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Technique of Optimizing Trajectories of Interplanetary Transfers with Gravity Assisted Maneuvers Using Low-Thrust Propulsion

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Abstract—A technique of searching for optimal trajectories with gravity assisted maneuvers (GAMs) for interplanetary transfers of spacecraft (SC) with an electric propulsion system (EPS) is proposed. In this case, the indirect optimization method is used. A distinctive feature of this technique is the combination of optimality conditions at the point of GAMs within a single boundary value problem for two cases, when the height of the flyby hyperbola with the GAM is less than or equal to the maximum one. This approach makes it possible to considerably reduce the volume of necessary calculations in optimizing SC interplanetary trajectories that include GAMs. It considers end-to-end trajectory optimization with an analysis of the full set of optimality conditions at the point of the GAM. The efficiency of the proposed approach is demonstrated by the example of optimization of interplanetary trajectories from Earth to Mercury with a GAM in the vicinity of Venus and from Earth to Jupiter with a GAM near Earth.

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INTRODUCTION

A search for optimal trajectories in problems of interplanetary transfers with gravity assisted maneuvers (GAMs) is appreciably more labor consuming in comparison to a search for a solution to problems of direct interplanetary transfer. This circumstance is explained by an appreciable increase in the order of the boundary value problem when GAMs are added to it. In the context of using indirect optimization methods, several techniques have been developed by now, which allow the indicated problems to be solved [1, 2]. These techniques have in common the fact that an addition of each GAM leads to the addition of nine boundary conditions to the boundary value problem (with fixed dates of launch and GAM), which results in a considerable increase in the problem dimension. Thus, construction of an effective algorithm for solving similar problems continues to be the pressing challenge. In connection with this, a technique of optimization of SC interplanetary trajectories is proposed, the application of which increases the number of boundary conditions of the boundary value problem by five with the addition of each GAM. As a consequence, the employment of the proposed approach makes it possible to ensure higher computational stability of the iterative process of the search for a solution. The relatively weak increase in the order of the boundary value problem, when GAMs are added to it, is provided by using a vector-rotation matrix in the mathematical model of the GAM, due to which a part of optimality conditions at the point of the GAM is fulfilled automatically.

1. EQUATIONS OF MOTION OF A SPACECRAFT WITH AN ELECTRIC PROPULSION SYSTEM

A heliocentric phase of the SC trajectory is considered. A mathematical model describing motion of a SC with an EPS, uses the following main assumptions: values of the EPS specific impulse and thrust are assumed constant at all EPS burns (a model of uncontrollable propulsion); no constraint is imposed on the EPS thrust vector direction; gravitational fields of planets and Sun are described by the Newton model; the assumptions of the method of zero-extension gravispheres are used: while the SC heliocentric trajectory is investigated, the extension of gravispheres of other planets is neglected. The time of the beginning of heliocentric trajectory is considered to be equal to the time of SC launch from the intermediate orbit of the artificial Earth satellite, the SC position in the heliocentric trajectory at the initial and final time moments coincides with the planet of departure and destination, respectively. In the analysis of the heliocentric phase, the SC velocity is considered equal to the sum of a velocity of the planet, from which it starts and the hyperbolic excess velocity (HEV), which the SC has at the escape from the gravisphere of this planet. This approach is justified for the preliminary design calculation, since it simplifies the mathematical model of SC motion and yields an error on the order of tenths of a fraction of a percent in propellant expenditures as compared with the solution to the n-body problem [3].

The problem of "zero docking" with a destination planet is considered (a problem of matching the SC position and velocity with the position and velocity of the destination planet).

A rectangular inertial coordinate system is used in calculations. Its role is fulfilled by the heliocentric ecliptic coordinate system, the International Celestial Reference Frame (ICRF) for the epoch J2000.0, which is a practical implementation within radio range of general demands and principles of the coordinate system construction: International Celestial Reference System (ICRS) [4]. The ICRS standard was accepted in August 1997 at the 23rd meeting of the International Astronomical Union (IAU) and came into force January 1, 1998. Positions and velocities of planets were determined using the DE405 ephemerides of Jet Propulsion Laboratory (JPL) [5].

With this in mind, the mathematical model of SC motion in the dimensionless form is described by the following system of differential equations:

$$\frac{d\mathbf{r}}{dt} = \mathbf{V},$$
$$\frac{d\mathbf{V}}{dt} = -\frac{1}{r^3}\cdot\mathbf{r} + \frac{\delta\cdot P_0}{m}\cdot\mathbf{e},$$
$$\frac{dm}{dt} = -\frac{\delta\cdot P_0}{w_0},$$

where $\mathbf{r}$ is the SC radius-vector, $\mathbf{V}$ is the vector of SC heliocentric velocity, $\mathbf{e}$ is the unit vector of EPS thrust, δ is the function of the engine startup–cutoff ($\delta \in [0,1]$), w_0 is the EPS exhaust velocity, P_0 is the magnitude of EPS thrust; m is the SC mass; and t is the time.

As the optimization criterion, the maximization of the SC final mass is considered, i.e., a functional of the form of $J = m(t_k) \to \max$, where t_k is the time of SC arrival at the destination.

The optimization problem is reduced to the boundary value problem using the Pontryagin maximum principle. The Hamiltonian of the optimal control problem has the form

$$H = \mathbf{p}_\mathbf{r}^T\cdot\mathbf{V} + \mathbf{p}_\mathbf{V}^T\left(-\frac{1}{r^3}\cdot\mathbf{r} + \frac{\delta\cdot P_0}{m}\cdot\mathbf{e}\right) - p_m\frac{\delta\cdot P_0}{w_0},$$

where $\mathbf{p}_\mathbf{r}^T = [p_{rx}, p_{ry}, p_{rz}]$, $\mathbf{p}_\mathbf{V}^T = [p_{Vx}, p_{Vy}, p_{Vz}]$, and p_m are the conjugate variables to the SC radius-vector, velocity vector, and mass, respectively.

In this case, differential equations of SC optimal motion are found from the following relations:

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}_r},\quad \frac{d\mathbf{V}}{dt} = \frac{\partial H}{\partial \mathbf{p}_\mathbf{V}},\quad \frac{dm}{dt} = \frac{\partial H}{\partial p_m},$$
$$\frac{d\mathbf{p}_\mathbf{r}}{dt} = -\frac{\partial H}{\partial \mathbf{r}},\quad \frac{d\mathbf{p}_\mathbf{V}}{dt} = -\frac{\partial H}{\partial \mathbf{V}},\quad \frac{dp_m}{dt} = -\frac{\partial H}{\partial m}.$$

From the condition of maximum of the Hamiltonian, we obtain the optimal law of control of the thrust unit vector:

$$\mathbf{e}^{\mathrm{opt}} = \frac{\mathbf{p}_\mathbf{V}}{p_V},\quad p_V = |\mathbf{p}_\mathbf{V}|.$$

A law of engine startup–cutoff has the form

$$\Psi = \frac{p_V}{m} - \frac{p_m}{w},\quad \delta^{\mathrm{opt}} = \begin{cases}1, & \text{if } \psi > 0;\\ 0, & \text{if } \psi \le 0,\end{cases}$$

special modes of control are not considered.

Thus, a system of differential equations of SC optimal motion is written as follows:

$$\frac{d\mathbf{r}}{dt} = \mathbf{V},\quad \frac{d\mathbf{V}}{dt} = -\frac{\mathbf{r}}{r^3} + \frac{\delta\cdot P_0}{m}\frac{\mathbf{p}_\mathbf{V}}{p_V},$$
$$\frac{dm}{dt} = -\frac{\delta\cdot P_0}{w_0},$$
$$\frac{d\mathbf{p}_\mathbf{r}}{dt} = \frac{\mathbf{p}_\mathbf{V}}{r^3} - \frac{3\cdot\mathbf{r}\cdot(p_{Vx}\cdot x + p_{Vy}\cdot y + p_{Vz}\cdot z)}{r^5},$$
$$\frac{d\mathbf{p}_\mathbf{V}}{dt} = -\mathbf{p}_\mathbf{r},\quad \frac{dp_m}{dt} = \delta\cdot\frac{P_0\cdot p_V}{m^2}.$$

Final conditions for the problem of direct transfer have form

$$\mathbf{r}(t_k) = \mathbf{r}_\mathbf{k},\quad \mathbf{V}(t_k) = \mathbf{V}_k,\quad p_m(t_k) = 1, \tag{1.1}$$

where t_k is the time of SC arrival at the destination.

Initial conditions for the analysis of the heliocentric trajectory of flight with allowance for optimality conditions for the direction of the initial HEV are written as

$$\mathbf{r}(t_0) = \mathbf{r}_0,\quad \mathbf{V}(t_0) = \mathbf{V}_0 + V_\infty(t_0)\cdot\frac{\mathbf{p}_\mathbf{V}(t_0)}{p_V(t_0)}, \tag{1.2}$$

where $V_\propto(t_0)$ is the SC HEV during escape from the planet of departure at moment t_0, $p_V(t_0) = |p_V(t_0)|$.

Values of conjugate variables at the initial point are the unknown parameters of the boundary value problem under study: $\mathbf{p}_\mathbf{r}(t_0)$, $\mathbf{p}_\mathbf{V}(t_0)$, and $p_m(t_0)$.

In calculations, the required conditions of optimality for the launch date and values of the initial HEV were used:

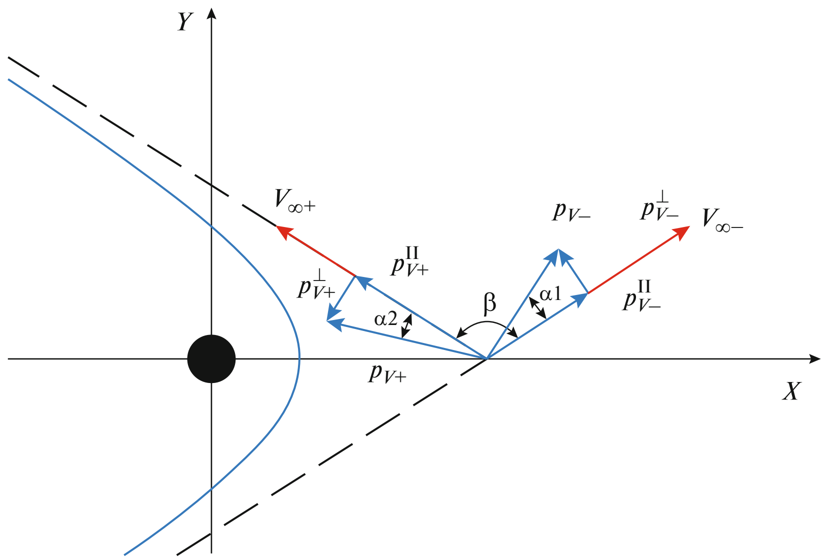

Fig. 1. Scheme of a GAM.

—The optimality condition for launch date t_0 with fixed total time of transfer has form:

$$\left(\frac{w_0 \cdot p_V(t_k)}{m(t_k)} - p_m(t_k)\right) \cdot \frac{\delta(t_k) \cdot P_0}{w_0} - \left(\frac{w_0 \cdot p_V(t_0)}{m(t_0)} - p_m(t_0)\right) \cdot \frac{\delta(t_0) \cdot P_0}{w_0} - \frac{V_\infty(t_0)}{p_V(t_0)} \cdot \mathbf{p}_\mathbf{r}(t_0)^T \cdot \mathbf{p}_\mathbf{V}(t_0) = 0. \quad (1.3)$$

– The optimality condition for the value of the initial HEV:

$$\mathbf{p}_\mathbf{V}(t_0) + p_m(t_0) \cdot \frac{\partial m_0}{\partial V_{\infty 0}} = 0. \quad (1.4)$$

A partial derivative of SC initial mass m_0 (mass after the separation of the chemical upper stage (CUS) that injects the SC into a heliocentric trajectory) with respect to the HEV during launch from the planet of departure $V_{\infty 0}$ is found as follows:

$$\frac{\partial m_0}{\partial V_{\infty 0}} = -\frac{m_{00} \cdot V_\infty(t_0)}{w_{\mathrm{CUS}} \cdot \sqrt{\dfrac{2 \cdot \mu_{\mathrm{pl}0}}{r_0} + V_\infty^2(t_0)}} \times \exp\left(\frac{\sqrt{\dfrac{2 \cdot \mu_{\mathrm{pl}0}}{r_0} + V_\infty^2(t_0)} - \sqrt{\dfrac{\mu_{\mathrm{pl}0}}{r_0}}}{w_{\mathrm{CUS}0}}\right),$$

where m_{00} is the mass, which is injected by the launcher into the base orbit of the planet of departure with a radius of r_0, $w_{\mathrm{CUS}0}$ is the exhaust velocity of the CUS propulsion, and $\mu_{\mathrm{pl}0}$ is the gravitational parameter of the planet of departure. When calculations are carried out, all quantities are used in the dimensionless form. It should be noted that the last relation is valid if velocity is lost during the burn of the CUS, which ensures launch from the circular orbit, are neglected.

Conditions (1.3) and (1.4) are added to boundary conditions of the boundary value problem, while to the varying parameters, launch date t_0 and magnitude $V_\infty(t_0)$ of HEV vector are added: t_0 and $V_\infty(t_0)$, respectively.

2. NECESSARY OPTIMALITY CONDITIONS AT THE POINT OF A GAM

In connection with the fact that the trajectory of the SC with EPS is analyzed, while the time duration of the planetocentric phase during the planet flyby is relatively short and the SC EPS has no time to provide a considerable velocity increment, we will consider a passive GAM.

According to the method of zero-extension gravispheres, the SC heliocentric velocity vector changes instantaneously during a passive GAM. This change implies a turn of the vector of HEV under action of a planet's gravity forces through angle β, the maximum value of which $\beta_{\max}$ is specified by the minimum height of the flyby hyperbola. The turn of the vector of HEV occurs in a certain plane, which is defined by vectors of the arrival and departure hyperbolic excess velocities and is called the GAM plane. A scheme of turning the vector of the SC hyperbolic velocity with a GAM is shown in Fig. 1.

The GAM plane can rotate through an arbitrary angle around the x-axis due to selection of a position of the point of SC trajectory intersection with the tangent plane of the planet flyby.

An angle between vectors of the arrival ($\mathbf{V}_{\infty-}$) and departure ($\mathbf{V}_{\infty+}$) HEVs during a GAM is determined by the following relation:

$$\beta = 2 \cdot \arcsin\left(\left(1 + \frac{r_p \cdot |\mathbf{V}_\infty|^2}{\mu_{pl}}\right)^{-1}\right),$$

where $|\mathbf{V}_\infty|$ is the magnitude of the vector of hyperbolic excess velocity (HEV), r_p is the radius of the pericenter of flyby hyperbola, and μ_{pl} is the gravitational parameter of the planet. Since the radius of the planet flyby is limited by a certain minimum admissible value $r_{p\min}$, then the angle of the HEV turn cannot exceed angle $\beta_{\max}$:

$$\beta \le \beta_{\max} = 2 \cdot \arcsin\left(\left(1 + \frac{r_p \cdot |\mathbf{V}_\infty|^2}{\mu_{pl}}\right)^{-1}\right).$$

The absolute value of HEV at the GAM remains unchanged:

$$|\mathbf{V}_{\infty-}| = |\mathbf{V}_{\infty+}|. \tag{2.1}$$

In Eq. (2.1) and below, index "–" denotes a value of any quantity before the GAM and index "+" refers to a quantity after the GAM.

In this case at the point of GAM, a number of necessary conditions of optimality should be satisfied, a form of which varies depending on the fact, whether the HEV angle of turn is equal to ($\beta = \beta_{\max}$) or less than ($\beta < \beta_{\max}$) the maximum one. The necessary conditions of optimality will be considered in the form proposed in [6, 7].

In the case when the HEV turn angle is less than the maximum value, $\beta < \beta_{\max}$, the necessary condition of optimality takes the following form:

—The basis-vector (a vector that is conjugate to the vector of SC heliocentric velocity) at arrival to the planet before GAM is collinear to the arrival HEV vector:

$$\mathbf{p}_{\mathbf{V}-} \parallel \mathbf{V}_{\infty-}. \tag{2.2}$$

—The basis-vector at departure from the planet after GAM is collinear to the departure HEV vector:

$$\mathbf{p}_{\mathbf{V}+} \parallel \mathbf{V}_{\infty+}. \tag{2.3}$$

—Magnitudes of the arrival and escape basis-vectors are equal to each other:

$$|\mathbf{p}_{\mathbf{V}-}| = |\mathbf{p}_{\mathbf{V}+}|. \tag{2.4}$$

When the HEV turn angle is equal to the maximum value, $\beta = \beta_{\max}$, the necessary conditions of optimality take the following form:

—The basis-vector at arrival to the planet for the GAM ($\mathbf{p}_{\mathbf{V}-}$) belongs to the GAM plane, created by the arrival ($\mathbf{V}_{\infty-}$) and departure ($\mathbf{V}_{\infty+}$) HEV vectors:

$$\mathbf{p}_{\mathbf{V}-} \in \Pi_{\mathbf{V}_{\infty-}\mathbf{V}_{\infty+}}. \tag{2.5}$$

—The basis-vector at departure from the planet after the GAM, $\mathbf{p}_{\mathrm{V}+}$ also belongs to the GAM plane:

$$\mathbf{p}_{\mathbf{V}+} \in \Pi_{\mathbf{V}_{\infty-}\mathbf{V}_{\infty+}}. \tag{2.6}$$

To insert the remaining GAM-optimality conditions, we introduce into consideration two components for each of the basis-vectors under study.

One of these components is the basis-vector projection on the direction of HEV vector. We call it collinear component $p_V^{\parallel}$, it can be both positive and negative. The second component $p_V^{\perp}$ is perpendicular to the HEV vector and lies in the GAM plane, and, accordingly, is called a perpendicular component (see Fig. 1).

The optimality condition of GAM has the following form:

—The magnitude of the perpendicular component of the basis-vector at arrival to the planet is equal to the magnitude of the perpendicular component of the basis-vector at departure from the planet, and they always are directed to the gravitational center:

$$p_{V-}^{\perp} = p_{V+}^{\perp}. \tag{2.7}$$

—A link between the magnitudes of the arrival and departure parallel components of basis-vectors has form

$$p_{V+}^{\parallel} = p_{V-}^{\parallel} + A \cdot p_{V-}^{\perp}, \tag{2.8}$$

where $A = 4 \cdot \tan\left(\frac{\beta}{2}\right) \cdot \left(1 - \sin\left(\frac{\beta}{2}\right)\right)$.

3. MATHEMATICAL MODEL OF A GRAVITY ASSISTED MANEUVER

For example, we consider the interplanetary transfer scheme that includes a single GAM. We conditionally divide the trajectory by the GAM point into two legs: the first leg is from the planet of launch to the planet in the vicinity of which the GAM is executed; the second leg is from the GAM planet to the destination planet.

After calculating the first leg of trajectory, the components of the HEV vector of arrival to the intermediate planet become known to us: $\mathbf{V}_{\infty-}$. Knowing the magnitude and direction of $\mathbf{V}_{\infty-}$, it is possible to unambiguously determine the magnitude and direction of $\mathbf{V}_{\infty+}$ by means of introducing two GAM parameters: the SC HEV turn angle during the GAM (β) and the angle of turn of the flyby hyperbola plane (γ) in the arbitrary, e.g., ecliptic coordinate system. Here, $\beta \in [0, \beta_{\max}]$, $\gamma \in [0, 2\pi]$.

Figure 2 [8] shows the GM plane, to which vectors $\mathbf{V}_{\infty-}$ and $\mathbf{V}_{\infty+}$ belong. A line of intersection of the tangent plane with ecliptic plane X–Y is denoted by l; a line of intersection of the tangent plane with the GAM plane is designated by m. Analyzed angle γ is the angle between l and m, counted from l to m in the counter-

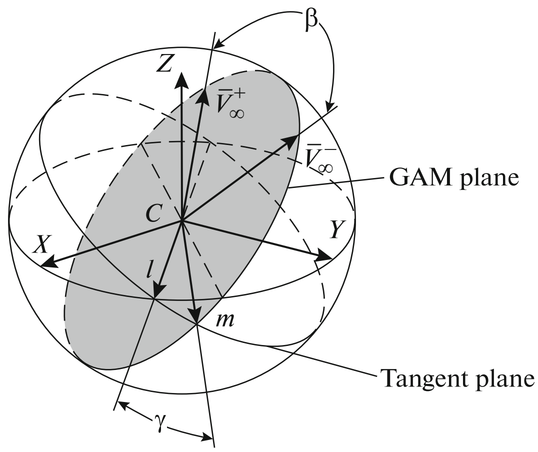

Fig. 2. Graphic image of angles β and γ.

clockwise direction, when looking from the end of vector $\mathbf{V}_{\infty-}$ (it is perpendicular to the tangent plane, in which l and m are located).

Components of vector $\mathbf{V}_{\infty+}$ are found from the following relation [9]:

$$\mathbf{V}_{\infty+} = \begin{pmatrix} V_{\infty-x} & \dfrac{-V_{\infty-y}\cdot|\mathbf{V}_{\infty-}|}{\sqrt{V_{\infty-x}^2+V_{\infty-y}^2}} & \dfrac{-V_{\infty-x}\cdot V_{\infty-z}}{\sqrt{V_{\infty-x}^2+V_{\infty-y}^2}} \\ V_{\infty-y} & \dfrac{V_{\infty-x}\cdot|\mathbf{V}_{\infty-}|}{\sqrt{V_{\infty-x}^2+V_{\infty-y}^2}} & \dfrac{-V_{\infty-y}\cdot V_{\infty-z}}{\sqrt{V_{\infty-x}^2+V_{\infty-y}^2}} \\ V_{\infty-z} & 0 & \sqrt{V_{\infty-x}^2+V_{\infty-y}^2} \end{pmatrix} \times \begin{pmatrix} \cos(\beta) \\ \sin(\beta)\cdot\cos(\gamma) \\ \sin(\beta)\cdot\sin(\gamma) \end{pmatrix}, \quad (3.1)$$

where $V_{\infty-x}$, $V_{\infty-y}$, and $V_{\infty-z}$ are the components of the HEV vector at arrival to the planet for the GAM.

Similarly, knowing components of arrival basis-vector $\mathbf{p}_{\mathbf{V}-}$and values of angles β and γ, it is possible to unambiguously determine components of vector $\mathbf{p}_{\mathbf{V}+}$.

When $\beta < \beta_{\max}$, components of vector $\mathbf{p}_{\mathbf{V}+}$ are found from the following relation:

$$\mathbf{p}_{\mathbf{V}+} = \begin{pmatrix} p_{V-x} & \dfrac{-p_{V-y}\cdot|\mathbf{p}_{\mathbf{V}-}|}{\sqrt{p_{V-x}^2+p_{V-y}^2}} & \dfrac{-p_{V-x}\cdot p_{V-z}}{\sqrt{p_{V-x}^2+p_{V-y}^2}} \\ p_{V-y} & \dfrac{p_{V-x}\cdot|\mathbf{p}_{\mathbf{V}-}|}{\sqrt{p_{V-x}^2+p_{V-y}^2}} & \dfrac{-p_{V-y}\cdot p_{V-z}}{\sqrt{p_{V-x}^2+p_{V-y}^2}} \\ p_{V-z} & 0 & \sqrt{p_{V-x}^2+p_{V-y}^2} \end{pmatrix} \times \begin{pmatrix} \cos(\beta) \\ \sin(\beta)\cos(\gamma) \\ \sin(\beta)\sin(\gamma) \end{pmatrix}, \quad (3.2)$$

where p_{V-x}, p_{V-y}, and p_{V-z} are the components of the basis-vector at arrival to the planet for a GAM.

In the case under study, necessary conditions of optimality (2.2) and (2.3) should be fulfilled. Conditions (2.1) and (2.4), according to (3.1) and (3.2), are satisfied automatically.

The mathematical model of a GAM unambiguously links collinearity conditions (2.2) and (2.3) through vector-rotation matrices (3.1) and (3.2). From this, it follows that if one of the two conditions (2.2) or (2.3) is fulfilled, the second condition will be satisfied automatically.

Thus, if relations (3.1) and (3.2) are used to find $\mathbf{V}_{\infty+}$ and $\mathbf{p}_{\mathbf{V}+}$, then boundary conditions at the GAM point can be considered for the case of $\beta < \beta_{\max}$ as

$$\begin{gathered} \mathbf{r}(t_i) - \mathbf{r}_{\text{пл}i}(t_i) = 0, \\ p^i_{V+y}\cdot\mathbf{V}^i_{\infty+x} - p^i_{V+x}\cdot\mathbf{V}^i_{\infty+y} = 0, \\ p^i_{V+z}\cdot\mathbf{V}^i_{\infty+x} - p^i_{V+x}\cdot\mathbf{V}^i_{\infty+z} = 0, \end{gathered} \quad (3.3)$$

where $\mathbf{r}(t_i)$ is the SC radius-vector, t_i is the time moment of GAM execution, $i = 1,\ldots,n$, where n is the number of GAMs, $p^i_{V+x}, p^i_{V+y}, p^i_{V+z}$ are components of basis-vector $\mathbf{p}^i_{\mathbf{V}+}$, and $\mathbf{V}^i_{\infty+x}, \mathbf{V}^i_{\infty+y}, \mathbf{V}^i_{\infty+z}$ are components of vector $\mathbf{V}^i_{\infty+}$. In total, five boundary conditions are obtained.

Unknown parameters at the GAM point are as follows:

$$\mathbf{p}_{\mathbf{r}}(t_i), \beta_i, \gamma_i. \quad (3.4)$$

Altogether five unknown parameters are obtained.

When a GAM is inserted into the transfer scheme, conditions (3.3) and (3.4) should be added to boundary conditions (1.1) and (1.2). In all, a problem of interplanetary transfer with a single GAM will have 11 boundary conditions and 11 unknowns; each additional GAM increases the order of the boundary value problem by 5 (if the GAM date is not considered to be unknown).

The last two terms in (3.3) are responsible for the fulfillment of necessary conditions of optimality (2.3); the collinearity of the departure HEV vector and the departure basis-vector. Numerous practical calculations have shown that this ensures much better convergence of the search for a solution to the boundary value problem as compared with the selection of condition (2.2); the collinearity of the arrival HEV vector and the arrival basis-vector. Indeed, adding collinearity conditions (2.2) (these are two scalar conditions) to the boundary value problem, it is necessary to add two variables (angles β and γ), but in this case, condition (2.2) is independent of these angles, and, consequently, we reduce the number of degrees of freedom for fulfillment of this constraint, which negatively affects the convergence of the iterative process.

Now we consider the case when $\beta = \beta_{max}$. At the GAM point, condition (2.1) and necessary optimality conditions (2.5)–(2.8) should be fulfilled. The same as for the case of $\beta < \beta_{max}$, components of vector $\mathbf{V}_{\infty+}$ are determined from Eq. (3.1) using known components of arrival HEV vector $\mathbf{V}_{\infty-}$, obtained after calculating the first leg of trajectory before the GAM; thus, we fulfill condition (2.1). Now, components of vector $\mathbf{p}_{\mathbf{V}+}$, which satisfy the necessary conditions of optimality, should be found. Vector $\mathbf{p}_{\mathbf{V}-}$ is considered to be known after the calculation of the trajectory leg preceding the GAM.

Components of the arrival and departure basis-vectors can be found from the following relations:

$$p_{V\pm}^{\perp} = \mathbf{p}_{\mathbf{V}\pm}^{T} \cdot \mathbf{e}_{\pm}^{\perp}, \quad p_{V\pm}^{\parallel} = \mathbf{p}_{\mathbf{V}\pm}^{T} \cdot \mathbf{e}_{\pm}^{\parallel}, \tag{3.5}$$

where $\mathbf{e}_{\pm}^{\parallel} = \dfrac{\mathbf{V}_{\infty}^{\pm}}{\left|\mathbf{V}_{\infty}^{\pm}\right|}$, $\mathbf{e}^{b} = \dfrac{\mathbf{V}_{\infty}^{-} \times \mathbf{V}_{\infty}^{+}}{\left|\mathbf{V}_{\infty}^{-} \times \mathbf{V}_{\infty}^{+}\right|}$, and $\mathbf{e}_{\pm}^{\perp} = \dfrac{\mathbf{e}^{b} \times \mathbf{V}_{\infty}^{\pm}}{\left|\mathbf{e}^{b} \times \mathbf{V}_{\infty}^{\pm}\right|}$.

From relations of (3.5), we determine the perpendicular ($p_{V-}^{\perp}$) and collinear ($p_{V-}^{\parallel}$) components of the arrival basis-vector. Using (2.7) and (2.8), we find the perpendicular ($p_{V+}^{\perp}$) and collinear ($p_{V+}^{\parallel}$) components of the departure basis-vector ($\mathbf{p}_{\mathbf{V}+}$) and calculate magnitudes of basis-vectors $\mathbf{p}_{\mathbf{V}-}$ and $\mathbf{p}_{\mathbf{V}+}$:

$$\left|\mathbf{p}_{V\pm}\right| = \sqrt{p_{V\pm}^{\parallel} + p_{V\pm}^{\perp}}.$$

Thus, we have found absolute values of vectors $\mathbf{p}_{V-}$ and $\mathbf{p}_{V+}$. As is known, a vector is the quantity characterized by its absolute value and direction. Therefore, it only remains to us to find a direction of basis-vector $\mathbf{p}_{V+}$ and its components will become known to us.

Let us introduce auxiliary angles $\alpha 1$ and $\alpha 2$, characterizing an angular position of the arrival and departure basis-vectors with respect to the arrival and departure HEVs, respectively (see Fig. 1).

We determine angles $\alpha 1$ and $\alpha 2$ from the following relations:

$$\alpha 1 = \arctan\left(\frac{p_{V-}^{\perp}}{p_{V-}}\right),$$

$$\alpha 2 = \arctan\left(\frac{p_{V+}^{\perp}}{p_{V+}}\right).$$

We find angle α, characterizing a difference in the angle position of the arrival and departure basis-vectors with respect to the relevant HEV vectors: $\alpha = \alpha 2 - \alpha 1$.

We turn vector $\mathbf{p}_{V-}$ through angles $\beta + \alpha$ and γ, thus defining the direction of vector $\mathbf{p}_{V+}$, and multiply the value obtained by a ratio of the magnitude of departure basis-vector to the magnitude of arrival basis-vector:

$$\mathbf{p}_{\mathbf{V}+} = \begin{pmatrix} p_{V-x} & \dfrac{-p_{V-y} \cdot \left|\mathbf{p}_{V-}\right|}{\sqrt{p_{V-x}^{2} + p_{V-y}^{2}}} & \dfrac{-p_{V-x} \cdot p_{V-z}}{\sqrt{p_{V-x}^{2} + p_{V-y}^{2}}} \\ p_{V-y} & \dfrac{p_{V-x} \cdot \left|\mathbf{p}_{V-}\right|}{\sqrt{p_{V-x}^{2} + p_{V-y}^{2}}} & \dfrac{-p_{V-y} \cdot p_{V-z}}{\sqrt{p_{V-x}^{2} + p_{V-y}^{2}}} \\ p_{V-z} & 0 & \sqrt{p_{V-x}^{2} + p_{V-y}^{2}} \end{pmatrix} \times \begin{pmatrix} \cos(\beta + \alpha) \\ \sin(\beta + \alpha) \cdot \cos(\gamma) \\ \sin(\beta + \alpha) \cdot \sin(\gamma) \end{pmatrix} \cdot \frac{\left|\mathbf{p}_{\mathbf{V}+}\right|}{\left|\mathbf{p}_{\mathbf{V}-}\right|}. \tag{3.6}$$

Thus, from Eq. (3.6) we have found components of basis-vector $\mathbf{p}_{\mathbf{V}+}$. In this case, necessary optimality conditions (2.7) and (2.8) are satisfied automatically. It will be sufficient to fulfill one of the two necessary optimality conditions (2.5) or (2.6) and the other will be satisfied automatically. Indeed, vectors $\mathbf{V}_{\infty-}$ and $\mathbf{V}_{\infty+}$ form a GAM plane and, if we turn vector $\mathbf{V}_{\infty-}$ through angles β and γ, by using Eq. (3.1), then vector $\mathbf{V}_{\infty+}$ falls on the GM plane by definition. Similarly, we assume that basis-vector $\mathbf{p}_{\mathbf{V}-}$ belongs to the GAM plane (i.e., condition (2.5) is fulfilled) and if it is turned through angles $\beta + \alpha$ and γ, by using Eq. (3.6), then the resulting vector $\mathbf{p}_{\mathbf{V}+}$ will also belong to the GAM plane, since angles β and α are in the same plane (GAM plane), while angle γ, used in Eqs. (3.1) and (3.6), certainly, is the same.

Practice has shown that it is desirable to choose Eq. (2.6) as a boundary condition, since two (out of three) vectors in it depend on angles β and γ (only one vector in condition (2.5)), which are the selected parameters; this ensures high flexibility of the process of solution and, consequently, better convergence.

Thus, at the GAM point for the case when the SC HEV turn angle with GAM $\beta = \beta_{max}$, we have the following boundary conditions:

$$\begin{gathered} \mathbf{r}(t_i) - \mathbf{r}_{\mathrm{pl}i}(t_i) = 0, \\ \mathbf{p}_{V+i}^{T} \cdot \left[\mathbf{V}_{\infty-i} \times \mathbf{V}_{\infty+i}\right] = 0, \\ \beta_{\max i} - \beta_i = 0, \end{gathered} \tag{3.7}$$

where $i = 1, \dots, n$, n is the number of GAMs. In total, five boundary conditions are obtained.

Parameters of (3.4) are the unknown parameters at the GAM point. Altogether, five unknown parameters are obtained.

When a GAM is included in the scheme of transfer, conditions (3.7) and (3.4) should be added to boundary conditions (1.1) and (1.2). In total, a problem of interplanetary transfer with a single GAM will have 11 boundary conditions and 11 unknowns; each additional GAM increases the order of the boundary-value problem by 5 (if the date of the GAM is considered known).

The presented technique allows optimization of the entire trajectory to be performed with the fulfillment of all necessary conditions of optimality at the GAM point.

In numerical examples presented below, necessary conditions of optimality were also used for the date of GAM execution t_i:

$$\mathbf{p}^i_{r+}(t_i)^T\mathbf{V}^i_{\infty+} - \mathbf{p}^i_{r-}(t_i)^T\mathbf{V}^i_{\infty-} = 0. \quad (3.8)$$

Condition (3.8) can be added to the remaining boundary conditions of the boundary-value problem, while the date of GAM execution will become, accordingly, an additional unknown variable.

4. USING MIXED-TYPE CONSTRAINTS AT THE POINT OF GRAVITY-ASSIST MANEUVERS

We have considered two separate cases at the GAM point: when $\beta < \beta_{\max}$ and $\beta = \beta_{\max}$. Combining these cases will allow the working hours of optimization of GAM-containing trajectories to be appreciably reduced. Indeed, with a single GAM, we need to consider two cases (when $\beta < \beta_{\max}$ and $\beta = \beta_{\max}$) and choose one of them, at which the boundary conditions of $\beta \le \beta_{\max}$ and the optimality conditions are not violated. While optimizing the transfer with an arbitrary number of GAMs, we need to consider a series of boundary-value problems, the number of which is determined from the following relation: $k = 2^n$, where n is the number of GAMs.

In the general case, HEV turn angle β during a GAM, according to (2.2) should satisfy condition

$$\beta_{\max} - \beta \ge 0. \quad (4.1)$$

We rewrite conditions of collinearity of the basis-vector and the SC HEV vector for the case of $\beta < \beta_{\max}$ (two last equalities in (3.3)) as follows:

$$\begin{aligned} &\mathbf{p}^T_{\mathbf{V}+i}\cdot[\mathbf{V}_{\infty-i}\times\mathbf{V}_{\infty+i}] = 0,\\ &p^{\perp}_{V+} = 0. \end{aligned} \quad (4.2)$$

Indeed, if departure basis-vector $\mathbf{p}_{\mathbf{V}+}$ and departure HEV vector $\mathbf{V}_{\infty+}$ lie in the same plane and, in this case, $p^{\perp}_{V+} = 0$ (see Fig. 1), then these vectors are collinear and, as is shown above, in the technique for calculation of the GAM point for the case of $\beta < \beta_{\max}$, vectors $\mathbf{p}_{\mathbf{V}-}$ and $\mathbf{V}_{\infty-}$ also become collinear due to Eqs. (3.1) and (3.2).

For the general case of $\beta \le \beta_{\max}$, we have four equality-type boundary conditions at the GAM point: the first vector equality in (3.3) or in (3.7) and the first equality in (4.2) or the second one in (3.7). Here, in the general case, the perpendicular component of basis-vector $p^{\perp}_{V+}$should satisfy inequality

$$p^{\perp}_{V+} \ge 0. \quad (4.3)$$

Eventually, in the general case at the GAM point, we have two inequality-type boundary conditions (4.1) and (4.3). Thus, we obtained the boundary value problem with constraints of mixed type (restrictions in the form of equalities and inequalities).

We take advantage of a method for introducing additional slack variables into boundary conditions and unknown parameters of the boundary value problem. This approach is widely used in the construction of mathematical models of different economic processes for the purpose of writing a stated problem in the canonical form of standard problems of linear or nonlinear programming. In these problems, the slack variables have a simple physical meaning of the unused quantity of raw material of one or another type. It is also applied in other fields of science, where the optimization problems with mixed-type constraints should be solved, e.g., in problems of planning (organization of works) or designing a technical object (automatic control systems, radio engineering complex, etc.) [10–13].

We apply this approach to the problem of optimization of SC interplanetary transfer. According to this method, we transform boundary conditions in the form of inequalities to the equality-type boundary conditions by adding to each of them nonnegative slack variable b_j^2 ($j = 1,\ldots,2$):

$$\begin{aligned} \beta_{\max i} - \beta_i - b_{1i}^2 &= 0,\\ p^{\perp}_{V+} - b_{2i}^2 &= 0, \end{aligned}$$

where $i = 1,\ldots,n$, n is the number of GAMs.

In this case, according to transversality conditions, either $b_{1i} = 0$ or $b_{2i} = 0$. Following this, we write one more boundary condition: $b_{1i}\cdot b_{2i} = 0$.

Eventually, at the GAM point for the case of $\beta \le \beta_{\max}$, the following boundary conditions should be satisfied:

$$\begin{aligned} \mathbf{r}(t_i) - r_{\text{пл}i}(t_i) &= 0,\\ \mathbf{p}^T_{V+i}\cdot[\mathbf{V}_{\infty-i}\times\mathbf{V}_{\infty+i}] &= 0,\\ \beta_{\max i} - \beta_i - b_{1i}^2 &= 0,\\ p^{\perp}_{V+} - b_{2i}^2 &= 0,\\ b_{1i}^2\cdot b_{2i}^2 &= 0, \end{aligned}$$

where $i = 1,\ldots,n$, n is the number of GAMs. In total, seven boundary conditions are obtained. There are also seven unknown parameters at the GAM point: $\mathbf{p}_{\mathbf{r}}(t_i), \beta_i, \gamma_i, b_{1i}, b_{2i}$.

Table 1. Variants of calculation schemes of SC trajectory optimization

	Variant no. 1	Variant no. 2	Variant no. 3
Condition for the angle β at the GAM	$\beta = \beta_{max}$	$\beta \in R$	$\beta \le \beta_{max}$
Condition for the perpendicular component of the basis-vector at the GAM	$p_V^{\perp} \in R$	$p_V^{\perp} = 0$	$p_V^{\perp} \ge 0$

Thus, applying the method of using the additional slack variables, we have managed to reduce the boundary value problem with mixed-type constraints to the boundary value problem with equality constraints.

Despite the fact that in the final analysis we obtained the boundary value problem with equality constraints, physically this problem corresponds completely to the boundary value problem with constraints of mixed type. This approach proved to be very efficient; it is important that it does not deteriorate the convergence of the process of solving the boundary value problem in comparison with solving a series of boundary value problems, in which one of the cases of $\beta < \beta_{max}$ or $\beta = \beta_{max}$ was fixed at GAM points.

5. NUMERICAL EXAMPLE OF INTERPLANETARY TRAJECTORY WITH A GRAVITY ASSISTED MANEUVER

Let us consider the employment of the proposed approach to a search for an optimal solution, corresponding to a boundary value problem with mixed-type constraints, in comparison with commonly used methods by examples of trajectory optimization of a SC with an EPS and a single GAM. Variant no. 1 corresponds to the case when during the GAM the height of the flyby hyperbola is equal to the minimum specified height, in this case, $\beta = \beta_{max}$. Variant no. 2 represents the case when the flyby hyperbola height is unlimited. Here, constraints on HEV turn angle β are not imposed. The problem formulation, using constraints of mixed type, will be called variant no. 3. Variants of calculation schemes for SC trajectory optimization are given in Table 1.

As a transport system, an Angara-A5 launcher, KVTK chemical upper stage (CUS), and a cruise EPS with ion engines of the RIT-22 type are used. The Angara-A5 launcher inserts the SC into a base orbit around the Earth. The KVTK CUS transfers the SC from the base orbit to the hyperbolic trajectory of departure. Then, the CUS is separated from the SC, and further motion of the SC in the heliocentric phase of trajectory is implemented using the EPS. A nuclear powerplant is used as a power source. The thrust, mass flowrate, and specific impulse of the burning engine were considered constant (independent of flight conditions) at all engine burns. The law of engine startup–cutoff, a program of thrust-vector control, magnitude of the initial HEV, and the launch and GAM dates are optimized. The total transfer time is fixed. A zero docking of the SC with the planet of destination is considered.

The Angara-A5 launcher ensures insertion into a low near-Earth orbit (a circular orbit with an altitude of 200 km) of the SC with a mass of 24235 kg.

The KVTK CUS has the following characteristics, which are used in this study: the CUS final mass, including the mass of the upper-stage adapter with the SC, was 3330 kg and specific thrust of the engine was 470 s.

The cruise EPS uses 12 engines of the RIT-22 type.

Characteristics of one engine of the RIT-22 type are assumed to be as follows [14]: a thrust of 0.2 N; specific impulse of 4650 s; and consumed electric power of 5.785 kW.

The total number of RIT-22 engines was assumed to be 13. A minimum height of flyby hyperbola with a GAM was assumed to be 400 km.

Let us consider a transfer from Earth to Mercury with a GAM near Venus. The total transfer time was assumed to be 720 days. A launch window was considered within a range from January 1, 2026 to December 31, 2026. Numerical results are presented in Table 2.

As can be seen from Table 2, it is optimal when the SC HEV turn angle during a GAM is equal to the maximum one. Here, the results of calculations for variants no. 1 and no. 3 coincided. Thus, the use of mixed-type constraints at the GAM point allows the optimal trajectory to be obtained by solving a single boundary value problem. In variant no. 2, the SC final mass proved to be the largest one; however, in this case, the HEV turn angle with a GAM proved to be more than the maximum admissible one, and the resulting trajectory is of no practical interest.

Figure 3 shows a projection of the SC interplanetary trajectory (Earth–Venus–Mercury) on the ecliptic plane.

Black thick solid arcs present active legs of the trajectory, at which the cruise EPS is started; black dotted lines correspond to its passive legs. Orbits of the planets (by convention, circular) are shown in gray color (thin circles). There are six active and five passive legs on the trajectory. The heliocentric transfer begins from the active leg and ends in a passive leg at arrival to Venus for the GAM execution; altogether there are two active and two passive legs at the stage of the Earth–Venus transfer. The Venus–Mercury phase of transfer begins from the SC passive motion at

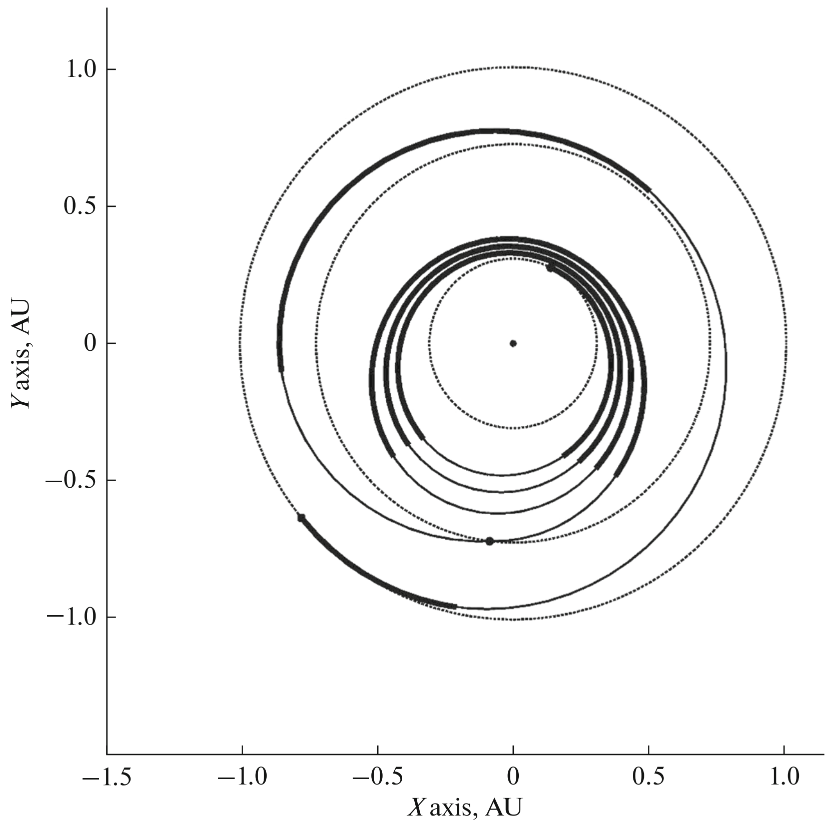

Fig. 3. Projection of the heliocentric trajectory of Earth–Venus–Mercury transfer on the ecliptic plane.

departure from Venus and ends in the active leg at arrival to Mercury; altogether there are four active and four passive legs in this phase.

Figure 4 shows dependences of the EPS switchover and startup–cutoff functions on the transfer time.

From Fig. 4, the sequence and duration of active and passive legs can be determined easily. An operation mode of the cruise EPS fully corresponds to an operation mode of the uncontrollable engine.

Let us consider a transfer from Earth to Jupiter with a GAM near the Earth. The total time of transfer is 1200 days. The launch window was considered within a range from January 1, 2018 to December 31, 2018. Numerical results are presented in Table 3.

Table 2. Results of trajectory optimization for different variants of calculation schemes (Earth–Venus–Mercury transfer)

	Variant no. 1	Variant no. 2	Variant no. 3
Launch date	April 29, 2026	May 2, 2026	May 29, 2026
Julian launch date	2461160.43	2461163.51	2461160.43
Initial HEV, m/s	1120.23	1229.92	1120.23
SC mass at the moment of launch from the Earth, kg	8549.52	8519.56	8549.52
Date of GAM	March 17, 2027	March 13, 2027	March 17, 2027
Julian date of GAM	2461482.19	2461478.37	2461482.19
HEV value at GM near Venus, m/s	6233.66	6266.13	6233.66
HEV turn angle at GAM β, deg	68.73	89.85	68.73
Maximum HEV turn angle at GAM β_{max}, deg	68.73	68.37	68.73
Date of arrival to Mercury	April 18, 2028	April 21, 2028	April 18, 2028
Julian date of arrival to Mercury	2461880.43	2461883.51	2461880.43
SC final mass, kg	6531.95	6597.05	6531.95

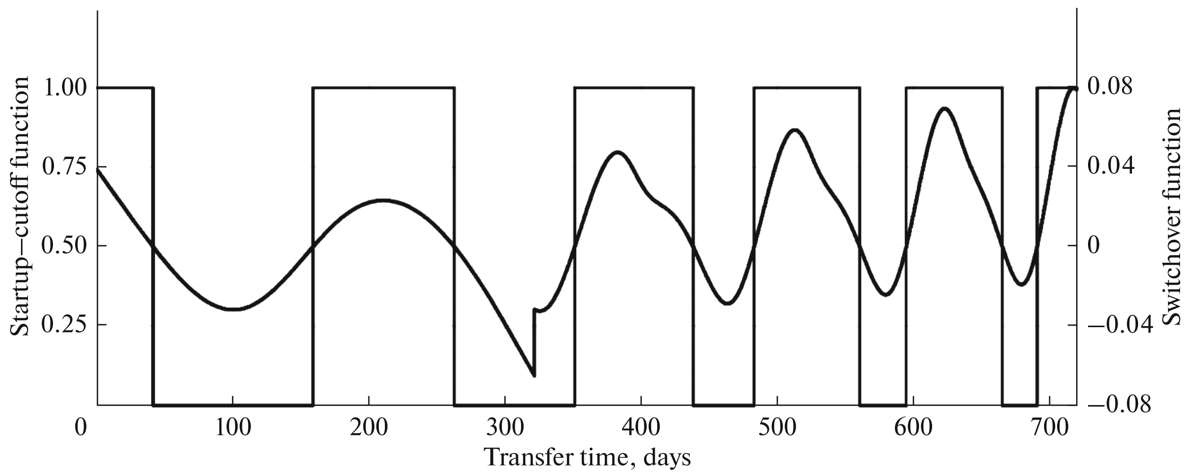

Fig. 4. Functions of engine switchover and startup–cutoff along the SC heliocentric trajectory with Earth–Venus–Mercury transfer.

From Table 3 it can be seen that the case proved to be optimal, when the SC HEV turn angle at the GAM is equal to the maximum one, the same as during the Earth-Venus–Mercury transfer. Results of calculations for variants no. 1 and no. 3 coincided. For this transfer scheme, the employment of mixed-type constraints at the GAM point also allowed the optimal trajectory to be obtained in the context of solving a single boundary value problem. In variant no. 2, the SC final mass proved to be the largest one, here the HEV turn angle at the GAM proved to be more than the maximum admissible one and the resulting trajectory is of no practical interest.

Figure 5 presents a projection of the Earth–Earth–Jupiter SC interplanetary trajectory on the ecliptic plane.

Active legs of the trajectory, at which the cruise EPS is started, are presented by black thick solid arcs; passive legs are shown by black dotted lines. Planet orbits are shown, by convention, circular in gray color (thin circles). There are five active and four passive legs on the trajectory. The heliocentric transfer begins from an active leg and ends in a passive leg at arrival to Earth for the GAM execution; altogether there are three active and three passive legs in the Earth–Earth phase of transfer. The Earth–Jupiter phase of transfer begins with the SC active motion and ends in the active leg at arrival to Jupiter; altogether there are two active legs and one passive leg in this phase.

Figure 6 shows the EPS switchover and startup–cutoff functions vs. transfer time.

From Fig. 6 it is easy to find the sequence and duration of active and passive legs. A mode of operation of the cruise EPS completely corresponds to the mode of operation of the uncontrollable engine.

Table 3. Results of trajectory optimization for different variants of calculation schemes (Earth–Earth–Jupiter transfer)

	Variant no. 1	Variant no. 2	Variant no. 3
Launch date	February 2, 2018	January 28, 2018	January 22, 2018
Julian launch date	2458172.95	2458147.49	2458172.95
Initial HEV, m/s	963.49	1044.17	963.49
SC mass at the time of launch from Earth, kg	8587.55	8568.64	8587.55
Date of GAM	February 25, 2019	February 19, 2019	February 25, 2019
Julian date of GAM	2458540.44	2458534.25	2458540.44
HEV magnitude at GAM near Earth, m/s	8317.17	9030.94	8317.17
HEV turn angle at GAM β, deg	54.71	74.83	54.71
Maximum HEV turn angle at GAM β_{max}, deg	54.71	49.54	54.71
Date of arrival to Jupiter	June 6, 2021	May 12, 2021	June 6, 2021
Julian date of arrival to Jupiter	2459372.95	2459347.49	2459372.95
SC final mass, kg	6442.99	6526.29	6442.99

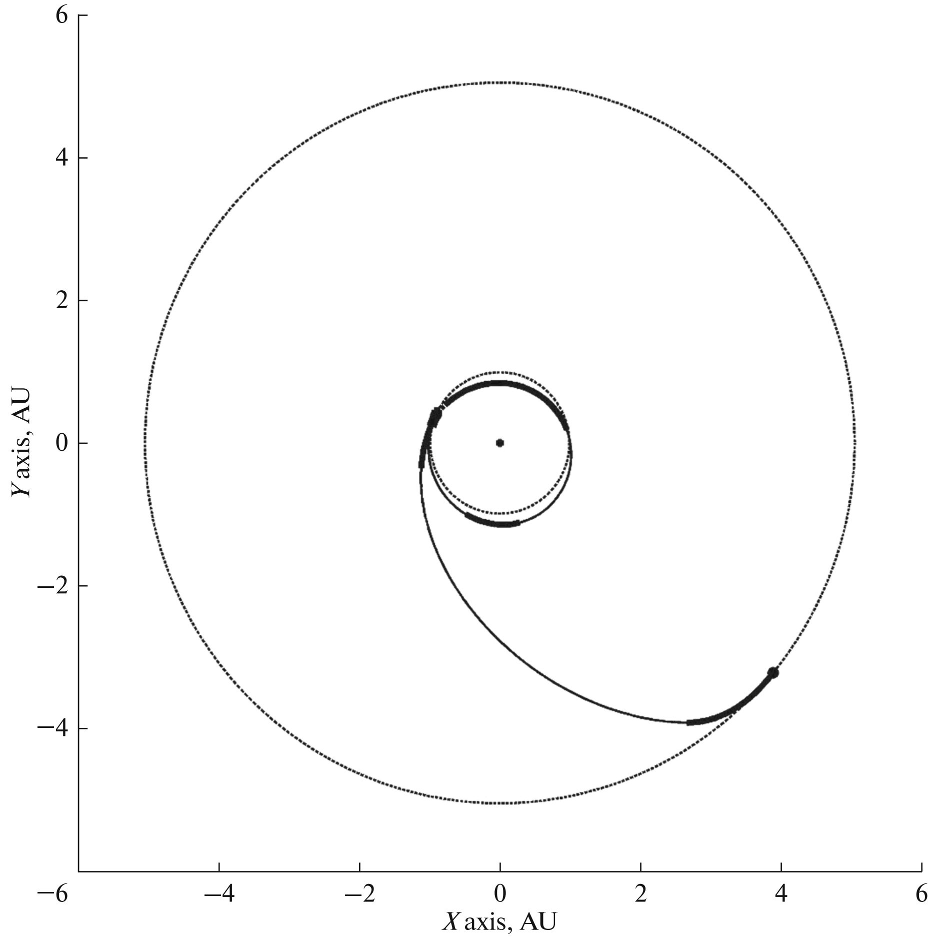

Fig. 5. Projection of the heliocentric trajectory of Earth–Earth–Jupiter transfer on the ecliptic plane.

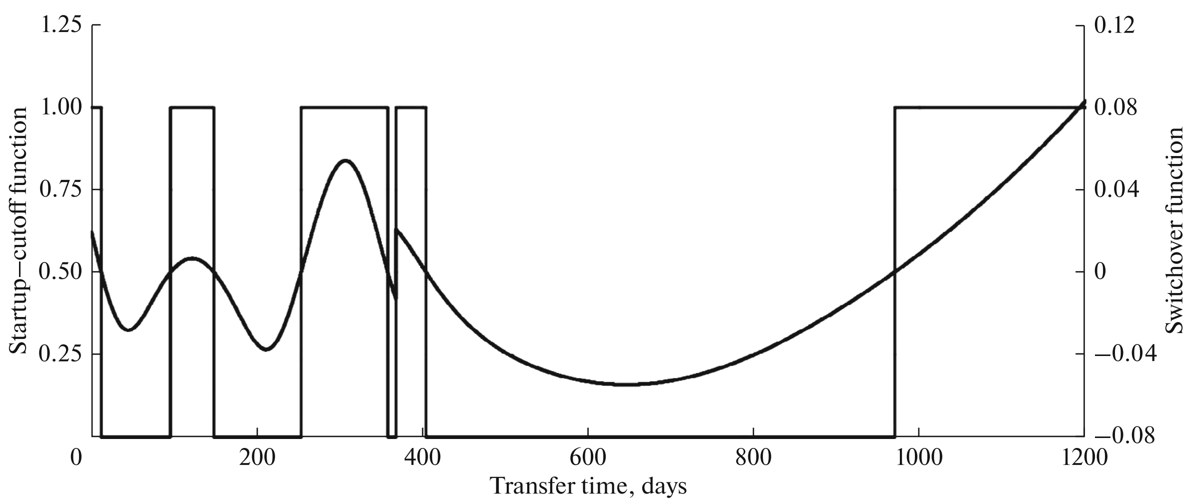

Fig. 6. Functions of engine switchover and startup–cutoff along the SC heliocentric trajectory with Earth–Earth–Jupiter transfer.

CONCLUSIONS

This study presents a new technique for calculating trajectories with GAMs. A distinctive feature of this technique is the relatively weak increase in the order of the boundary value problem due to the technique application. This result is achieved by using the mathematical model of GAM, in which a part of optimality conditions is fulfilled automatically and they need not be written into the boundary conditions. A decrease in the order of the boundary value problem makes it possible to enhance the stability of iterative process of solution search and to reduce demands for computational resources, which are necessary to carry out the calculation. The presented technique allows optimization of trajectories to be performed with the fulfillment of all necessary conditions of optimality at the GAM point.

Based on the new proposed technique of GAM calculation, both cases, when $\beta < \beta_{max}$ and $\beta = \beta_{max}$, are combined. As a result of this combination, the boundary value problem with mixed-type constraints is obtained, which is offered to be solved by introducing additional slack variables into the boundary conditions. This approach allows the optimal trajectories to be obtained for SC interplanetary transfers with GAMs in the context of solving a single boundary value problem.

The efficiency of the presented technique is shown by the example of a search for optimal trajectories of the transfer from Earth to Mercury with a GAM near Venus and the transfer from Earth to Jupiter with a GAM in the vicinity of Earth.

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