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Evolution of Rotational Motion of the Planet Earth under the Influence of Internal Dissipative Forces

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Abstract—The influence of internal dissipation on the rotational motion of the Earth in the gravitational field of the Sun and Moon is studied within the model of M.A. Lavrentiev. The averaged equations of second approximation describing the evolution of the Earth's rotation axis and the magnitude of its angular velocity are obtained. The dependence of the rate of evolution on the values of the model parameters is studied. Phase trajectories of the evolutionary process are constructed for different parameter values. It is shown that the observed drift of the Earth's magnetic poles can be explained within the framework of a mechanical model by the angular acceleration of the Earth.

Keywords: gravitational field, satellite with a spherical damper (Lavrentiev model), dissipative forces, evolutionary equations, evolution of the rotational motion of the Earth, drift of magnetic poles

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INTRODUCTION

As is known [1], for a satellite modeled by one solid body and moving in a central gravitational field, the only evolutionary effect is precession around the normal to the orbit plane.

Accounting for internal dissipative forces makes it possible to reveal additional evolutionary effects, such as a change (decrease) in the angular velocity with time and a secular shift of the rotation axis along the nutation angle.

The problem of the influence of internal dissipative forces on the rotational motion of satellites in a gravitational field has been considered in various formulations by many authors. In most works on this topic, one of three satellite models was used to model internal dissipation:

1) a solid body with a cavity filled with a viscous fluid [2–4];

2) a solid body with a spherical damper (model of M.A. Lavrentiev) [5–10]; and

3) a model of tidal friction, including a viscoelastic body [1, 11–14].

All of the above models give, as a rule, qualitative conclusions that agree with each other about the main evolutionary effects in the rotational motion of the satellite. However, the Lavrentiev model is distinguished by the greatest simplicity, which makes it possible to study in detail the quantitative characteristics of the evolution of the satellite's rotational motion in a wide range of parameter values and initial conditions.

In this paper, within the framework of the Lavrentiev model, we study the evolution of the rotational motion of the planet Earth in the gravitational field of the Sun and Moon under the influence of internal dissipative forces.

The planet Earth, as is known, has a complex structure [15]. Its main components are a hard shell (crust) covered with oceans, which occupies about 1% of the volume of the Earth, a layer of viscous mantle (about 84% of the Earth's volume), an outer liquid core (about 14% of the Earth's volume), and an inner solid core (about 1% of the Earth's volume).

In the Lavrentiev model used below, the tidal influence of the oceans on the evolution of the Earth's rotational motion is not taken into account. The main purpose of this paper is to study the influence of dissipative forces in the inner part of the planet (between the shell, mantle, and core) on the rotational motion of the shell.

1. EQUATIONS OF ROTATIONAL MOTION OF A PLANET IN THE GRAVITATIONAL FIELD OF TWO ATTRACTING CENTERS

In the Lavrentiev model, the planet consists of two solid bodies: a solid shell and a homogeneous spherical core, which plays the role of a damper [5]. Within the framework of this model, the planet is represented by a gyrostat, i.e. its inertia tensor in the basis associated with the shell remains unchanged. Let O be the center of mass of the planet and $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ be an orthonormal basis with the origin at point O and axes associated with the shell. Let us denote by $\mathbf{J}$ the tensor of inertia of the entire system (the shell together with the damper) in this basis, and by I the moment of inertia of the damper relative to its central axis.

Below, we will assume that the moment of dissipative forces depends linearly on the relative angular velocity of the damper and is determined by the formula

$$\mathbf{M}_d = -\sigma I(\boldsymbol{\Omega} - \boldsymbol{\omega}), \tag{1.1}$$

where $\boldsymbol{\omega}$ is the absolute angular velocity vector of the shell, $\boldsymbol{\Omega}$ is the vector of the absolute angular velocity of the damper, and factor σ will be called the "coefficient of viscous friction between the shell and the damper."

In addition, we will assume that the plane of the Moon's orbit coincides with the plane of the ecliptic, and the orbits of the Earth around the Sun and the Moon around the Earth are circular.

We choose the Koenig basis $O\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3$ respect to which the rotational motion of the planet is studied so that axis $\mathbf{i}_3$ coincides with normal $\mathbf{n}$ to the ecliptic plane and axis $\mathbf{i}_1$ coincides with some fixed direction in the ecliptic plane (Fig. 1).

The gravitational moment acting on the planet is determined by the formula [1]

$$\mathbf{M}_g = \frac{3GM_1}{R_1^5}\mathbf{R}_1 \times \mathbf{J}\mathbf{R}_1 + \frac{3GM_2}{R_2^5}\mathbf{R}_2 \times \mathbf{J}\mathbf{R}_2. \tag{1.2}$$

Here, G is the gravitational constant, M_1 is the mass of the Sun, M_2 is the mass of the Moon, and $\mathbf{R}_1$ $\mathbf{R}_2$ are vectors connecting the centers of the Sun and the Moon with the Earth's center of mass.

Let us denote by M the mass of the Earth, by ω_1 the angular velocity of the orbital basis in the Sun–Earth problem, and by ω_2 the angular velocity of the orbital basis in the Moon–Earth problem. These angular velocities are expressed by the formulas

$$\omega_k^2 = \frac{G(M + M_k)}{R_k^3} \Rightarrow \omega_1^2 \approx \frac{GM}{R_1^3}, \quad \omega_2^2 \approx \frac{82GM_2}{R_2^3}. \tag{1.3}$$

Let us introduce dimensionless variables according to the formulas

$$\mathbf{r}_1 = \frac{\mathbf{R}_1}{R_1}, \quad \mathbf{r}_2 = \frac{\mathbf{R}_2}{R_2}, \quad \mathbf{u} = \frac{\boldsymbol{\omega}}{\omega_1}, \quad \mathbf{w} = \frac{\boldsymbol{\Omega} - \boldsymbol{\omega}}{\omega_1}. \tag{1.4}$$

Here, $\mathbf{r}_1$ and $\mathbf{r}_2$ are unit vectors, $\mathbf{u}$ is the reduced angular velocity of the shell, and $\mathbf{w}$ is the reduced relative angular velocity of the damper.

Let us denote by μ and $\mathbf{m}_g$ the dimensionless coefficient of viscous friction and the reduced gravitational moment

$$\mu = \frac{\sigma}{\omega_1}, \tag{1.5}$$

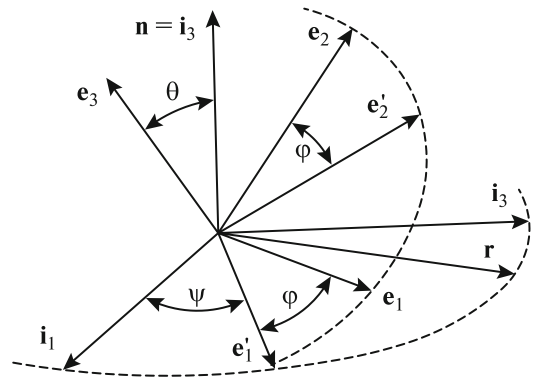

Fig. 1.

$$\mathbf{m}_g = \frac{\mathbf{M}_g}{\omega_1^2} = 3(\mathbf{r}_1 \times \mathbf{J}\mathbf{r}_1 + \alpha\mathbf{r}_2 \times \mathbf{J}\mathbf{r}_2), \quad \alpha = \frac{\beta^2}{82}. \tag{1.6}$$

Here, $\beta = \omega_2/\omega_1 \approx 13.37$ and $\alpha \approx 2.17$.

As an independent variable that plays the role of time, we choose the true anomaly in the Sun–Earth problem $\tau = \omega_1 t$. Then the dynamic equations of the rotational motion of the planet can be written in the following form [8]:

$$\left.\begin{aligned} (\mathbf{J} - I\mathbf{E})\dot{\mathbf{u}} + \mathbf{u} \times \mathbf{J}\mathbf{u} &= \mu I\mathbf{w} + \mathbf{m}_g, \\ (\mathbf{J} - I\mathbf{E})(\dot{\mathbf{w}} + \mathbf{u} \times \mathbf{w}) &= -\mu\mathbf{J}\mathbf{w} + \mathbf{u} \times \mathbf{J}\mathbf{u} - \mathbf{m}_g. \end{aligned}\right\} \tag{1.7}$$

Here, $\mathbf{E}$ is the identity matrix and the dot denotes dimensionless time derivative $\tau = \omega_1 t$. In the written equations, all vectors are given by their components in the basis $O\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ associated with the shell.

We note that Eqs. (1.7) differ from the equations obtained in [8] only by the expression for reduced gravitational moment $\mathbf{m}_g$ (1.6). In publication [8], the motion of a planet in the field of one attracting center was considered and moment $\mathbf{m}_g$ was expressed by only one (first) term from expression (1.6).

Equations (1.7) are supplemented to a closed system by kinematic equations for the rotational motion of the planet's shell. For the purposes of numerical integration of these equations, it is advisable to use the Poisson equations in quaternions:

$$2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \mathbf{u}. \tag{1.8}$$

Here, $\boldsymbol{\Lambda}$ is a quaternion of the unit norm, which specifies the position of the basis of the main axes of inertia of the planet $O\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ associated with the shell relative to the Koenig basis $O\mathbf{i}_1\mathbf{i}_2\mathbf{i}_3$. For the analytical study of the rotational motion of the planet, we will use kinematic equations in Euler angles ψ, θ, and φ (Fig. 1).

Considering further the case of a dynamically symmetric planet oblate along the axis of symmetry ($A = B < C$), we introduce the following parameters characterizing the geometry of the masses of the planet:

$$\varepsilon = \frac{C - A}{A - I}, \quad \gamma = \frac{I}{A - I}. \tag{1.9}$$

Considering that the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are given in the Koenig basis by the formulas

$$\mathbf{r}_1 = \mathbf{i}_1\cos\tau + \mathbf{i}_2\sin\tau, \quad \mathbf{r}_2 = \mathbf{i}_1\cos\beta\tau + \mathbf{i}_2\sin\beta\tau,$$

and, projecting Eqs. (1.7) onto the axes of the Resal basis $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3$ ($\mathbf{e}_3$ is the axis of symmetry of the planet), given by angles ψ and θ (Fig. 1), we obtain the following closed system of eight equations [8]:

$$\left.\begin{aligned}
\dot{u}_1 &= -(1+\varepsilon)u_3u_2 + u_2^2\cot\theta + \mu\gamma w_1 \\
&\quad + F_1\left[\cos 2(\tau-\psi) - 1\right] \\
&\quad + \alpha F_1\left[\cos 2(\beta\tau-\psi) - 1\right], \\
\dot{u}_2 &= (1+\varepsilon)u_3u_1 - u_2u_1\cot\theta + \mu\gamma w_2 \\
&\quad + F_2\sin 2(\tau-\psi) + \alpha F_2\sin 2(\beta\tau-\psi), \\
\dot{w}_1 &= \varepsilon u_2u_3 + u_2w_2\cot\theta - u_2w_3 \\
&\quad - \mu(1+\gamma)w_1 - F_1\left[\cos 2(\tau-\psi) - 1\right] \\
&\quad - \alpha F_1\left[\cos 2(\beta\tau-\psi) - 1\right], \\
\dot{w}_2 &= -\varepsilon u_1u_3 + u_1w_3 - u_2w_1\cot\theta \\
&\quad - \mu(1+\gamma)w_2 - F_2\sin 2(\tau-\psi) \\
&\quad - \alpha F_2\sin 2(\beta\tau-\psi), \\
\dot{w}_3 &= u_2w_1 - u_1w_2 - \mu(1+\gamma+\varepsilon)\,w_3/(1+\varepsilon), \\
\dot{u}_3 &= \mu\gamma w_3/(1+\varepsilon), \quad \dot{\theta} = u_1, \quad \dot{\psi}\sin\theta = u_2.
\end{aligned}\right\} \tag{1.10}$$

Here, functions F_1 and F_2 are defined by the formulas

$$F_2 = \frac{3\varepsilon\sin\theta}{2}, \quad F_1 = \frac{3\varepsilon\sin 2\theta}{4} = F_2\cos\theta. \tag{1.11}$$

Equations (1.10) differ from the equations in [8] only in those additional terms on the right-hand sides that contain factor α.

2. EVOLUTIONARY EQUATIONS

In the problem of the evolution of the planet's rotational motion, the main interest is the behavior of the planet's rotation axis and the value of its angular velocity. For an oblate planet in the steady state of slow evolution, the motion is close to rotation around the axis of symmetry. Therefore, the analysis of evolution reduces to the study of the behavior of phase variables u_3, θ, and ψ.

Evolutionary equations for a dynamically symmetric planet with a spherical damper, close to spherically symmetric ($0 < \varepsilon \ll 1$ is a small parameter), moving in the gravitational field of one attracting center, were obtained earlier in [8]. When deriving these equations, the averaging method [16, 17] was used, but without reducing the system to the standard form.

The method described in [8] is also applicable to the problem of the motion of a planet in the field of two attracting centers. In the problem under consideration, to obtain averaged equations, the following change of variables is used

$$u_k = U_k + S_k, \quad w_k = W_k + S_{k+2}, \quad k = 1, 2, \tag{2.1}$$

where U_k and W_k are the evolutionary components and S_j are oscillatory components, which are defined as solutions of the system of equations

$$\begin{cases}
\dfrac{\partial S_1}{\partial\tau} = -(1+\varepsilon)US_2 + \mu\gamma S_3 + F_1\cos 2(\tau-\psi) \\
+ \alpha F_1\cos 2(\beta\tau-\psi), \\
\dfrac{\partial S_2}{\partial\tau} = (1+\varepsilon)US_1 + \mu\gamma S_4 + F_2\sin 2(\tau-\psi) \\
+ \alpha F_2\sin 2(\beta\tau-\psi), \\
\dfrac{\partial S_3}{\partial\tau} = \varepsilon US_2 - mS_3 - F_1\cos 2(\tau-\psi) \\
- \alpha F_1\cos 2(\beta\tau-\psi), \\
\dfrac{\partial S_4}{\partial\tau} = -\varepsilon US_1 - mS_4 - F_2\sin 2(\tau-\psi) \\
- \alpha F_2\sin 2(\beta\tau-\psi).
\end{cases} \tag{2.2}$$

Here and below, we use the notation

$$m = \mu(1+\gamma), \quad U = U_3. \tag{2.3}$$

Solutions to system (2.2) are written as harmonic functions of the form

$$\begin{aligned}
S_k &= p_{1k}\sin 2(\tau-\psi) + q_{1k}\cos 2(\tau-\psi) \\
&\quad + p_{2k}\sin 2(\beta\tau-\psi) + q_{2k}\cos 2(\beta\tau-\psi),
\end{aligned}$$

where coefficients p_{1k}, q_{1k}, p_{2k}, and q_{2k} are bounded functions of small parameter ε and depend only on variables U and θ.

The change of variables (2.1) leads to the fact that in the equations for evolutionary components U_k and W_k, the dependence on the time τ "goes" into terms of the second order and higher in ε. After substituting this replacement into equations (1.10), we obtain, after averaging over τ, the following system of averaged equations of the second approximation:

$$\begin{cases}
\dot{U}_1 = -(1+\varepsilon)UU_2 + U_2^2\cot\theta + \mu\gamma W_1 \\
- (1+\alpha)F_1 + O(\varepsilon^2), \\
\dot{W}_1 = \varepsilon UU_2 + U_2W_2\cot\theta - U_2W_3 - mW_1 \\
+ (1+\alpha)F_1 + O(\varepsilon^2), \\
\dot{U}_2 = (1+\varepsilon)UU_1 - U_2U_1\cot\theta + \mu\gamma W_2 + O(\varepsilon^3), \\
\dot{W}_2 = -\varepsilon UU_1 + U_1W_3 - mW_2 - U_2W_1\cot\theta \\
- \dfrac{\mu F_2^2}{\sin\theta}(G_1 + \alpha^2 G_2) + O(\varepsilon^3), \\
\dot{W}_3 = -\mu(1+\gamma+\varepsilon)W_3/(1+\varepsilon) + U_2W_1 - U_1W_2 \\
- \mu F_2^2\left(H_1 + \alpha^2 H_2\right) + O(\varepsilon^3).
\end{cases} \tag{2.4}$$

$$\dot{U} = \mu\gamma W_3/(1+\varepsilon), \quad \dot{\theta} = U_1, \quad \dot{\psi}\sin\theta = U_2. \tag{2.5}$$

In Eqs. (2.4), we have used the notation

$$G_1 = \frac{U(3-\cos^2\theta)\cos\theta + 4}{2(4+m^2)(U^2-4)}, \quad G_2 = \frac{U(3-\cos^2\theta)\cos\theta + 4\beta}{2(4\beta^2+m^2)(U^2-4\beta^2)}. \tag{2.6}$$

$$H_1 = \frac{U(1+\cos^2\theta) + 4\cos\theta}{2(4+m^2)(U^2-4)}, \quad H_2 = \frac{U(1+\cos^2\theta) + 4\beta\cos\theta}{2(4\beta^2+m^2)(U^2-4\beta^2)}. \tag{2.7}$$

Equations (2.4) and (2.5) differ from the averaged equations in [8] only in those additional terms on the right-hand sides that contain factor α. When deriving these equations, it was assumed that the values of parameter m are bounded by the inequality $m = \mu(1+\gamma) \geq \sqrt{\varepsilon}$.

System (2.4) has stationary solutions that are asymptotically stable in variables U_1, U_2, W_1, W_2, and W_3 for fixed U and θ. These solutions are determined from the conditions that the right-hand sides of Eqs. (2.4) are equal to zero and are expressed by the following formulas:

$$W_1 = \frac{(1+\alpha)F_2\cos\theta}{\mu(1+\gamma)} + O(\varepsilon^2), \quad U_2 = -\frac{(1+\alpha)F_2\cos\theta}{U(1+\gamma)} + O(\varepsilon^2), \tag{2.8}$$

$$U_1 = \frac{\mu\gamma F_2^2}{(1+\gamma)U\sin\theta} \times \left(G_1 + \alpha^2 G_2 - (1+\alpha)^2\frac{\cos^3\theta}{m^2 U}\right) + O(\varepsilon^3), \tag{2.9}$$

$$W_2 = -\frac{F_2^2}{(1+\gamma)\sin\theta} \times \left(G_1 + \alpha^2 G_2 - (1+\alpha)^2\frac{\cos^3\theta}{m^2 U}\right) + O(\varepsilon^3), \tag{2.10}$$

$$W_3 = -\frac{F_2^2}{(1+\gamma)} \times \left(H_1 + \alpha^2 H_2 + (1+\alpha)^2\frac{\cos^2\theta}{m^2 U}\right) + O(\varepsilon^3). \tag{2.11}$$

Thereafter, to obtain equations describing the evolution of variables ψ, θ, and U, we use Tikhonov's theorem [18] on reduction conditions in a system of differential equations with a small parameter. It follows from Eqs. (2.4), (2.5), and formulas (2.8)–(2.11) that, in the steady-state mode of slow evolution, variables U_1, U_2, W_1, W_2, and W_3 are "fast" in comparison with variables ψ, θ, and U (due to the second of formulas (2.8), the right-hand side of the equation $\dot{\psi}\sin\theta = U_2$ will be a bounded by the function ε, and, due to formulas (2.9) and (2.11), the right-hand sides of the first two equations (2.5) will be bounded by functions ε^2). Therefore, according to Tikhonov's theorem, the system of differential equations (2.4) can be replaced by system of algebraic equations (2.8)–(2.11) and solved jointly with differential equations (2.5).

Substituting solutions (2.8), (2.9), and (2.11) into Eqs. (2.5), we obtain the following equations describing the evolution of variables ψ, θ, and U:

$$\dot{\psi} = -\frac{(1+\alpha)F_2\cos\theta}{U(1+\gamma)\sin\theta} + O(\varepsilon^2), \tag{2.12}$$

$$\dot{U} = -\frac{\mu\gamma F_2^2}{(1+\gamma)} \times \left(H_1 + \alpha^2 H_2 + (1+\alpha)^2\frac{\cos^2\theta}{m^2 U}\right) + O(\varepsilon^3), \tag{2.13}$$

$$\dot{\theta} = \frac{\mu\gamma F_2^2}{(1+\gamma)U\sin\theta} \times \left(G_1 + \alpha^2 G_2 - (1+\alpha)^2\frac{\cos^3\theta}{m^2 U}\right) + O(\varepsilon^3). \tag{2.14}$$

It should be noted that, if Eqs. (1.10) are immediately averaged over time τ, without using substitution (2.1), then, from these averaged equations of the first approximation, it would be impossible to obtain adequate expressions for solutions (2.9)–(2.11) and the right-hand sides of Eqs. (2.13) and (2.14) up to the accuracy of $O(\varepsilon^3)$. These averaged equations of the first approximation would describe the motions of the original system (1.10) only up to the accuracy of $O(\varepsilon^2)$.

Equations (2.13) and (2.14) form a closed system of evolution equations with respect to variables U and θ. From these equations, one can eliminate the time and the problem of determining the phase trajectories of the system and reduce to the integration of a single equation:

$$\frac{d\theta}{dU} = -\frac{1}{U\sin\theta} \times \left(\frac{m^2 U(G_1 + \alpha^2 G_2) - (1+\alpha)^2\cos^3\theta}{m^2 U(H_1 + \alpha^2 H_2) + (1+\alpha)^2\cos^2\theta}\right). \tag{2.15}$$

Evolution equations (2.13), (2.14), and (2.15) differ from the evolution equations in [8] only in those additional terms on the right-hand sides that contain factor α.

It follows from Eqs. (2.13), (2.14) and formulas (1.11) that the rate of evolution with respect to variables U and θ is proportional to ε^2 and inversely proportional to U^2. Since the rate of numerical integration of differential equations is proportional to the rate of evolution, then, at $\varepsilon \ll 1 \ll U$, to construct phase trajectories with acceptable accuracy over large time intervals, an unacceptably long running time of the program for numerical integration of exact equations (1.10) may be required. As for Eq. (2.15), its numerical integration does not cause the above difficulties, and the required program running time for calculating phase trajectories with acceptable accuracy turns out to be many orders of magnitude less than when integrating Eqs. (1.10).

In [8], the adequacy of the evolution equations was confirmed by the results of numerical integration of the exact equations. It was found that. in the problem of the motion of a planet in the field of one attracting center for the values of the parameters $0.02 \le \varepsilon \le 0.2$, $\mu \le 10$ and under the initial conditions $U(0) \le 10$, the phase trajectories of the exact and evolutionary equations are practically indistinguishable throughout the entire process of the planet's evolution.

In the considered problem for the planet Earth, where $\varepsilon \approx 0.01$ and $U(0) \approx 365$, to obtain one complete evolutionary trajectory using exact equations (1.10), an unacceptably long running time of the numerical integration program will be required (thousands of times longer than for the initial conditions $U(0) \le 10$). Therefore, Eqs. (2.13)–(2.15) are the only effective means of studying the evolution of the rotational motion of the Earth over large time intervals (millions and billions of years).

Below, the adequacy of evolution equations (2.13), (2.14) will be confirmed by the results of numerical integration of exact equations (1.10) over relatively short time intervals (several thousands of years).

Hereinafter, to write Eqs. (2.12)–(2.14) and subsequent equations, the following parameters will be used instead of parameters (1.9):

$$\delta = \frac{C - A}{A}, \quad \lambda = \frac{I}{A}. \tag{2.16}$$

Here, $\delta \approx 0.0033$ *is the coefficient of the Earth's dynamic compression*.

The relationship between parameters (1.9) and (2.16) is expressed by the formulas

$$\varepsilon = \frac{\delta}{1-\lambda}, \quad \gamma = \frac{\lambda}{1-\lambda}, \quad m = \mu(1+\gamma) = \frac{\mu}{1-\lambda}. \tag{2.17}$$

In this case, the restrictions $\varepsilon \ll 1$ and $m \ge \sqrt{\varepsilon}$ specified in the derivation of the averaged equations will be satisfied for $\lambda \le 1 - \sqrt{\delta}$, $\mu \ge \sqrt{\delta}$.

Below, to analyze the rate of the Earth's evolution in terms of variables ψ, θ, and U, we will proceed from the following values of the parameters and initial conditions:

$$\delta = 0.00328, \quad \beta = 13.37, \quad \alpha = 2.17, \quad \theta_0 = 23.5°, \quad U_0 = 365.3. \tag{2.18}$$

At the same time, since all these variables are "slow," in the first approximation, to calculate their changes over time intervals up to 10^4 years, values θ and U in the right parts of Eqs. (2.12)–(2.14) can be considered unchanged, coinciding with initial values θ_0 and U_0.

For the angular velocity of precession, on the basis of Eqs. (2.12) and formulas (1.11) and (2.17), we obtain the following expression, up to the accuracy of $O(\delta^2)$:

$$\dot{\psi} = -(1+\alpha)\frac{3\delta\cos\theta}{2U} \approx -3.9 \times 10^{-5}. \tag{2.19}$$

Since $\dot{\psi}$ is the time derivative $\tau = \omega_1 t$, where ω_1 is the angular velocity of the orbital basis in the Earth–Sun problem, then, according to formula (2.19), the change in the precession angle for 1 Earth year is $\Delta\psi = -2\pi 3.9 \times 10^{-5}$ rad, i.e. about 1.4° per century. The value inverse to expression (2.19) is the period of precession of the Earth's axis expressed in years and is about 25600 years.

Thus, the planet's precession rate determined by formula (2.19) is proportional to small parameter δ and, up to the accuracy of $O(\delta^2)$, does not depend on the action of internal dissipative forces.

The equations for the magnitude of the Earth's angular velocity and the nutation angle based on Eqs. (2.13), (2.14) are written up to an accuracy of $O(\delta^3)$ in the form

$$\dot{U} = -\frac{9\mu\lambda\delta^2\sin^2\theta}{4} \times \left(H_1^* + \alpha^2 H_2^* + (1+\alpha)^2\frac{\cos^2\theta}{\mu^2 U}\right), \tag{2.20}$$

$$\dot{\theta} = \frac{9\mu\lambda\delta^2\sin\theta}{4U}\left(G_1^* + \alpha^2 G_2^* - (1+\alpha)^2\frac{\cos^3\theta}{\mu^2 U}\right). \tag{2.21}$$

Here, we use the notation

$$G_1^* = \frac{U(3-\cos^2\theta)\cos\theta + 4}{2\left[4(1-\lambda)^2 + \mu^2\right](U^2-4)}, \quad G_2^* = \frac{U(3-\cos^2\theta)\cos\theta + 4\beta}{2\left[4\beta^2(1-\lambda)^2 + \mu^2\right](U^2 - 4\beta^2)}, \tag{2.22}$$

$$H_1^* = \frac{U(1+\cos^2\theta)+4\cos\theta}{2[4(1-\lambda)^2+\mu^2](U^2-4)},$$
$$H_2^* = \frac{U(1+\cos^2\theta)+4\beta\cos\theta}{2[4\beta^2(1-\lambda)^2+\mu^2](U^2-4\beta^2)}. \quad (2.23)$$

Eliminating the time from Eqs. (2.20), (2.21), we obtain one Eq. (2.15), which is rewritten in terms of parameters μ and λ in the following form:

$$\frac{d\theta}{dU} = -\frac{1}{U\sin\theta} \times \left(\frac{\mu^2 U(G_1^* + \alpha^2 G_2^*) - (1+\alpha)^2\cos^3\theta}{\mu^2 U(H_1^* + \alpha^2 H_2^*) + (1+\alpha)^2\cos^2\theta}\right). \quad (2.24)$$

It follows from Eqs. (2.20) and (2.21) that the rate of evolution with respect to variables U and θ is proportional to δ^2, i.e., much less than precession rate (2.19).

The rate of evolution depends in a complex way on parameters μ and λ, the values of which are unknown. Taking into account that, in the considered problem, $U \gg \beta$, to estimate the magnitude of this velocity, instead of Eqs. (2.20), (2.21), we can use the approximate equations

$$\dot{U} = -\frac{9\mu\lambda\delta^2\sin^2\theta}{8U}\left(\frac{1+\cos^2\theta}{4(1-\lambda)^2+\mu^2} + \frac{\alpha^2(1+\cos^2\theta)}{4\beta^2(1-\lambda)^2+\mu^2} + \frac{2\cos^2\theta}{\mu^2}(1+\alpha)^2\right), \quad (2.25)$$

$$\dot{\theta} = \frac{9\mu\lambda\delta^2\sin 2\theta}{16U^2}\left(\frac{3-\cos^2\theta}{4(1-\lambda)^2+\mu^2} + \frac{\alpha^2(3-\cos^2\theta)}{4\beta^2(1-\lambda)^2+\mu^2} - 2(1+\alpha)^2\frac{\cos^2\theta}{\mu^2}\right). \quad (2.26)$$

It follows from Eq. (2.26) that, for the above values α and β and initial conditions (2.18) the angle between the Earth's axis of rotation and the normal to the ecliptic plane decreases for any values of parameters μ and λ from the region

$$\mu \geq \sqrt{\delta}, \quad \lambda \leq 1-\sqrt{\delta}. \quad (2.27)$$

Moreover, in region (2.27), the maximum possible change in angle θ in 1 year is limited by the inequality

$$|\Delta\theta|_{max} < \frac{9\pi(1+\alpha)^2\delta^{3/2}\sin 2\theta\cos^2\theta}{4U^2} \approx 6.15\times 10^{-8}; \quad (2.28)$$

that is, it is no more than 1.3 arcsec per century.

Based on Eq. (2.25), the change in the Earth's angular velocity over time is estimated within the framework of the model under consideration. Let us denote by ΔU the change in the value U over 1 year. Taking into account that this change is determined by the formula $\Delta U = 2\pi\dot{U}$, we obtain that, in the region (2.27),

$$\frac{|\Delta U|_{max}}{U} < \frac{9\pi\delta^{3/2}\sin^2\theta\left[(1+\alpha^2)(1+3\cos^2\theta)+4\alpha\cos^2\theta\right]}{4U^2} \approx 4.3\times 10^{-8}. \quad (2.29)$$

The change in length of the day ΔT for 1 year is determined by the formula

$$\Delta T = -\frac{\Delta U}{U}T = -\frac{2\pi\dot{U}}{U}T, \quad (2.30)$$

where T is the current duration of the day. From formulas (2.25), (2.29), and (2.30), it follows that, in the range of values of parameters (2.27), the maximum possible increase in duration of a day ΔT_{max} cannot exceed 0.37 s per century.

Below are two tables that indicate the increments of the duration of the day in milliseconds per century calculated on the basis of Eq. (2.20) and formula (2.30) (Table 1) and the increments of the nutation angle calculated on the basis of Eq. (2.21) in arc milliseconds per century (Table 2) depending on the values of parameters μ and λ.

If we proceed from the fact that, at present, the lengthening of the day is estimated at 2 ms per century, i.e., $\Delta T = 2\times 10^{-5}$ s, then, according to Table 1, in the region $\mu \geq \sqrt{\delta} \approx 0.06$ and $\lambda \leq 1-\sqrt{\delta} \approx 0.94$, where the evolution equations used are applicable, there are many values μ and λ, for which Eq. (2.20) gives the observed value. According to Table 1, the relationship between μ and λ on this set is close to linear and is approximated by the formula $\mu = 7.3\lambda$. In turn, according to the data in Table 2, increments of the nutation angle at different points of this set do not differ significantly from each other and do not exceed 12.5 marcsec per century.

Note that, for $\Delta T = 2\times 10^{-5}$ s, value $\dot{U}$ is determined by the formula

$$\dot{U} = \frac{\Delta U}{2\pi} = -\frac{\Delta T}{2\pi T}U \approx -\frac{2.3\times 10^{-10}U}{2\pi}. \quad (2.31)$$

To check the adequacy of evolution equations (2.20), (2.21), we compared the data obtained from these equations in Tables 1 and 2 with the results of numerical integration of exact equations (1.10).

In Figs. 2–4, we present plots of changes in the length of the day and the angle of nutation as a function of time obtained by numerical integration of exact equations (1.10) for the following three combinations of parameter values:

Table 1. Day-length increment (in milliseconds per century)

Parameters	$\mu = 0.1$	$\mu = 0.6$	$\mu = 1.4$	$\mu = 2.2$	$\mu = 3.0$	$\mu = 3.8$	$\mu = 4.6$
$\lambda = 0.1$	13.3	2.25	0.99	0.64	0.48	0.39	0.32
$\lambda = 0.2$	26.7	4.51	2.00*	1.31	0.97	0.78	0.65
$\lambda = 0.3$	40.0	6.78	3.02	1.97	1.47	1.18	0.98
$\lambda = 0.4$	53.4	9.09	4.07	2.65	1.98	1.58	1.33
$\lambda = 0.5$	66.7*	11.4	5.14	3.35	2.50	2.01*	1.69
$\lambda = 0.6$	80.1	13.9	6.24	4.07	3.05	2.46	2.07
$\lambda = 0.7$	93.6	16.5	7.39	4.84	3.65	2.97	2.53

Table 2. Increment of nutation angle (in arc milliseconds per century)

Parameters	$\mu = 0.1$	$\mu = 0.6$	$\mu = 1.4$	$\mu = 2.2$	$\mu = 3.0$	$\mu = 3.8$	$\mu = 4.6$
$\lambda = 0.1$	–76.2	–12.5	–5.17	–3.18	–2.27	–1.76	–1.43
$\lambda = 0.2$	–152	–25.0	–10.2*	–6.30	–4.51	–3.49	–2.84
$\lambda = 0.3$	–228	–37.3	–15.2	–9.35	–6.69	–5.18	–4.20
$\lambda = 0.4$	–304	–49.4	–20.0	–12.3	–8.81	–6.89	–5.50
$\lambda = 0.5$	–380*	–61.3	–24.7	–15.2	–10.8	–8.33*	–6.69
$\lambda = 0.6$	–456	–72.5	–29.1	–17.9	–12.7	–9.67	–7.69
$\lambda = 0.7$	–532	–82.9	–33.2	–20.2	–14.2	–10.6	–8.26

$$\begin{aligned} &\text{Fig. 1: } \mu = 0.1, \quad \lambda = 0.5; \\ &\text{Fig. 2: } \mu = 1.4, \quad \lambda = 0.2; \\ &\text{Fig. 3: } \mu = 3.8, \quad \lambda = 0.5. \end{aligned} \tag{2.32}$$

In these plots, N denotes the number of years; ΔT indicates the increment of the duration of the day, ms; and $\Delta\theta$ denotes the increment of the nutation angle, marcsec.

As can be seen from the presented plots, there is an oscillatory component of large amplitude in the behavior of the nutation angle. Therefore, in order to identify with acceptable accuracy the evolutionary component in the behavior of the nutation angle, the numerical integration of equations (1.10) was carried out over a time interval of 1000 years (Figs. 2, 3) and 2000 years (Fig. 4). In the latter case, the evolutionary component in the change in the nutation angle was only about 0.17 arcsec for 2000 years and the running time of the program for the numerical integration of equations (1.10) was about 1 h. This example shows that exact equations (1.10) are unsuitable for analyzing the evolution of a system over long time intervals (millions or billions of years), since their numerical integration requires an unacceptably large amount of time.

Comparison of the results of numerical integration of exact equations (1.10) presented in Figs. 2–4 and those given in Tables 1 and 2 of the results of calculations based on evolution equations (2.20) and (2.21) for the values of the parameters (2.32) shows (the cells of the tables corresponding to the values of the parameters (2.32) are marked with *) that Eqs. (2.20), (2.21) are adequate and describe the evolution of system (1.10) with high accuracy. This circumstance gives grounds for using Eqs. (2.20), (2.21), and (2.24) in the analysis of the evolution of a system over large time intervals, where the use of exact equations is problematic.

In Fig. 5, we present the phase trajectories of the evolutionary process in the rotational motion of the Earth calculated on the basis of Eq. (2.24). Here, angular velocity U was expressed as the number of revolutions of the Earth around its axis in 1 year and U_x and U_z denote the projections of this angular velocity onto the orbital plane and onto the normal to the orbital plane, respectively.

In the left part of the figure, $\lambda = 0.5$ and different phase trajectories correspond to different values of parameter μ, from $\mu = 0.1$ (upper curve) to 10.1 (lower curve). In the right part of the figure $\mu = 2$, and different trajectories correspond to different values of parameter λ, from $\lambda = 0.1$ (upper curve) to $\lambda = 0.9$ (lower curve).

As can be seen from the presented plots, on each trajectory, the angular velocity of the Earth and the nutation angle decrease monotonically, and the end of the evolutionary process in each case is rotation around the normal to the orbital plane with a constant angular velocity.

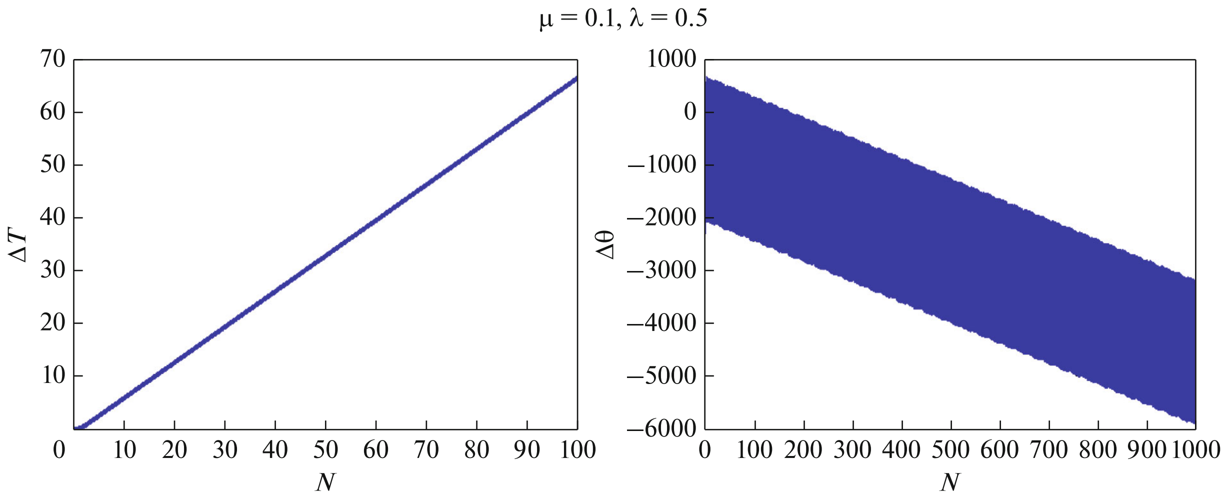

Fig. 2.

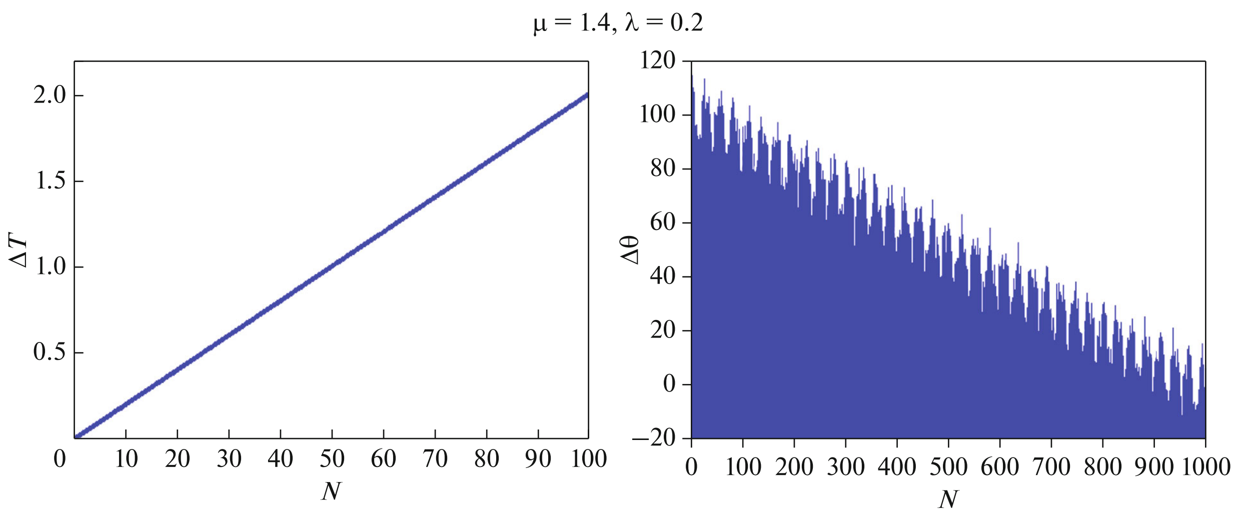

Fig. 3.

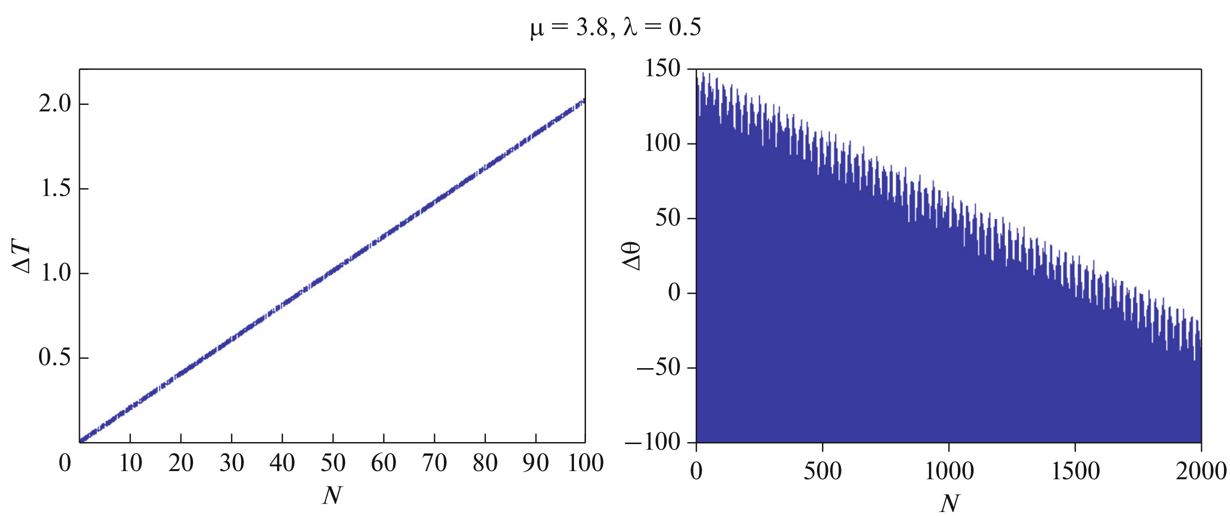

Fig. 4.

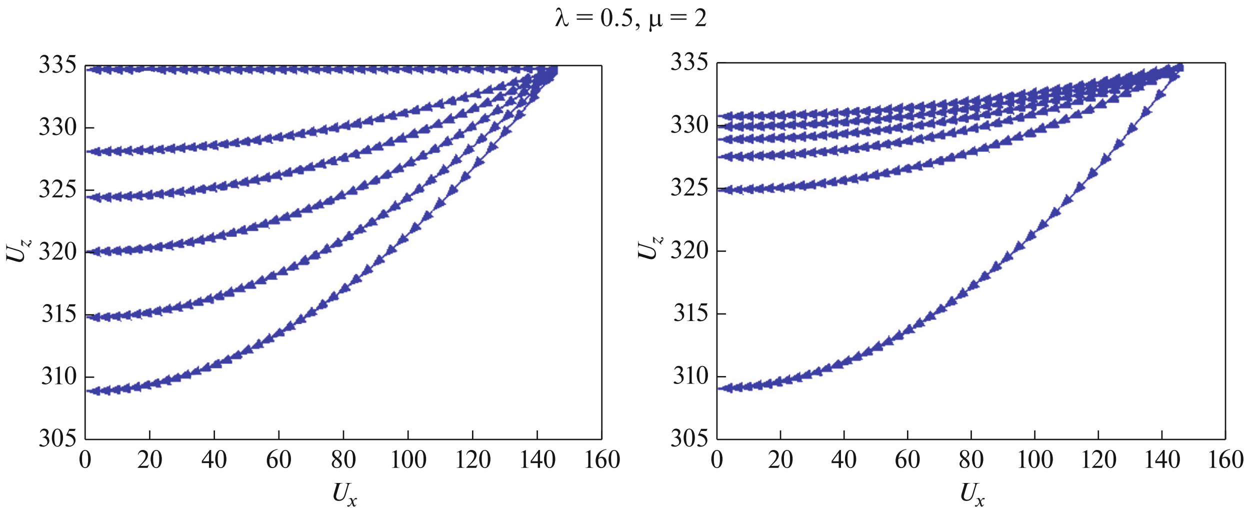

Fig. 5.

3. ON THE DRIFT OF THE EARTH'S MAGNETIC POLES

According to observational data [19], the Earth's north magnetic pole is at an angle of about 10° to the Earth's rotation axis and drifts westward at a speed of almost 50 km per year.

Let us show that the observed drift of the magnetic poles can be explained within the framework of the mechanical model by the angular acceleration of the Earth. Without delving into the physics of the magnetic-field generation process in detail, we will proceed from the assumption that the direction of the magnetic field is determined by the orientation of the magnetic dipole associated *with the inner solid core of the Earth*. In this formulation, the drift of the magnetic poles can be explained by the rotations of the Earth's inner core relative to its shell (crust).

To solve the problem, the planet Earth will be modeled by a system of three bodies: a shell, a damper, and a core. By designate as the "shell" the Earth's crust and the part of the viscous mantle layer adjacent to it, the rest of the viscous mantle layer and the outer liquid as the "damper," and the spherical homogeneous *internal solid core of the Earth* as the "core." Within the framework of such a model, the inner solid core, due to the relative smallness of its moment of inertia (according to rough estimates, the ratio of the moment of inertia of the inner solid core to the moment of inertia of the entire Earth is no more than 5×10^{-3}) will not have a significant effect on the motion of the shell and damper. Therefore, to describe the rotational motion of the shell and damper, we will use averaged equations (2.19)–(2.21) obtained in the previous section in the framework of the "shell–damper" problem, and we will determine its angular displacement relative to the shell from the equations of rotational motion of the internal solid core.

Let us denote by $\mathbf{V} = (\boldsymbol{\omega}^* - \boldsymbol{\omega})/\omega_l$ the vector of the dimensionless angular velocity of the core relative to the shell. Here, as before, $\boldsymbol{\omega}$ is the absolute angular velocity of the shell, $\boldsymbol{\omega}^*$ is the absolute angular velocity of the nucleus, and ω_l is the angular velocity of the orbital basis in the Sun–Earth problem. Using, as before, dimensionless time $\tau = \omega_l t$, we obtain the following equations for the rotational motion of the core relative to the shell, written in projections on the axes of the Resal basis $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3$:

$$\dot{\mathbf{V}} = -\dot{\mathbf{U}} - \mathbf{U}'\,\mathbf{V} - \mu^*(\mathbf{V} - \mathbf{W}). \tag{3.1}$$

Here, $\mathbf{U} = \boldsymbol{\omega}/\omega_l$ is the dimensionless angular velocity of the shell,

$$\mathbf{U}' = \dot{\theta}\mathbf{e}_1' + \dot{\psi}\sin\theta\mathbf{e}_2' + \dot{\psi}\cos\theta\mathbf{e}_3 \tag{3.2}$$

is the dimensionless angular velocity of the Resal basis, $\mathbf{W}$ is the dimensionless relative angular velocity of the damper, and $\mu^* = \sigma^*/\omega_l$ is the dimensionless coefficient of viscous friction between the damper and the inner core. In the written equations, the behavior of variables $\mathbf{U}$, $\mathbf{U}'$, and $\mathbf{W}$ will be described by solutions of evolution equations (2.19)–(2.21).

Further, we will proceed from the assumption that $\mu^* \ll \mu$, where μ is the dimensionless coefficient of viscous friction between the shell and the damper. Under these conditions, in the steady state of slow evolution, we will have $|\mathbf{V}| \gg |\mathbf{W}|$, and Eqs. (3.1) can be replaced by the equations

$$\dot{\mathbf{V}} = -\dot{\mathbf{U}} - \mathbf{U}'\,\mathbf{V} - \mu^*\mathbf{V}. \tag{3.3}$$

The angular velocity of the shell in projections on the axis of the Resal basis is expressed by the formula

$$\mathbf{U} = \dot{\psi}\sin\theta\mathbf{e}_2' + \dot{\theta}\mathbf{e}_1' + (U + \dot{\psi}\cos\theta)\mathbf{e}_3, \tag{3.4}$$

where $U \approx 365.3$; $\dot\psi$ and $\dot\theta$ are expressed by formulas (2.19) and (2.21). The main terms in the expression for the angular acceleration of the shell are written as

$$\dot{\mathbf{U}} = \mathbf{U}'U\mathbf{e}_3 + \dot{U}\mathbf{e}_3 = \dot\psi U\sin\theta\mathbf{e}_1' - \dot\theta U\mathbf{e}_2' + \dot{U}\mathbf{e}_3, \quad (3.5)$$

where $\dot{U}$ is expressed by formula (2.31).

As a result, from vector equation (3.3), taking into account formulas (3.2), (3.4), and (3.5), we obtain the following system of equations in projections on the axes $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3$:

$$\begin{cases} \dot{V}_1 = -\dot\psi U\sin\theta - V_3\dot\psi\sin\theta + V_2\dot\psi\cos\theta - \mu^* V_1, \\ \dot{V}_2 = \dot\theta U + \dot\theta V_3 - V_1\dot\psi\cos\theta - \mu^* V_2, \\ \dot{V}_3 = -\dot{U} + V_1\dot\psi\sin\theta - V_2\dot\theta - \mu^* V_3. \end{cases} \quad (3.6)$$

System (3.6) admits a stationary solution, which corresponds to the steady rotation of the nucleus. This solution is determined by the condition that the right-hand sides of Eqs. (3.6) are equal to zero and is expressed by the following formulas:

$$V_1 = -\frac{(\mu^* U - \dot{U})(\mu^* \dot\psi\sin\theta - \dot\psi\dot\theta\cos\theta)}{\mu^*(\mu^{*2} + \dot\psi^2 + \dot\theta^2)}, \quad (3.7)$$

$$V_2 = \frac{(\mu^* U - \dot{U})(\mu^* \dot\theta + \dot\psi^2\sin\theta\cos\theta)}{\mu^*(\mu^{*2} + \dot\psi^2 + \dot\theta^2)}, \quad (3.8)$$

$$V_3 = -\frac{\mu^* U\dot\theta^2 + \mu^{*2}\dot{U} + \dot\psi^2\dot{U}\cos^2\theta + \mu^* \dot\psi^2 U\sin^2\theta}{\mu^*(\mu^{*2} + \dot\psi^2 + \dot\theta^2)}. \quad (3.9)$$

Given that $|\dot\theta| \ll |\dot\psi|$, $|\dot{U}| \ll |\dot\psi|$, and assuming that $|\dot\psi| \ll \mu^*$, we obtain

$$V_1 \approx -\frac{U\dot\psi\sin\theta}{\mu^*}, \quad |V_2| \ll |V_1| \Rightarrow V_2 \approx 0, \qquad V_3 \approx -\frac{\dot\psi^2 U\sin^2\theta}{\mu^{*2}} - \frac{\dot{U}}{\mu^*}. \quad (3.10)$$

Thus, in the steady state, the vector of the relative angular velocity of the nucleus is approximately expressed by the formula

$$\mathbf{V} = V_1\mathbf{e}_1' + V_3\mathbf{e}_3 = -\frac{U\dot\psi\sin\theta}{\mu^*}\mathbf{e}_1' - \left(\frac{\dot\psi^2 U\sin^2\theta}{\mu^{*2}} + \frac{\dot{U}}{\mu^*}\right)\mathbf{e}_3. \quad (3.11)$$

Since the projections of this velocity on the axes of the Rezal basis are unchanged, such a motion of the nucleus relative to the shell is a regular precession. It can be represented as a combination of two rotations:

$$\mathbf{V} = \mathbf{\Omega}_1 + \mathbf{\Omega}_2, \quad \mathbf{\Omega}_1 = -U\mathbf{e}_3, \qquad \mathbf{\Omega}_2 = (U + V_3)\mathbf{e}_3 + V_1\mathbf{e}_1', \quad (3.12)$$

where the first is the rotation of the Resal basis around axis $\mathbf{e}_3$ with angular velocity $\mathbf{\Omega}_1 = -U\mathbf{e}_3$ (precession) and the second is the proper rotation relative to the Resal basis with angular velocity $\mathbf{\Omega}_2$. In this case, the value of the angular velocity of the intrinsic rotation is expressed by the formula

$$\Omega_2 = \sqrt{(U + V_3)^2 + V_1^2} \quad (3.13)$$

and the axis of proper rotation is determined by the unit vector

$$\boldsymbol{\xi}' = \frac{(U + V_3)\mathbf{e}_3 + V_1\mathbf{e}_1'}{\Omega_2} = \mathbf{e}_3\cos\alpha + \mathbf{e}_1'\sin\alpha. \quad (3.14)$$

Let us pass to new dimensionless time $t = U\tau$ and introduce the notation

$$h = \frac{\Omega_2}{U} - 1 \approx \frac{V_3}{U} + \frac{V_1^2}{2U^2}. \quad (3.15)$$

Then. for the precessional rotation of the nucleus described by formulas (3.11)–(3.14), the quaternion of the final rotation is expressed as the product $\Lambda = \Lambda_1 \circ \Lambda_2$, where

$$\Lambda_1 = \cos\frac{t}{2} - \mathbf{e}_3\sin\frac{t}{2}, \qquad \Lambda_2 = \cos\frac{(1+h)t}{2} + (\mathbf{e}_3\cos\alpha + \mathbf{e}_1\sin\alpha)\sin\frac{(1+h)t}{2}. \quad (3.16)$$

Computing this product, we obtain

$$\Lambda = \cos\frac{ht}{2} + \mathbf{e}_3\sin\frac{ht}{2} + (\cos\alpha - 1)\sin\frac{t}{2} \times \sin\frac{(1+h)t}{2} + \mathbf{e}_1\sin\alpha\sin\frac{(1+h)t}{2}\cos\frac{t}{2} + \mathbf{e}_3(\cos\alpha - 1)\cos\frac{t}{2}\sin\frac{(1+h)t}{2} - \mathbf{e}_2\sin\alpha\sin\frac{(1+h)t}{2}\sin\frac{t}{2}. \quad (3.17)$$

It follows that, during time $\Delta t = 2\pi/(1+h)$, which corresponds to one revolution of the planet around its axis, the quaternion of the final rotation of the nucleus is expressed by the formula

$$\Lambda(\Delta t) = \cos\left(\frac{2\pi h}{2(1+h)}\right) + \mathbf{e}_3\sin\left(\frac{2\pi h}{2(1+h)}\right). \quad (3.18)$$

This means that, in 1 day, the core will rotate around axis $\mathbf{e}_3$ by angle $2\pi h$, and, in 1 year, the angle of rotation of the core around axis $\mathbf{e}_3$ will be

$$\begin{aligned}\Delta\varphi &\approx 2\pi U h \approx 2\pi\left(V_3 + \frac{V_1^2}{2U}\right)\\ &= 2\pi\left(-\frac{\dot{\psi}^2 U \sin^2\theta}{2\mu^{*2}} - \frac{\dot{U}}{\mu^*}\right).\end{aligned} \tag{3.19}$$

For the observed displacement of the Earth's north magnetic pole, the angle of rotation of the core around axis $\mathbf{e}_3$ in one year should be $\Delta\varphi \approx -0.05$. Substituting this value into formula (3.19), as well as the previously calculated values $\dot{\psi}$ (2.19), $\dot{U}$ (2.31), $\theta = 23.5°$, and $U = 365.3$, we obtain the following equation:

$$-\frac{2.77\times10^{-7}}{\mu^{*2}} + \frac{0.84\times10^{-7}}{\mu^*} = -0.05. \tag{3.20}$$

The solution of this equation,

$$\mu^* \approx 2.35\times10^{-3}\vec{r}, \tag{3.21}$$

determines the value of the coefficient of viscous friction, at which, within the framework of the model under consideration, the rotational motion of the inner core relative to the shell is realized, which adequately explains the observed drift of the Earth's magnetic poles. The low value of the coefficient (3.21) can be explained by the high temperature of the Earth's inner solid core, as a result of which the layer of the outer liquid core adjacent to it can have the property of superfluidity.

Note that the value of the coefficient (3.21) satisfies the previously accepted assumptions $\mu^* \ll \mu$ and $|\dot{\psi}| \gg \mu^*$.

Within the framework of the considered model, the drift of the magnetic poles is explained only by the presence of the angular acceleration of the Earth, which is due to the precession around the normal to the plane of the orbit and the change in the value of the angular velocity.

CONCLUSIONS

In the framework of the Lavrentiev model, a study was carried out of the influence of internal dissipative forces on the evolution of the rotational motion of the Earth in the gravitational field of the Sun and Moon. Evolutionary equations are obtained that describe the behavior of the Earth's rotation axis and the value of its angular velocity. The dependences of the evolution rate for different variables on the values of the model parameters are determined. It is shown that the observed change in the length of the day can be explained in terms of the Lavrentiev model. Phase trajectories are constructed that describe the evolution of the rotational motion of the Earth from the current state to the final point, at which the Earth will rotate around the normal to the orbital plane with a constant angular velocity.

It is shown that the observed drift of the Earth's magnetic poles can be explained within the framework of a mechanical model by the angular acceleration of the Earth.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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