))$ for all admissible processes (x, u) in some neighborhood of $(\bar{x}, \bar{u})$. Here a neighborhood is understood in the sense of the topology of a linear finite-dimensional space. The component $\bar{u}$ of an optimal process $(\bar{x}, \bar{u})$ is called an *optimal control*, and the component $\bar{x}$ is called the *corresponding trajectory*.

First-order necessary optimality conditions in problems with continuous time (the Pontryagin maximum principle) were obtained in the 1950s (see [1, p. 23]). For some time, it was expected that a similar result would hold for the discrete problem as well, and even some erroneous proofs were published (for example, [2]). A result in the form of a maximum principle was obtained in [3] under the assumption that the set of admissible velocities is convex. Later, the convexity assumption was replaced by the weaker condition of directional convexity (see [4]) or by local convexity (see [5]). The monograph [6] deals with optimal control problems with discrete time.

The theory of second-order conditions for optimal control problems with discrete time started to develop relatively recently (e.g., see [7–10]). It was assumed in these papers that the function f specifying the dynamics of the system is differentiable with respect to the control and that the constraints on u are either absent [7, 9, 10] or given by equalities [8]. In the present paper, we do not assume a priori any smoothness of the function f with respect to u and consider general control constraints.

To study problems with discrete time, we use the finite-dimensional approximation method suggested and developed in [11–15] for problems with continuous time. In the general case, this method was used in the derivation of the maximum principle for an optimal control problem with delay in the control (see [16]) and for problems with state constraints (see [14, p. 141]). In addition, local controllability conditions were obtained in [12] by the finite-dimensional approximation method.

The use of this method is simplified significantly for problems with discrete time, which are finite-dimensional by their nature. This permitted us not only to obtain new second-order necessary conditions but also to strengthen well-known first-order necessary conditions. (For details, see below.)

For a vector $p \in \mathbb{R}^n$, set

$$H(t,x,u,p) := p^{\mathrm{T}} f(t,x,u).$$

Let us present first-order necessary optimality conditions for problem (1).

Theorem 1. *Let* $(\overline{x},\overline{u})$ *be an optimal process in problem* (1). *In addition, let the function* φ *satisfy the Lipschitz condition in a neighborhood of the point* $\bar{x}(N+1)$, *be differentiable at that point, and satisfy the following conditions for all* $t \in [0,N]$: $f(t,\cdot,\cdot)$ *is a continuous function, the function* $f(t,\cdot,\bar{u}(t))$ *is differentiable at the point* $x = \bar{x}(t)$, *the sets* $U(t)$ *are closed, and the sets* $f(t,\bar{x}(t),U(t))$ *are convex. Then there exists a solution* $p\colon\ [0,N] \to \mathbb{R}^n$ *of the adjoint system*

$$p(t) = H_x(t,\overline{x}(t),\overline{u}(t),p(t+1)), \qquad t \in [0,N], \qquad p(N+1) = -\varphi_x(\overline{x}(N+1)) \tag{2}$$

for which one has the maximum condition

$$H(t,\overline{x}(t),\overline{u}(t),p(t+1)) = \max_{u\in U(t)} H(t,\overline{x}(t),u,p(t+1)), \qquad t \in [0,N]. \tag{3}$$

Note that this theorem strengthens the well-known first-order optimality conditions (see [17]) under the assumption that the function f is differentiable with respect to x at the point $\bar{x}(t)$ rather than in an entire neighborhood of the point $\bar{x}(t)$ and only for $u = \bar{u}(t)$. The sets of admissible velocities $f(t,\bar{x}(t),U(t))$ are as well assumed to be only convex for $x = \bar{x}(t)$ rather than in a neighborhood of the point $\bar{x}(t)$.

Theorem 1 may fail to provide useful information; i.e., there exists a problem in which an admissible pair (x,u) is not optimal, even though the first-order necessary optimality conditions (Theorem 1) hold for it. Let us illustrate this by the following example.

Example 1. Consider the one-step problem

$$-x^2(1) \to \min, \qquad x(1) = |u|, \qquad u \in \mathbb{R}. \tag{4}$$

Here $N = 0$, $\varphi(x) = -x^2$, $f(t,x,u) = |u|$, and $U = \mathbb{R}$. Obviously, $\overline{u} = 0$ is not a solution. However, $\overline{u} = 0$ satisfies the condition of maximum (3). Indeed, if $\overline{x}(1) = 0$, then $p(1) = 0$. Consequently, $H(0,\overline{x}(0),\overline{u},p(1)) = p(1)|\overline{u}| = 0 = \max_{u\in\mathbb{R}} p(1)|u|$.

Let $\Psi(t)$, $t \in [1,N+1]$, be the matrix function determined from the system

$$\Psi(t) = f_x^{\mathrm{T}}(t)\Psi(t+1)f_x(t) + H_{xx}(t), \qquad t \in [0,N], \qquad \Psi(N+1) = \varphi_{xx}(\overline{x}(N+1)), \tag{5}$$

where $f_x(t) = f_x(t,\overline{x}(t),\overline{u}(t))$ and $H_x(t) = H_x(t,\overline{x}(t),\overline{u}(t),p(t+1))$.

The following theorem is the main result of the present paper. It strengthens Theorem 1 and makes it possible to exclude some nonoptimal processes satisfying Theorem 1.

Theorem 2. *Let $(\overline{x}, \overline{u})$ be an optimal process in problem* (1). *In addition, let the function φ be continuously differentiable in a neighborhood of the point $\bar{x}(N+1)$ and twice differentiable at that point. Next, let the following assumptions be satisfied for all $t \in [0, N]$: $f(t, \cdot, \cdot)$ is a continuous function, the function $f(t, \cdot, \bar{u}(t))$ is continuously differentiable in a neighborhood of the point $x = \bar{x}(t)$ and twice differentiable at that point, the sets $U(t)$ are closed, and the sets $f(t, \bar{x}(t), U(t))$ are convex.*

Let the relation

$$H(\theta, \overline{x}(\theta), \overline{u}(\theta), p(\theta+1)) = H(\theta, \overline{x}(\theta), v, p(\theta+1)) \tag{6}$$

be valid for $\theta \in [0, N]$ and $v \in U(\theta)$, where $p(\cdot)$ is a solution of the adjoint system (2) *satisfying the maximum condition* (3). *Then*

$$\Psi(\theta+1)[f(\theta, \overline{x}(\theta), v) - f(\theta, \overline{x}(\theta), \overline{u}(\theta))]^2 \geq 0. \tag{7}$$

Here $\Psi[w]^2 = w^{\mathrm{T}}\Psi w$ is a quadratic form or, in a more general case, $\Psi[w_1, w_2] = w_1^{\mathrm{T}}\Psi w_2$.

In Example 1, let us apply Theorem 2 to the control $\bar{u} = 0$ satisfying Theorem 1. Obviously, condition (6) is valid at the point $\theta = 0$ for all $v \in \mathbb{R}$. From system (5), we find $\Psi(1) = -2$; therefore, inequality (7) has the form $-2v^2 \geq 0$ for all v, which is not the case. Therefore, the control $\overline{u} = 0$ is not optimal. Consequently, Theorem 2 makes it possible to exclude some nonoptimal processes satisfying Theorem 1. Note that other well-known results (see [7–10]) cannot be applied to this example, because the function f is not differentiable with respect to the variable u at the point $\bar{u} = 0$.

2. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. Fix an arbitrary number $\theta \in [0, N]$ and a vector $y \in f(\theta, \overline{x}(\theta), U(\theta)) =: A$ and take a vector $v \in U(\theta)$ such that $y = f(\theta, \overline{x}(\theta), v)$. Set $\overline{y} = f(\theta, \overline{x}(\theta), \overline{u}(\theta))$, and let $x(t, y)$, $t \in [0, N]$, be the trajectory of problem (1) corresponding to the parameter y, or, equivalently, to the control

$$u(t) = \begin{cases} \overline{u}(t) & \text{for} \quad t \neq \theta, \\ v & \text{for} \quad t = \theta. \end{cases}$$

Note that $x(t, \overline{y}) = \overline{x}(t)$ for all t in $[0, N]$.

For arbitrary indices $t, s \in [1, N+1]$, we introduce the $n \times n$ matrix $\Phi(t, s)$,

$$\Phi(t, s) = \begin{cases} 0 & \text{for} \quad t < s, \\ I & \text{for} \quad t = s, \\ f_x(t-1) f_x(t-2) \cdots f_x(s) & \text{for} \quad t > s. \end{cases} \tag{8}$$

First, let us show that the function $y \mapsto x(t, y)$ is differentiable with respect to the set A at the point $\overline{y}$ for all $t \in [1, N+1]$. Recall the related definition introduced in the monograph [13, p. 107]. A function $y \mapsto x(t, y)$ is *differentiable with respect to a set* A at a point $\overline{y}$ if there exists a matrix $D_y x(t, \bar{y})$ such that

$$x(t, y) - x(t, \bar{y}) = D_y x(t, \bar{y})(y - \bar{y}) + o(|y - \bar{y}|), \qquad y \in A.$$

In addition, let us show that such a derivative of the function $y \mapsto x(t, y)$ is computed by the rule

$$D_y x(t, \overline{y}) = \Phi(t, \theta+1), \qquad t \in [1, N+1]. \tag{9}$$

Indeed, this assertion holds for all $t \leq \theta$, because, in this case, both the right- and left-hand sides in relation (9) are zero. For $t \geq \theta + 1$, we prove this assertion by induction.

If $t = \theta + 1$, then $x(\theta+1, y) = f(\theta, \overline{x}(\theta), v) = y$, and consequently,

$$D_y x(\theta+1, \overline{y}) = I = \Phi(\theta+1, \theta+1).$$

Therefore, formula (9) holds for $t = \theta + 1$.

Now assume that relation (9) holds for all $\tau \in [1, t]$ with some $t > \theta + 1$. It remains to show that

$$D_y x(t+1, \overline{y}) = \Phi(t+1, \theta+1). \tag{10}$$

By differentiating both sides of the equation

$$x(t+1, y) = f(t, x(t, y), \overline{u}(t)), \qquad t \geq \theta + 1,$$

with respect to y, we obtain

$$D_y x(t+1, \bar{y}) = f_x(t, x(t, \bar{y}), \overline{u}(t)) D_y x(t, \bar{y}), \qquad t \geq \theta + 1. \tag{11}$$

By the inductive assumption and definition (8) of the matrix Φ, we have the relation

$$D_y x(t+1, \overline{y}) = f_x(t)\Phi(t, \theta+1) = \Phi(t+1, \theta+1).$$

The proof of relation (9) is complete.

By construction, $y = \overline{y}$ is a solution of the problem

$$\varphi(x(N+1, y)) \to \min, \qquad y \in A. \tag{12}$$

Let us write out the first-order necessary optimality conditions for this problem,

$$0 \in D_y \varphi(x(N+1, \overline{y})) + N(\overline{y}, A) = \varphi_x^{\mathrm{T}}(\overline{x}(N+1)) D_y x(N+1, \overline{y}) + N(\overline{y}, A); \tag{13}$$

here $N(\cdot\,,\cdot)$ is the normal cone in the sense of convex analysis, and $D_y\varphi$ is treated as a row vector. By applying relation (13) to $(y - \overline{y})$ and by taking into account (9), we obtain

$$\varphi_x^{\mathrm{T}}(\overline{x}(N+1))\Phi(N+1, \theta+1)(y - \overline{y}) \geq 0. \tag{14}$$

Set

$$p^{\mathrm{T}}(t) = -\varphi_x^{\mathrm{T}}(\overline{x}(N+1))\Phi(N+1, t), \qquad t \in [0, N].$$

Since $\Phi(N+1, t) = \Phi(N+1, t+1) f_x(t)$, it follows that the function p satisfies the equation

$$p^{\mathrm{T}}(t) = p^{\mathrm{T}}(t+1) f_x(t) = H_x(t, \overline{x}(t), \overline{u}(t), p(t+1)), \qquad t \in [0, N], \tag{15}$$

with the condition

$$p(N+1) = -\varphi_x(\overline{x}(N+1)).$$

Inequality (14) can be rewritten as follows:

$$p^{\mathrm{T}}(\theta+1)(y - f(\theta, \overline{x}(\theta), \overline{u}(\theta))) \leq 0,$$

or

$$H(\theta, \overline{x}(\theta), \overline{u}(\theta), p(\theta+1)) \geq H(\theta, \overline{x}(\theta), v, p(\theta+1)).$$

By virtue of the arbitrary choice of θ and v, the maximum condition (3) is satisfied, which completes the proof of the theorem.

Proof of Theorem 2. Fix the parameters θ, y, and v as at the beginning of the proof of Theorem 1 and consider problem (12). Assume that the derivative of the function φ in the direction $\sigma = f(\theta, \overline{x}(\theta), v) - \overline{y}$ at the point $\overline{y}$ is zero; i.e.,

$$D_y \varphi(x(N+1, \overline{y}))\sigma = 0. \tag{16}$$

Then the second-order necessary condition is given by the nonnegativity of the corresponding quadratic form,

$$D_{yy}\varphi(x(N+1, \overline{y}))[\sigma, \sigma] \geq 0. \tag{17}$$

It follows from relation (16) [see (14)] that

$$\varphi_x^{\mathrm{T}}(\overline{x}(N+1))\Phi(N+1,\theta+1)(f(\theta,\overline{x}(\theta),v)-\overline{y})=0,$$

which is equivalent to condition (6). Condition (17) for the vectors $y=f(\theta,\overline{x}(\theta),v)$ and $\sigma=y-\overline{y}$ implies that $D_{yy}\varphi(x(N+1,\overline{y}))[y-\overline{y},y-\overline{y}]\geq 0$, or

$$\sum_{i,j=1}^{n}\frac{\partial^2\varphi}{\partial y_i\,\partial y_j}(x(N+1,\overline{y}))(y_i-\overline{y}_i)(y_j-\overline{y}_j)\geq 0. \tag{18}$$

Let us compute the derivatives $\dfrac{\partial^2\varphi}{\partial y_i\,\partial y_j}(x(N+1,\overline{y}))$, $i,j\in\{1,\dots,n\}$. Take arbitrary indices i and j. By differentiating both sides of the relation

$$\frac{\partial\varphi}{\partial y_i}(x(N+1,y))=\varphi_x^{\mathrm{T}}(x(N+1,y))\frac{\partial x}{\partial y_i}(N+1,y)$$

with respect to the variable y_j at the point $y=\overline{y}$, we obtain the representation

$$\frac{\partial^2\varphi}{\partial y_i\,\partial y_j}(x(N+1,\overline{y}))$$
$$=\varphi_{xx}(\overline{x}(N+1))\left[\frac{\partial x}{\partial y_i}(N+1,\overline{y}),\frac{\partial x}{\partial y_j}(N+1,\overline{y})\right]+\varphi_x^{\mathrm{T}}(\overline{x}(N+1))\frac{\partial^2 x}{\partial y_i\,\partial y_j}(N+1,\overline{y}). \tag{19}$$

Let us show that

$$\frac{\partial^2 x}{\partial y_i\,\partial y_j}(t,\bar{y})=\sum_{\tau=\theta+1}^{t-1}\Phi(t,\tau+1)\sum_{k,l=1}^{n}\frac{\partial^2 f}{\partial x_k\,\partial x_l}(\tau)\frac{\partial x_k}{\partial y_i}(\tau,\bar{y})\frac{\partial x_l}{\partial y_j}(\tau,\bar{y}), \tag{20}$$

where $\dfrac{\partial^2 f}{\partial x_k\,\partial x_l}(t):=\dfrac{\partial^2 f}{\partial x_k\partial x_l}(t,\bar{x}(t),\bar{u}(t))$.

Indeed, by differentiating both sides of the relation

$$x(t+1,y)=f(t,x(t,y),\bar{u}(t)),\qquad t\geq\theta+1,$$

with respect to the variable y_i, we obtain

$$\frac{\partial x}{\partial y_i}(t+1,y)=f_x(t,x(t,y),\bar{u}(t))\frac{\partial x}{\partial y_i}(t,y)=\sum_{k=1}^{n}\frac{\partial f}{\partial x_k}(t,x(t,y),\bar{u}(t))\frac{\partial x_k}{\partial y_i}(t,y),\qquad t\geq\theta+1.$$

By again differentiating both sides of the resulting relation with respect to y_j at the point $y=\bar{y}$, we obtain the equation

$$\frac{\partial^2 x}{\partial y_i\partial y_j}(t+1,\bar{y})=f_x(t)\frac{\partial^2 x}{\partial y_i\,\partial y_j}(t,\bar{y})+\sum_{k,l=1}^{n}\frac{\partial^2 f}{\partial x_k\,\partial x_l}(t)\frac{\partial x_k}{\partial y_i}(t,\bar{y})\frac{\partial x_l}{\partial y_j}(t,\bar{y}),\qquad t\geq\theta+1. \tag{21}$$

By virtue of the relation $x(\theta+1,y)=f(\theta,\bar{x}(\theta),v)=y$, we have

$$\frac{\partial^2 x}{\partial y_i\,\partial y_j}(\theta+1,\bar{y})=0. \tag{22}$$

We have thereby found that, for all $i,j\in\{1,\dots,n\}$, the function $\dfrac{\partial^2 x}{\partial y_i\,\partial y_j}(\cdot,\bar{y})$ satisfies the linear equation (21) with the initial condition (22).

Recall that the solution of the equation $z(t+1) = f_x(t)z(t)+h(t)$, $t \geq 0$, is given by the function $z(t) = \Phi(t,0)z(0) + \sum_{\tau=0}^{t-1} \Phi(t,\tau+1)h(\tau)$. Consequently, the solution (22) of problem (21) satisfies the relation

$$\frac{\partial^2 x}{\partial y_i\,\partial y_j}(t,\bar{y}) = \Phi(t,\theta+1)\frac{\partial^2 x}{\partial y_i\,\partial y_j}(\theta+1,\bar{y}) + \sum_{\tau=\theta+1}^{t-1} \Phi(t,\tau+1) \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial x_k\,\partial x_l}(\tau)\frac{\partial x_k}{\partial y_i}(\tau,\bar{y})\frac{\partial x_l}{\partial y_j}(\tau,\bar{y})$$
$$= \sum_{\tau=\theta+1}^{t-1} \Phi(t,\tau+1) \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial x_k\,\partial x_l}(\tau)\frac{\partial x_k}{\partial y_i}(\tau,\bar{y})\frac{\partial x_l}{\partial y_j}(\tau,\bar{y}).$$

This completes the proof of relation (20).

By virtue of relation (20), the last term in (19) can be represented in the form

$$\begin{aligned}
\varphi_x^{\mathrm{T}}(\bar{x}(N+1))&\frac{\partial^2 x}{\partial y_i\,\partial y_j}(N+1,\bar{y}) \\
&= \varphi_x^{\mathrm{T}}(\bar{x}(N+1)) \sum_{\tau=\theta+1}^{N} \Phi(N+1,\tau+1) \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial x_k\,\partial x_l}(\tau)\frac{\partial x_k}{\partial y_i}(\tau,\bar{y})\frac{\partial x_l}{\partial y_j}(\tau,\bar{y}) \\
&= \sum_{\tau=\theta+1}^{N} p^{\mathrm{T}}(\tau+1) \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial x_k\,\partial x_l}(\tau)\frac{\partial x_k}{\partial y_i}(\tau,\bar{y})\frac{\partial x_l}{\partial y_j}(\tau,\bar{y}) \\
&= \sum_{\tau=\theta+1}^{N} H_{xx}(\tau)\left[\frac{\partial x}{\partial y_i}(\tau,\bar{y}), \frac{\partial x}{\partial y_j}(\tau,\bar{y})\right]. \qquad (23)
\end{aligned}$$

By virtue of relations (19), (9), and (23), condition (18) has the form

$$\begin{aligned}
0 &\leq \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial y_i\,\partial y_j}(x(N+1,\bar{y}))(y_i-\bar{y}_i)(y_j-\bar{y}_j) \\
&= \varphi_{xx}(\bar{x}(N+1))[\Phi(N+1,\theta+1)(y-\bar{y})]^2 + \sum_{\tau=\theta+1}^{N} H_{xx}(\tau)[\Phi(\tau,\theta+1)(y-\bar{y})]^2 \\
&= \Phi^{\mathrm{T}}(N+1,\theta+1)\varphi_{xx}(\bar{x}(N+1))\Phi(N+1,\theta+1)[y-\bar{y}]^2 \\
&\quad + \sum_{\tau=\theta+1}^{N} \Phi^{\mathrm{T}}(\tau,\theta+1)H_{xx}(\tau)\Phi(\tau,\theta+1)[y-\bar{y}]^2. \qquad (24)
\end{aligned}$$

Let $\Psi(t)$, $t \in [1, N+1]$, be the symmetric matrix function determined by the system

$$\Psi(t) = \Phi^{\mathrm{T}}(N+1,t)\varphi_{xx}(\bar{x}(N+1))\Phi(N+1,t) + \sum_{\tau=t}^{N} \Phi^{\mathrm{T}}(\tau,t)H_{xx}(\tau)\Phi(\tau,t), \quad t \in [0,N],$$
$$\Psi(N+1) = \varphi_{xx}(\bar{x}(N+1)).$$

By virtue of definition (8) of the matrix Φ, we have

$$\begin{aligned}
\Psi(t) &= f_x^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(N+1,t+1)\varphi_{xx}(\bar{x}(N+1))\Phi(N+1,t+1)f_x(t) \\
&\quad + f_x^{\mathrm{T}}(t)\left\{\sum_{\tau=t+1}^{N} \Phi^{\mathrm{T}}(\tau,t+1)H_{xx}(\tau)\Phi(\tau,t+1)\right\}f_x(t) + H_{xx}(t) \\
&= f_x^{\mathrm{T}}(t)\Psi(t+1)f_x(t) + H_{xx}(t), \qquad t \in [0,N];
\end{aligned}$$

i.e., the function Ψ satisfies system (5). Condition (24) can be represented in terms of the function Ψ in the form

$$\Psi(\theta + 1)[y - f(\theta, \bar{x}(\theta), \bar{u}(\theta))]^2 \geq 0,$$

which completes the proof of the theorem.

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