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PARTIAL DIFFERENTIAL EQUATIONS

Classical Solution of a Problem with an Integral Condition for the One-Dimensional Wave Equation

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Abstract—We find a closed-form classical solution of the homogeneous wave equation with Cauchy conditions, a boundary condition on the lateral boundary, and a nonlocal integral condition involving the values of the solution at interior points of the domain. A classical solution is understood as a function that is defined everywhere in the closure of the domain and has all classical derivatives occurring in the equation and conditions of the problem. The derivatives are defined via the limit values of finite-difference ratios of the function and corresponding increments of the arguments.

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1. INTRODUCTION

Numerous papers deal with problems for the one-dimensional wave equation (e.g., see [1–15]). Most of them study classical solutions. Note that numerical methods in the form of difference schemes or finite elements for solving boundary value problems for differential equations are based on the assumptions of existence of classical solutions of these problems. However, under the same boundary conditions without a correct choice of the functions in these conditions satisfying so-called matching conditions, the problem will have no classical solution: the solution does not have the same smoothness in the entire domain, which is required by the numerical method applied to the case. This aspect was analyzed in detail in [9, 10], where a classical solution of the mixed problem was found for the equation of small transverse vibrations of a string.

In the mathematical modeling of processes in physics and other fields of science, there arise problems in which the values of the unknown function are defined in integral form [16]. Numerous papers deal with such nonlocal problems (see [17–24]). However, in most of those papers, the problems are considered in a generalized statement. As was mentioned above, classical solutions play an important role from the viewpoint of practical use. It turns out that the method of finding classical solutions suggested in the case of boundary value problems for hyperbolic equations [1–10] can also be used for nonlocal problems with integral conditions. In the present paper, we use this approach for the simplest problem for the one-dimensional wave equation in the case of an integral condition. It can also be used for hyperbolic equations in the case of problems with more general integral conditions without any essential modification with regard of the specific statement.

Note also the papers [4, 25, 26] in which solutions were found by the above-mentioned method in the case of integral boundary conditions. However, the presence of singularities in these conditions does not permit one to find a classical solution in the entire domain where the unknown functions are to be defined.

2. STATEMENT OF THE PROBLEM

In the closure $\overline{Q} = [0,\infty) \times [0,l]$ of the domain $Q = (0,\infty) \times (0,l)$ of two independent variables $(t,x) \in \overline{Q} \subset \mathbb{R}^2$, consider the wave equation

$$(\partial_{tt} - a^2\partial_{xx})u(t,x) = f(t,x), \qquad (t,x) \in \overline{Q}, \tag{2.1}$$

where a^2 and l are positive real numbers, $(0,l) \subset [0,l]$, $(0,\infty) \subset [0,\infty)$, and $\partial_{tt} = \partial^2/\partial t^2$ and $\partial_{xx} = \partial^2/\partial x^2$ are second partial derivatives with respect to t and x. Equation (2.1) is supplemented with the Cauchy condition

$$u(0,x) = \varphi(x), \qquad \partial_t u(0,x) = \psi(x), \qquad x \in [0,l], \tag{2.2}$$

on the lower base $\Omega^{(0)} = \{(t,x) \in \overline{Q} | t = 0\}$ of the half-strip $\overline{Q}$, the boundary condition

$$u(t,0) = \mu(t), \qquad t \in [0,\infty), \tag{2.3}$$

on the lateral line $\Gamma = \{(t,x) \in \overline{Q},\ x = 0\}$, and the integral condition

$$\int\limits_0^l u(t,x)\,dx = s(t), \qquad t \in [0,l]. \tag{2.4}$$

Here $f:\ \overline{Q} \ni (t,x) \to f(t,x)$ is a given function on $\overline{Q}$, $\varphi:\ [0,l] \ni x \to \varphi(x) \in \mathbb{R}$ and $\psi:\ [0,l] \ni x \to \psi(x) \in \mathbb{R}$ are functions on $[0,l]$, and $\mu:\ [0,\infty) \ni t \to \mu(t) \in \mathbb{R}$ and $s:\ [0,\infty) \ni t \to s(t) \in \mathbb{R}$ are given functions on $[0,\infty)$ whose smoothness will be specified below.

The functions f, φ, ψ, μ, and s satisfy the matching conditions

$$\int\limits_0^l \varphi(x)\,dx - s(0) = 0, \tag{2.5}$$

$$\mu(0) - \varphi(0) = \delta^{(1)}, \qquad \frac{1}{a}(\psi(0) - \mu'(0)) = \delta^{(2)}, \qquad \frac{1}{a^2}(\mu''(0) - \varphi''(0) - f(0,0)) = \delta^{(3)}, \tag{2.6}$$

$$\begin{gathered}\varphi(0) - \frac{1}{a}\int\limits_0^l \psi(x)\,dx - \mu(0) + \frac{1}{a}s'(0) = \sigma^{(1)},\\
\varphi'(0) - \varphi'(l) + \frac{1}{a}\psi(0) + \frac{1}{a^2}s''(0) - \frac{1}{a}\mu'(0) - \frac{1}{a^2}\int\limits_0^l f(0,x)\,dx = \sigma^{(2)},\\
\varphi''(0) + \frac{1}{a}\psi'(0) - \frac{1}{a}\psi'(l) - \frac{1}{a^2}\mu''(0) + \frac{1}{a^3}s'''(0) + \frac{1}{a^2}f(0,0)\\
- \frac{1}{a^3}\int\limits_0^l \partial_t f(0,x)\,dx - \frac{1}{a^2}\int\limits_0^l \partial_x f(0,x)\,dx = \sigma^{(3)}.\end{gathered} \tag{2.7}$$

If all numbers $\sigma^{(0)}$, $\delta^{(j)}$, and $\sigma^{(j)}$, $j = 1,2,3$, in the matching conditions (2.5)–(2.7) are zero, then conditions (2.5)–(2.7) are said to be *homogeneous* with respect to the given functions in problem (2.1)–(2.4).

Note that, for sufficiently smooth given functions occurring in Eq. (2.1) and conditions (2.2)–(2.4), on the set $\overline{Q}$ there exists a unique classical solution of this problem if and only if the matching conditions (2.5)–(2.7) for these functions are homogeneous. Otherwise, on certain characteristics in the domain Q, the solution u of problem (2.1)–(2.7), together with its derivatives, has discontinuities. These discontinuities can be represented as matching conditions, which will be stated below. Therefore, in the general case, problem (2.1)–(2.7) can be replaced by problem (2.1)–(2.4) with matching conditions on the characteristics, where the jumps of the function and its derivatives are expressed via given real numbers $\delta^{(j)}$, $j = 1, 2, 3$, and $\sigma^{(j)}$, $j = 0, \ldots, 3$. We obtain a solution of problem (2.1)–(2.7) in closed form in terms of the functions f, φ, ψ, μ, and s.

3. SOLUTION OF EQUATION (2.1)

By using the reduction of Eq. (2.1) to the second canonical form, one can readily find a general solution of the homogeneous equation (2.1) [3], and it is given by the sum of two functions

$$g^{(1)}(x + at) + g^{(2)}(x - at), \tag{3.1}$$

where $g^{(i)}$ are arbitrary twice continuously differentiable functions on the corresponding domains $D(g^{(i)})$, where $D(g^{(1)}) = [0, \infty)$ and $D(g^{(2)}) = (-\infty, l]$ if $(t, x) \in \overline{Q}$ and $a > 0$.

By $C^k(G)$ we denote the set of continuous functions that are defined on G and have continuous derivatives of order $k \geq 0$, where for G one can take the interval $[0, l]$, the half-line $[0, \infty]$, the closure $\overline{Q}$ of the domain Q, and other sets.

Let a function $v : \mathbb{R}^2 \supset \overline{Q} \ni (t, x) \to v(t, x) \in \mathbb{R}$ belong to the class $C^2(\overline{Q})$ and satisfy Eq. (2.1). Such a function is referred to as a particular solution of this equation.

Lemma 3.1. *The general solution of Eq.* (2.1) *is the sum*

$$u(t, x) = \widetilde{u}(t, x) + v(t, x) = g^{(1)}(x + at) + g^{(2)}(x - at) + v(t, x), \tag{3.2}$$

where $\widetilde{u}$ is the general solution (3.1) *of the homogeneous equation* (2.1) *and v is a particular solution of the inhomogeneous equation* (2.1).

Proof. Suppose that, in addition to the solution (3.2), Eq. (2.1) has a solution U. Then the difference $u - U$ satisfies the homogeneous equation (2.1) and has the form (3.1). Therefore it follows that the assumed other solution U of Eq. (2.1) can also be represented in the form (3.2). The proof of the lemma is complete.

It follows from the general solution $\widetilde{u}$ that $\widetilde{u}(t, x) = \widetilde{u}^{(1)}(t, x) + \widetilde{u}^{(2)}(t, x)$. In addition to the function $\widetilde{u}$, we introduce the function $\overline{\widetilde{u}} = \widetilde{u}^{(1)} - \widetilde{u}^{(2)}$. From the representation of the general solution (3.2), we obtain the relations

$$a\,\partial_x \widetilde{u} = \partial_t \overline{\widetilde{u}}, \qquad a\,\partial_x \overline{\widetilde{u}} = \partial_t \widetilde{u}. \tag{3.3}$$

Let us construct a particular solution v of the inhomogeneous equation (2.1) with the properties (3.3).

Let a function $w : (t, \tau, x) \to w(t, \tau, x)$ be a solution of the homogeneous equation (2.1) with respect to the independent variables t and x and depend on a parameter $\tau \in [0, \infty)$; i.e., w satisfies the equation

$$\partial_{tt} w(t, \tau, x) - a^2 \partial_{xx} w(t, \tau, x) = 0. \tag{3.4}$$

In addition to Eq. (3.4), the function w satisfies the Cauchy conditions

$$w(0, \tau, x) = 0, \qquad \partial_t w(0, \tau, x) = f(\tau, x), \tag{3.5}$$

where the function f occurring in (3.5) is the right-hand side of Eq. (2.1). We solve problem (3.4), (3.5).

It is well known that the general solution of Eq. (3.4) is a sum of two functions,

$$w(t, \tau, x) = G^{(1)}(x + at, \tau) + G^{(2)}(x - at, \tau), \tag{3.6}$$

where the $G^{(j)}$ are arbitrary functions of the class C^2 with respect to the first argument with domains $D(G^{(j)}) = D(g^{(j)}) \times [0,\infty)$. By requiring that the function (3.6) satisfies the Cauchy conditions (3.5), we obtain the system of equations

$$G^{(1)}(x,\tau) + G^{(2)}(x,\tau) = 0, \tag{3.7}$$
$$a\,\partial_{x+at}G^{(1)}(x+at,\tau) - a\,\partial_{x-at}G^{(2)}(x-at,\tau) = f(\tau,x) \quad \text{for} \quad t=0,$$

or

$$a\,\partial_x G^{(1)}(x,\tau) - a\,\partial_x G^{(2)}(x,\tau) = f(\tau,x). \tag{3.8}$$

By integrating Eq. (3.7), we obtain

$$G^{(1,0)}(x,\tau) - G^{(2,0)}(x,\tau) = \frac{1}{a}\int_0^x f(\tau,\xi)\,d\xi + 2C(\tau), \tag{3.9}$$

where $C(\tau)$ is an arbitrary function in $C^2([0,\infty))$. From system (3.7), (3.9), we have

$$G^{(j,0)}(x,\tau) = \frac{(-1)^{j+1}}{2a}\int_0^x f(\tau,\xi)\,d\xi + (-1)^{j+1}C(\tau), \qquad j=1,2. \tag{3.10}$$

Since the function f is defined on $\overline{Q}$, it follows that the functions $G^{(j,0)}$ $(j=1,2)$ are defined by relations (3.10) only on the interval $[0,l]$ with respect to the first argument. Set

$$G^{(j)}(x,\tau) = \begin{cases} G^{(j,0)}(x,\tau) & \text{for} \quad x\in[0,l], \\ \widetilde{G}^{(j)}(x,\tau) & \text{for} \quad x\in D(g^{(j)})\backslash[0,l]. \end{cases} \tag{3.11}$$

Obviously, for the functions $G^{(j)}$ defined by relations (3.11) to belong to the class $C^2(D(g^{(i)}))$ with respect to the first argument, they should satisfy the matching conditions

$$\frac{\partial^p}{\partial x^p}\widetilde{G}^{(1)}(x,\tau) = \frac{\partial^p}{\partial x^p}G^{(1,0)}(x,\tau) \quad \text{for} \quad x=l,$$
$$\frac{\partial^p}{\partial x^p}\widetilde{G}^{(2)}(x,\tau) = \frac{\partial^p}{\partial x^p}G^{(2,0)}(x,\tau) \quad \text{for} \quad x=0, \quad p=0,1,2.$$

Therefore, the function $\widetilde{G}^{(1)}$ satisfies the conditions

$$\begin{gathered} \widetilde{G}^{(1)}(l,\tau) = \frac{1}{2a}\int_0^l f(\tau,\xi)\,d\xi + C(\tau), \\ \frac{\partial\widetilde{G}^{(1)}}{\partial x}(l,\tau) = \frac{1}{2a}f(\tau,l), \qquad \frac{\partial^2\widetilde{G}^{(1)}}{\partial x^2}(l,\tau) = \frac{1}{2a}\frac{\partial}{\partial x}f(\tau,l), \end{gathered} \tag{3.12}$$

and the function $\widetilde{G}^{(2)}$ satisfies the conditions

$$\widetilde{G}^{(2)}(0,\tau) = -C(\tau), \quad \frac{\partial G^{(2)}}{\partial x}(0,\tau) = -\frac{1}{2a}f(\tau,0), \quad \frac{\partial^2 G^{(2)}}{\partial x^2}(0,\tau) = -\frac{1}{2a}\frac{\partial}{\partial x}f(\tau,0). \tag{3.13}$$

On the basis of the function w defined by relations (3.6), (3.7), and (3.10)–(3.13), we introduce the function $v:\ \overline{Q} \ni (t,x) \to v(t,x) \in \mathbb{R}$ by the relation

$$\begin{aligned} v(t,x) = \int_0^t w(t-\tau,\tau,x)\,d\tau &= \int_0^t G^{(1)}(x+a(t-\tau),\tau)\,d\tau \\ &+ \int_0^t G^{(2)}(x-a(t-\tau),\tau)\,d\tau = v^{(1)}(t,x) + v^{(2)}(t,x). \end{aligned} \tag{3.14}$$

The function v defined via w by relation (3.14) is a solution of Eq. (2.1). Indeed, let us compute their derivatives,

$$\partial_t v(t,x) = w(0,t,x) + \int\limits_0^t \partial_{t-\tau} w(t-\tau,\tau,x)\,d\tau = \int\limits_0^t \partial_{t-\tau} w(t-\tau,\tau,x)\,d\tau,$$

$$\partial_{tt} v(t,x) = f(t,x) + \int\limits_0^t \partial_{(t-\tau)(t-\tau)} w(t-\tau,\tau,x)\,d\tau,$$

$$\partial_{xx} v(t,x) = \int\limits_0^t \partial_{xx} w(t-\tau,\tau,x)\,d\tau.$$

By substituting the derivatives $\partial_{tt}v$ and $\partial_{xx}v$ into Eq. (2.1), we find that, by virtue of (3.4), the function v defined by relation (3.14) is a solution of this inhomogeneous equation.

It follows from (3.14) that the first derivatives of the function v satisfy the relations

$$a\,\partial_x v = \partial_t \overline{v}, \qquad a\,\partial_x \overline{v} = \partial_t v. \tag{3.15}$$

We have thereby proved the following assertion.

Lemma 3.2. *If a function f belongs to the class $C^1(\overline{Q})$ and the $G^{(j)}$, $j = 1, 2$, satisfy the matching conditions* (3.12) *and* (3.13), *then the function v given by relations* (3.14), (3.6), (3.11), *and* (3.10) *belongs to the class $C^2(\overline{Q})$, is a solution of Eq.* (2.1), *and satisfies relations* (3.15).

It follows from Lemmas 3.1 and 3.2 that the general solution (3.2) satisfies the inhomogeneous equation (2.1), where the function v is defined by relation (3.14), has the properties listed in Lemma 3.2, and satisfies the relations

$$a\,\partial_x u = \partial_t \overline{u}, \qquad a\,\partial_x \overline{u} = \partial_t u. \tag{3.16}$$

In what follows, we consider only the general solution (3.2), (3.14) of Eq. (2.1). For this solution, the integral condition (2.4) can be replaced by the nonlocal boundary condition

$$a\overline{u}(t,l) - a\overline{u}(t,0) = s'(t). \tag{3.17}$$

Lemma 3.3. *If the given function s belongs to the class $C^1[0,\infty)$, then the integral condition* (2.4) *is equivalent to conditions* (3.17) *and* (2.5).

Proof. The proof is by a straightforward verification in direct and inverse order. The boundary condition (3.17) is obtained by the differentiation of relation (2.4) with respect to t and by the use of relation (3.16). Condition (2.4) is obtained in the opposite order with the use of integration of relation (3.17), relations (3.16), and the matching condition (2.5).

4. PROBLEM FOR THE HOMOGENEOUS WAVE EQUATION WITH HOMOGENEOUS MATCHING CONDITIONS

We seek a particular solution v of problem (2.1)–(2.7) in the class $C^2(\overline{Q})$. It is defined by relation (3.14) via the function w, and w is a solution of the homogeneous equation (3.4). The smoothness of the solution v depends on the matching conditions (2.5)–(2.7). If w belongs to the class C^2 with respect to the variables t and x, then $v \in C^2(\overline{Q})$. However, this is the case if and only if the matching conditions are homogeneous. Therefore, consider the problem for the function $u : \overline{Q} \ni (t,x) \to u(t,x)$ that is the homogeneous equation

$$\partial_{tt} u - a^2 \partial_{xx} u = 0, \qquad (t,x) \in \overline{Q}, \tag{4.1}$$

with the Cauchy conditions (2.2), the boundary condition (2.3), the nonlocal boundary condition (3.17), and the homogeneous matching conditions (2.6) and (2.7), $\delta^{(j)} = \sigma^{(j)} = 0$, $j = 1, 2, 3$.

It is well known that the general solution u of Eq. (4.1) can be represented in the form

$$u(t, x) = g^{(1)}(x + at) + g^{(2)}(x - at), \tag{4.2}$$

where the $g^{(j)}$ $(j = 1, 2)$ are twice continuously differentiable functions with domains $D(g^{(1)}) = [0, \infty)$ and $D(g^{(2)}) = (-\infty, l]$.

By subjecting the solution (4.2) to the Cauchy conditions (2.2), we obtain the following system for the functions $g^{(j)}$ $(j = 1, 2)$ defined on $[0, l]$:

$$g^{(1)}(x) + g^{(2)}(x) = \varphi(x), \qquad -a{g^{(1)}}'(x) + a{g^{(2)}}'(x) = \psi(x), \qquad x \in [0, l], \tag{4.3}$$

where ${g^{(j)}}'$ are the first derivatives of the functions $g^{(j)}$, $j = 1, 2$. By solving system (4.3), we find the functions $g^{(j)}$ given by the relations

$$g^{(j)}(z) = g^{(j,0)}(z) = \frac{1}{2}\varphi(z) + (-1)^j \frac{1}{2a} \int\limits_0^z \psi(\xi)\, d\xi + (-1)^j C \tag{4.4}$$

for $z \in [0, l]$, where C is an arbitrary constant appearing after the integration of the second equation in system (4.3).

For other values of the argument z, the functions $g^{(j)}$ are defined step by step by subjecting the desired solution (4.2) to the boundary conditions (2.3) and (3.17). By satisfying condition (2.3), we obtain the equation

$$g^{(2,1)}(z) = \mu\left(-\frac{z}{a}\right) - g^{(1,0)}(-z). \tag{4.5}$$

Since the function $g^{(1)}(z) = g^{(1,0)}(z)$ is already determined by relation (4.4), by the expression (4.5), we find $g^{(2)}(z) = g^{(2,1)}(z)$ via given functions for $z \in [-l, 0]$. Next, we use condition (3.17). By substituting the function (4.2) into relation (3.17) and by using relations (4.4) and (4.5), we obtain

$$g^{(1)}(z) = g^{(1,1)}(z) = g^{(2,0)}(2l - z) - g^{(2,1)}(l - z) + g^{(1,0)}(z - l) + \frac{1}{a}s'\left(\frac{z - l}{a}\right) \tag{4.6}$$

for $z \in [l, 2l]$.

By returning to condition (2.3), we find the values $g^{(2,2)}(z)$ of the function $g^{(2)}$; then from condition (3.17), we find the values $g^{(1,2)}(z)$ of the function $g^{(1)}$ for $z \in [2l, 3l]$, and so on. In the general case, the above-performed procedure can be represented in the form

$$g^{(2,k)}(z) = \mu\left(-\frac{z}{a}\right) - g^{(1,k-1)}(-z), \qquad z \in [-kl, -(k-1)l], \qquad k = 1, 2, \ldots, \tag{4.7}$$

$$g^{(1,k)}(z) = g^{(2,k-1)}(2l - z) - g^{(2,k)}(l - z) + g^{(1,k-1)}(z - l) + \frac{1}{a}s'\left(\frac{z - l}{a}\right), \tag{4.8}$$
$$z \in [kl, (k+1)l], \qquad k = 1, 2, 3, \ldots$$

Let us compute the first and second derivatives of the functions (4.7) and (4.8),

$$\begin{gathered} {g^{(2,k)}}'(z) = -\frac{1}{a}\mu'\left(-\frac{z}{a}\right) + {g^{(1,k-1)}}'(-z), \qquad {g^{(2,k)}}''(z) = \frac{1}{a^2}\mu''\left(-\frac{z}{a}\right) - {g^{(1,k-1)}}''(-z), \\ z \in [-kl, -(k-1)l], \qquad k = 1, 2, 3, \ldots, \end{gathered} \tag{4.9}$$

$$\begin{gathered} {g^{(1,k)}}'(z) = \frac{1}{a^2}s''\left(\frac{z - l}{a}\right) - {g^{(2,k-1)}}'(2l - z) + {g^{(2,k)}}'(l - z) + {g^{(1,k-1)}}'(z - l), \\ {g^{(1,k)}}''(z) = \frac{1}{a^3}s'''\left(\frac{z - l}{a}\right) + {g^{(2,k-1)}}''(2l - z) - {g^{(2,k)}}''(l - z) + {g^{(1,k-1)}}''(z - l), \\ z \in [kl, (k+1)l], \qquad k = 1, 2, 3, \ldots \end{gathered} \tag{4.10}$$

For the functions $g^{(1)}$ to belong to the class $C^2[0,\infty)$, and for the functions $g^{(2)}$ to belong to the class $C^2(-\infty,l]$, in addition to the smoothness requirements for given functions of problem (4.1), (2.2), (2.3), (3.17), one should require that the values of the functions $g^{(j,k)}(z)$, $j=1,2$, $k=1,2,3,\ldots$, and values of their first and second derivatives should coincide at their common points; i.e., the following relations should be valid:

$$\begin{aligned} &g^{(1,k)}(kl)=g^{(1,k-1)}(kl), \qquad g^{(1,k)\prime}(kl)=g^{(1,k-1)\prime}(kl),\\ &g^{(1,k)\prime\prime}(kl)=g^{(1,k-1)\prime\prime}(kl), \qquad k=1,2,3,\ldots, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &g^{(2,k)}(l-kl)=g^{(2,k-1)}(l-kl), \qquad g^{(2,k)\prime}(l-kl)=g^{(2,k-1)\prime}(l-kl),\\ &g^{(2,k)\prime\prime}(l-kl)=g^{(2,k-1)\prime\prime}(l-kl), \qquad k=1,2,3,\ldots \end{aligned} \tag{4.12}$$

Lemma 4.1. *Relations* (4.11) *and* (4.12) *hold if and only if they hold for* $k=1$.

Proof. We rewrite relations (4.11) for $k=1$ via the values of the given functions,

$$\begin{aligned} g^{(1,1)}(l)&=g^{(2,0)}(l)-g^{(2,1)}(0)+g^{(1,0)}(0)+\frac{1}{a}s'(0)\\ &=\frac{1}{a}s'(0)+\frac{1}{2}\varphi(l)-\frac{1}{2a}\int\limits_0^l\psi(\xi)\,d\xi+C-\mu(0)+\frac{1}{2}\varphi(0)+C+\frac{1}{2}\varphi(0)-C\\ &=g^{(1,0)}(l)=\frac{1}{2}\varphi(l)+\frac{1}{2a}\int\limits_0^l\psi(\xi)\,d\xi+C,\\ g^{(1,1)\prime}(l)&=g^{(2,0)\prime}(l)+g^{(2,1)\prime}(0)+g^{(1,0)\prime}(0)+\frac{1}{a}s''(0)\\ &=-\frac{1}{2}\varphi'(l)+\frac{1}{2a}\psi(l)-\frac{1}{a}\mu'(0)+\frac{1}{2}\varphi'(0)+\frac{1}{2a}\psi(0)+\frac{1}{2}\varphi'(0)+\frac{1}{2a}\psi(0)+\frac{1}{a^2}s''(0)\\ &=g^{(1,0)\prime}(l)=\frac{1}{2}\varphi'(l)+\frac{1}{2a}\psi(l),\\ g^{(1,1)\prime\prime}(l)&=\frac{1}{2}\varphi''(l)-\frac{1}{2a}\psi'(l)-\frac{1}{a^2}\mu''(0)+\varphi''(0)+\frac{1}{a}\psi'(0)+\frac{1}{a^3}s'''(0)\\ &=g^{(1,0)\prime\prime}(l)=\frac{1}{2}\varphi''(l)+\frac{1}{2a}\psi'(l). \end{aligned} \tag{4.13}$$

Relations (4.13) hold by virtue of the homogeneous matching conditions (2.7).

In a similar way, we consider relations (4.12) for $k=1$,

$$\begin{aligned} &g^{(2,1)}(0)=\mu(0)-\frac{1}{2}\varphi(0)+C=g^{(2,0)}(0)=\frac{1}{2}\varphi(0)+C,\\ &g^{(2,1)\prime}(0)=-\frac{1}{a}\mu'(0)+\frac{1}{2}\varphi'(0)+\frac{1}{2a}\psi(0)=g^{(2,0)\prime}(0)=\frac{1}{2}\varphi'(0)-\frac{1}{2a}\psi(0),\\ &g^{(2,1)\prime\prime}(l)=\frac{1}{a^2}\mu''(0)-\frac{1}{2}\varphi''(0)-\frac{1}{2a}\psi'(0)=g^{(2,0)\prime\prime}(0)=\frac{1}{2}\varphi''(0)-\frac{1}{2a}\psi'(0). \end{aligned} \tag{4.14}$$

Relations (4.14) hold by virtue of the homogeneous matching conditions (2.6). The proof of the lemma is complete.

The constructed functions $g^{(j,k)}:\ D(g^{(j,k)})=\{z|\ z\in[(-1)^{j+1}kl,(-1)^{j+1}kl+l]\}\ni z\to g^{(j,k)}(z)$ find the functions $g^{(j)}$, where

$$g^{(j)}(z)=g^{(j,k)}(z), \qquad z\in D(g^{(j,k)}(z)), \qquad j=1,2, \qquad k=0,1,2,\ldots \tag{4.15}$$

Lemma 4.2. *If* $\varphi \in C^2[0,l]$, $\psi \in C^1[0,l]$, $\mu \in C^2[0,\infty)$, $s \in C^3[0,\infty)$, *and the homogeneous matching conditions* (2.6) *and* (2.7) *are satisfied, then the functions* $g^{(j)}$ $(j = 1,2)$ *given by relations* (4.15) *and* (4.4)–(4.8) *have the form*

$$g^{(j)}(z) = \widetilde{g^{(j)}}(z) + (-1)^j C, \qquad j = 1,2, \tag{4.16}$$

where C *is an arbitrary constant. In addition, the functions* $\widetilde{g}^{(j)}$ *are uniquely determined,* $g^{(1)} \in C^2[0,\infty)$, *and* $g^{(2)} \in C^2(-\infty,l]$.

Proof. By the assumptions of Lemma 4.2 and relations (4.4), (4.7), and (4.8), it follows that the functions $g^{(j,k)}$ belong to the classes $C^2(D(g^{(j,k)}))$, $j = 1,2$, $k = 0,1,2,\ldots$ Definition (4.15) of the functions $g^{(j)}$, Lemma 4.1, relations (4.11) and (4.12), and the homogeneous matching conditions (2.6) and (2.7) imply that $g^{(1)} \in C^2[0,\infty)$ and $g^{(2)} \in C^2(-\infty,l]$.

Starting from the representation (4.4), the functions $g^{(j,k)}$ for each $k = 1,2,\ldots$ are determined by relations (4.7) and (4.8). By induction, one can readily show that

$$g^{(j,k)}(z) = \widetilde{g}^{(j,k)}(z) + (-1)^j C, \tag{4.17}$$

where the $\widetilde{g}^{(j,k)}$ are uniquely determined. The proof of the lemma is complete.

The result of these considerations can be stated as the following assertion.

Theorem 4.1. *If* $\varphi \in C^2[0,l]$, $\psi \in C^1[0,l]$, $\mu \in C^2[0,\infty)$, $s \in C^3[0,\infty)$, *and these functions satisfy the homogeneous matching conditions* (2.6) *and* (2.7), *then the function* (4.2) *belongs to the class* $C^2(\overline{Q})$ *and is the unique classical solution of problem* (4.1), (2.2), (2.3), (3.17), *where the* $g^{(j)}$ $(j = 1,2)$ *are the functions defined in relations* (4.15) *and* (4.4)–(4.8).

Let the function $w:\ \mathbb{R}^2 \supset \overline{G} = [0,\infty) \times [0,\infty) \times [0,l] \ni (t,\tau,x) \to w(t,\tau,x) \in \mathbb{R}$ be a solution of the homogeneous equation (3.4), satisfy the Cauchy condition (3.5) on $[0,\infty) \times [0,l]$, the boundary condition

$$w(t,\tau,0) = tf(\tau,0) + \frac{t^2}{2} f(0,0), \qquad t,\tau \in [0,\infty), \tag{4.18}$$

and the integral condition

$$\int_0^l w(t,\tau,x)\,dx$$
$$= t\int_0^l f(\tau,x)\,dx + \frac{t^2}{2!}\int_0^l f(0,x)\,dx + \frac{t^3}{3!}\int_0^l [\partial_t f(0,x) + a\,\partial_x f(0,x)]\,dx, \quad t,\tau \in [0,\infty). \tag{4.19}$$

One can readily see that the functions occurring in conditions (3.5), (4.18), and (4.19) satisfy the homogeneous matching conditions (2.5)–(2.7).

We seek the solution of problem (3.4), (3.5), (4.18), (4.19) in the form (3.6). The Cauchy conditions (3.5) find the functions $G^{(j)}$ on the closed interval $z \in [0,l]$ by relations (3.10). Then, by using conditions (4.18) and (4.19), we find the functions $\widetilde{G}^{(j)}$ $(j = 1,2)$ for other values of z; more precisely, $\widetilde{G}^{(1)}(z,\tau)$ for $z \in [l,\infty)$ and $\tau \in [0,\infty)$ and $\widetilde{G}^{(2)}(z,\tau)$ for $z \in (-\infty,0]$ and $\tau \in [0,\infty)$. By reproducing the argument in the proof of Theorem 4.1 for this case, we obtain a solution w of problem (3.4), (3.5), (4.18), (4.19) in the class $C^2(\overline{G})$ if $f \in C^1(\overline{Q})$.

On the basis of the function w, we define the function $v:\ \overline{Q} \ni (t,x) \to v(t,x) \in \mathbb{R}$ by relation (3.14). The function v belongs to the class $C^2(\overline{Q})$, is a solution of Eq. (2.1), and satisfies the homogeneous Cauchy conditions

$$v(0,x) = \partial_t v(0,x) = 0, \tag{4.20}$$

the boundary condition

$$v(t,0) = \int_0^t (t-\tau) f(\tau,0)\,d\tau + \frac{t^3}{3!} f(0,0) = \widetilde{\mu}(t), \tag{4.21}$$

and the integral condition

$$\int_0^l v(t,x)\,dx$$
$$= \int_0^t \int_0^l (t-\tau) f(\tau,x)\,dx\,d\tau + \frac{t^3}{3!} \int_0^l f(0,x)\,dx + \frac{t^4}{4!} \int_0^l [\partial_t f(0,x) + a\,\partial_x f(0,x)]\,dx = \widetilde{s}(t). \tag{4.22}$$

Let $f \in C^1(\overline{Q})$. Then f, $\widetilde{\mu}$, and $\widetilde{s}$ satisfy the homogeneous matching conditions (2.5)–(2.7).

Lemma 4.3. *If the function f occurring in Eq.* (2.1) *belongs to the class $C^1(\overline{Q})$, then problem* (2.1), (4.20)–(4.22) *has exactly one solution in the class $C^2(\overline{Q})$, and relations* (3.15) *hold.*

5. PROBLEM (2.1)–(2.7)

We seek a solution of problem (2.1)–(2.7) in the form of a general solution (3.2) of Eq. (2.1), where v is its particular solution, that is, a solution of problem (2.1), (4.20)–(4.22). It follows that, to find a solution u of the form (3.2) of problem (2.1)–(2.7), one should find a solution $\widetilde{u} = u - v$ of the homogeneous equation (4.1) satisfying the Cauchy conditions (2.2), the boundary condition

$$\widetilde{u}(t,0) = u(t,0) - v(t,0) = \mu(t) - \widetilde{\mu}(t) = \widetilde{\widetilde{\mu}}(t) \tag{5.1}$$

and the integral condition

$$\int_0^l \widetilde{u}(t,x)\,dx = \int_0^l u(t,x)\,dx - \int_0^l v(t,x)\,dx = s(t) - \widetilde{s}(t) = \widetilde{\widetilde{s}}(t). \tag{5.2}$$

Since the function v satisfies the homogeneous conditions (2.6) and (2.7), it follows that the function $\widetilde{u}$ satisfies conditions of the form (2.6) and (2.7), where the numbers $\delta^{(m)}$ and $\sigma^{(m)}$, $m = 1,2,3$, can be nonzero. In addition, since the functions $\widetilde{u}$ and v satisfy relations (3.3) and (3.15), it follows from Lemma 4.1 that conditions (5.2) can be replaced for $\widetilde{u}$ by the nonlocal boundary condition

$$a\overline{\widetilde{u}}(t,l) - a\overline{\widetilde{u}}(t,0) = \widetilde{\widetilde{s}}'(t), \qquad t \in [0,\infty). \tag{5.3}$$

Therefore, we find a classical solution $\widetilde{u}$ of problem (4.1), (2.2), (5.1), (5.3) satisfying the inhomogeneous matching conditions (2.6) and (2.7).

To find a solution of this problem, we use the scheme and some results presented in Section 4. By reproducing the argument for the solution (4.2), we obtain the representations (4.4)–(4.8) for the functions $g^{(i)}$ $(i = 1,2)$ and their derivatives, which form the function $\widetilde{u}$ in formula (4.2). Here, instead of μ and s and their derivatives, we take the functions $\widetilde{\widetilde{\mu}} = \mu - \widetilde{\mu}$ and $\widetilde{\widetilde{s}} = s - \widetilde{s}$. The partially defined functions (4.4)–(4.8) satisfy the relations

$$\begin{aligned} g^{(1,k)}(kl) - g^{(1,k-1)}(kl) &= g^{(2,k-1)}(2l-kl) - g^{(2,k)}(l-kl) + g^{(1,k-1)}(kl-l) - g^{(2,k-2)}(2l-kl) \\ &\quad + g^{(2,k-1)}(l-kl) - g^{(1,k-2)}(kl-l) \\ &= g^{(2,k-1)}(2l-kl) - g^{(2,k-2)}(2l-kl) + 2[g^{(1,k-1)}((k-1)l) - g^{(1,k-2)}((k-1)l)], \end{aligned} \tag{5.4}$$

$$g^{(2,k)}(l-kl) - g^{(2,k-1)}(l-kl) = -[g^{(1,k-1)}((k-1)l) - g^{(1,k-2)}((k-1)l)], \tag{5.5}$$

$k = 2,3,4,\dots$, at their common points.

Let us compute the functions (5.4) and (5.5) via the values of given functions of the considered problem with regard of the matching conditions (2.6) and (2.7). By using the representations (4.4)–(4.6), for $k = 1$, we obtain

$$g^{(2,1)}(0) - g^{(2,0)}(0) = \mu(0) - \varphi(0) = \delta^{(1)},$$

$$\begin{aligned} g^{(1,1)}(l) - g^{(1,0)}(l) &= \frac{1}{2}\varphi(l) - \frac{1}{2a}\int\limits_0^l \psi(x)\,dx - \mu(0) + \varphi(0) + \frac{1}{a}s'(0) - \frac{1}{2}\varphi(l) - \frac{1}{2a}\int\limits_0^l \psi(x)\,dx \\ &= \varphi(0) - \frac{1}{a}\int\limits_0^l \psi(x)\,dx - \mu(0) + \frac{1}{a}s'(0) = \sigma^{(1)}. \end{aligned} \tag{5.6}$$

Next, by induction, we obtain the relations

$$\begin{gathered} g^{(2,k)}(l-kl) - g^{(2,k-1)}(l-kl) = -(k-2)\delta^{(1)} - (k-1)\sigma^{(1)}, \qquad k = 2,3,\dots, \\ g^{(1,k)}(kl) - g^{(1,k-1)}(kl) = (k-1)\delta^{(1)} + k\sigma^{(1)}, \qquad k = 1,2,\dots \end{gathered} \tag{5.7}$$

Likewise, by using the representations (4.9) and (4.10), for the derivatives, we obtain the relations

$$\begin{aligned} &\frac{d^p}{dz^p}[g^{(2,k)}(z) - g^{(2,k-1)}(z)]_{z=l-kl} \\ &\quad = (-1)^{p+1}\frac{d^p}{dz^p}[g^{(1,k-1)}(z) - g^{(1,k-2)}(z)]_{z=kl-l}, \qquad k = 2,3,\dots, \qquad p = 1,2, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\frac{d^p}{dz^p}[g^{(1,k)}(z) - g^{(1,k-1)}(z)]_{z=kl} \\ &\quad = (-1)^p\frac{d^p}{dz^p}[g^{(2,k-1)}(z) - g^{(2,k-2)}(z)]_{z=2l-kl} \\ &\qquad + 2\frac{d^p}{dz^p}[g^{(1,k-1)}(z) - g^{(1,k-2)}(z)]_{z=kl-l}, \qquad k = 2,3,\dots, \qquad p = 1,2. \end{aligned} \tag{5.9}$$

For $k = 1$, the values of the derivatives are defined as follows:

$$\begin{aligned} &g^{(2,1)\prime}(0) - g^{(2,0)\prime}(0) = \frac{1}{2}\varphi'(0) + \frac{1}{2a}\psi(0) - \frac{1}{a}\mu'(0) - \frac{1}{2}\varphi'(0) + \frac{1}{2a}\psi(0) = \frac{1}{a}(\psi(0) - \mu'(0)) = \delta^{(2)}, \\ &g^{(2,1)\prime\prime}(0) - g^{(2,0)\prime\prime}(0) = \frac{1}{a^2}\mu''(0) - \varphi''(0) = \delta^{(3)}, \\ &g^{(1,1)\prime}(l) - g^{(1,0)\prime}(l) = \varphi'(0) - \varphi'(l) + \frac{1}{a}\psi(0) + \frac{1}{a^2}s''(0) - \frac{1}{a}\mu'(0) - \frac{1}{a^2}\int\limits_0^l f(0,x)\,dx = \sigma^{(2)}, \\ &g^{(1,1)\prime\prime}(l) - g^{(1,0)\prime\prime}(l) = \varphi''(0) + \frac{1}{a}\psi'(0) - \frac{1}{a}\psi'(l) - \frac{1}{a^2}\mu''(0) + \frac{1}{a^3}s'''(0) \\ &\qquad + \frac{1}{a^2}f(0,0) - \frac{1}{a^3}\int\limits_0^l \partial_t f(0,x)\,dx - \frac{1}{a^2}\int\limits_0^l \partial_x f(0,x)\,dx = \sigma^{(3)}. \end{aligned} \tag{5.10}$$

In the general case, for any index k, the values of the derivatives (5.8) and (5.9) can be represented in the form

$$\begin{aligned} &\frac{d^p}{dz^p}[g^{(2,k)}(z) - g^{(2,k-1)}(z)](-(k-1)l) \\ &\quad = -(k-2)\delta^{(p+1)} + (-1)^{p+1}\sigma^{(p+1)}, \qquad k = 2,3,\dots, \qquad p = 1,2, \end{aligned} \tag{5.11}$$

$$\frac{d^p}{dz^p}[g^{(1,k)}(z) - g^{(1,k-1)}(z)](kl) = (-1)^p\delta^{(p+1)} + k\sigma^{(p+1)}, \quad k = 1,2,3,\dots, \quad p = 1,2. \tag{5.12}$$

Relations (5.7), (5.11), and (5.12) imply the following assertion.

Assertion 5.1. *If the homogeneous matching conditions* (2.6) *and* (2.7) *fail for given functions* f, φ, ψ, μ, *and* s, *then, for any smoothness of these functions, problem* (2.1)–(2.7) *does not have a classical solution defined on* $\overline{Q} = [0, \infty) \times [0, l]$.

If the inhomogeneous matching conditions (2.6) and (2.7) are valid, $f \in C^1(\overline{Q})$, $\varphi \in C^2[0, l]$, $\psi \in C^1[0, l]$, $\mu \in C^2[0, \infty)$, and $s \in C^3[0, \infty)$, then there exists a subset $\widetilde{Q} \subset \overline{Q}$ on which the function u is a solution of problem (2.1)–(2.7).

The set $\widetilde{Q}$ is the union of the following disjoint subsets $Q^{(k,m)} \subset \overline{Q}$.

The empty set $Q^{(0,-1)} = \varnothing$.

$Q^{(0,0)} = \{(t, x) \in \overline{Q} |\ x \in [0, l/2],\ 0 \le at < x\} \cup \{(t, x) \in \overline{Q} |\ x \in [l/2, l],\ 0 \le at < l - x\}$.

$Q^{(k,k-1)} = \{(t, x) \in \overline{Q} |\ x \in [0, l/2],\ x + (k-1)l < at < kl - x\}$.

$Q^{(k,k)} = \{(t, x) \in \overline{Q} |\ x \in [0, l/2],\ kl - x < at < x + kl\} \cup \{(t, x) \in \overline{Q} |\ x \in [l/2, l],\ x + (k-1)l < at < (k+1)l - x\}$.

$Q^{(k-1,k)} = \{(t, x) \in \overline{Q} |\ x \in [l/2, l],\ kl - x \le at < x + (k-1)l\}$, $k = 1, 2, \ldots$; i.e. (see the figure),

$$\widetilde{Q} = \bigcup_{k=0}^{\infty} \bigcup_{m=k-1}^{k+1} Q^{(k,m)}.$$

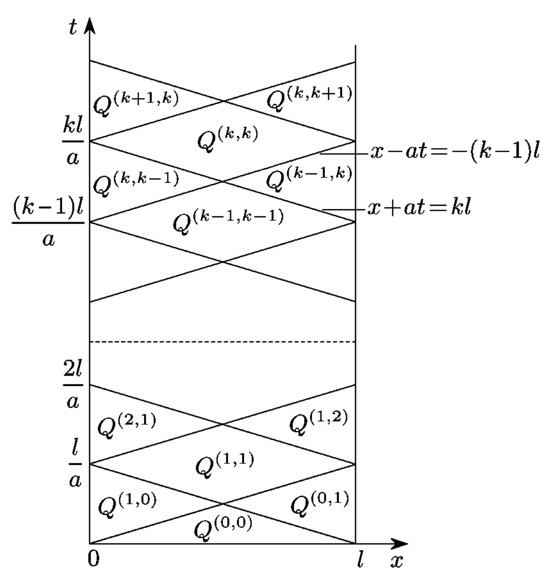
The solution

$$u(t, x) = \widetilde{u}(t, x) + v(t, x) \tag{5.13}$$

of problem (2.1)–(2.7) on the set $\widetilde{Q}$ is defined as follows. The function v is a solution of problem (2.1), (4.20)–(4.22) on $\overline{Q}$; consequently, it is a solution of the same problem on $\widetilde{Q}$. One has

$$\begin{aligned} &\widetilde{u}(t, x) = \widetilde{u}^{(k,m)}(t, x) = g^{(1,k)}(x + at) + g^{(2,m)}(x - at), \\ &(t, x) \in Q^{(k,m)}, \end{aligned}$$

where $g^{(1,k)}$ and $g^{(2,m)}$ are defined via the given functions f, φ, ψ, μ, and s in the course of the solution of problem (4.1), (2.2), (5.1), (5.2), (2.6), (2.7).



The partition of the domain Q into subdomains $Q^{(k,m)}$.

Theorem 5.1. *Let the functions* $f \in C^1(\overline{Q})$, $\varphi \in C^2[0,l]$, $\psi \in C^1[0,l]$, $\mu \in C^2[0,\infty)$, *and* $s \in C^3[0,\infty)$ *satisfy the inhomogeneous matching conditions* (2.5)–(2.7). *Then the function* u *defined by relation* (5.13) *belongs to the class* $C^2(\widetilde{Q})$ *and is the unique classical solution of problem* (2.1)–(2.7) *on the set* $\widetilde{Q}$.

If conditions (2.6) and (2.7) are homogeneous, then it follows from relations (5.7)–(5.12) that $\widetilde{u} \in C^2(\overline{Q})$, where $\delta^{(j)} = \sigma^{(j)} = 0$, $j = 1,2,3$. In this case, Theorem 5.1 is stated as follows.

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *be satisfied for the smoothness of given functions, and let these functions satisfy the homogeneous matching conditions* (2.5)–(2.7) ($\delta^{(j)} = \sigma^{(j)} = 0$, $j = 1,2,3$). *Then the function* (5.13) *belongs to the class* $C^2(\overline{Q})$ *and is the unique classical solution of problem* (2.1)–(2.7) *on* $\overline{Q}$.

Remark 5.1. If $\delta^{(1)} = \sigma^{(1)} = 0$, then it follows from the representations (5.6) and (5.7) and Theorem 5.1 that the solution u of problem (2.1)–(2.7) belongs to the class $C(\overline{Q}) \cap C^2(\widetilde{Q})$. But if $\delta^{(j)} = \sigma^{(j)} = 0$ for $j = 1,2$, then $u \in C^1(\overline{Q}) \cap C^2(\widetilde{Q})$.

Remark 5.2. If, in general, the given functions in problem (2.1)–(2.4) satisfy the inhomogeneous matching conditions (2.6) and (2.7) (at least one of the numbers $\delta^{(i)}$ and $\sigma^{(i)}$, $i = 1,2,3$, is nonzero), then the solution of problem (2.1)–(2.7) can be reduced to the solution of the corresponding transmission problem, where the transmission conditions are posed on the characteristics $x - at = -(k-1)l$ and $x + at = kl$, $k = 1,2,\ldots$ For the transmission conditions, one can take the conditions

$$\left[\left(\frac{\partial^p u}{\partial x^p}\right)^- - \left(\frac{\partial^p u}{\partial x^p}\right)^+\right](t,x)|_{x=at} = \delta^{(p+1)}, \quad at \in [0,l],$$

$$\left[\left(\frac{\partial^p u}{\partial x^p}\right)^- - \left(\frac{\partial^p u}{\partial x^p}\right)^+\right](t,x)|_{x=at-(k-1)l} = -(k-2)\delta^{(p+1)} + (-1)^{p+1}(k-1)\sigma^{(p+1)}, \tag{5.14}$$
$$at \in [kl,(k-1)l], \qquad k = 2,3,4,\ldots,$$

$$\left[\left(\frac{\partial^p u}{\partial x^p}\right)^- - \left(\frac{\partial^p u}{\partial x^p}\right)^+\right](t,x)|_{x=-at+kl} = -(-1)^p(k-1)\delta^{(p+1)} + k\sigma^{(p+1)}, \tag{5.15}$$
$$at \in [kl,(k-1)l], \qquad k = 1,2,3,\ldots,$$

where the $(\cdot)^\pm$ are the limit values of the function u and its derivatives $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ on different sides from the characteristics $x - at = -(k-1)l$ and $x + at = kl$; i.e.,

$$\left(\frac{\partial^p u}{\partial x^p}\right)^\pm (t,x)|_{x=at-(k-1)l} = \lim_{\substack{\Delta x>0\\ \Delta x\to 0}} \left(\frac{\partial^p u}{\partial x^p}\right)(t,x)|_{x=\pm\Delta x+at-(k-1)l},$$
$$\left(\frac{\partial^p u}{\partial x^p}\right)^\pm (t,x)|_{x=-at+kl} = \lim_{\substack{\Delta x>0\\ \Delta x\to 0}} \left(\frac{\partial^p u}{\partial x^p}\right)(t,x)|_{x=\pm\Delta x+(-at+kl)}.$$

Now one can state problem (2.1)–(2.7) by using the transmission conditions (5.14) and (5.15).

Problem. Find a classical solution of Eq. (2.1) satisfying the Cauchy conditions (2.2), the boundary condition (2.3), the integral condition (2.4), and the transmission conditions (5.14) and (5.15), where $\delta^{(j)}$ and $\sigma^{(j)}$, $j = 1,2,3$, are the numbers defined in relations (2.6) and (2.7).

Note that such a statement of the considered problem with transmission conditions is more convenient for its numerical implementation.

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