ISSN 0010-9525, Cosmic Research, 2015, Vol. 53, No. 5, pp. 385–393. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © S.M. Poleshchikov, 2015, published in Kosmicheskie Issledovaniya, 2015, Vol. 53, No. 5, pp. 421–429.

Regularization of Equations of the Planar Restricted Problem of Three Bodies with *L*-Transformations

S. M. Poleshchikov
Syktyvkar Institute of Forestry, Syktyvkar, Russia
e-mail: polsm@syktsu.ru
Received March 25, 2013

Abstract—The regularization of the equations of motion of the planar circular restricted three-body problem is considered. The regularization is performed in the canonical variables in the movable coordinate system. Two different L-matrices of the second order are used in the regularization. The constructed equations have a polynomial structure. The obtained system is numerically integrated by the Runge–Kutta–Felberg method. The results of numerical experiments with the parameters of the Earth–Moon system are presented for various L-matrices.

DOI: 10.1134/S0010952515040061

1. INTRODUCTION

The restricted three-body problem consists in studying the motion of a passively gravitating point M in the Newtonian field of attraction of two material points M_1, M_2 having masses m_1, m_2, respectively. The points move in Keplerian orbits around their centers of mass [1]. One distinguishes planar and spatial problems, which have corresponding names, such as "the spatial–elliptic restricted problem of three bodies." In this work we consider the planar circular restricted three-body problem, where point M moves in a plane determined by the circular orbits of points M_1, M_2.

The restricted three-body problem has practical application in space dynamics. It is sufficient to mention [2], which summarizes the applications of this problem for calculating the trajectories to reach the Moon. Note also papers [3–5], in which the periodic orbits of the restricted three-body problems are investigated. These papers contain extensive lists of references on this subject.

In this paper the regularization (elimination of singularities) of the equations of motion in the planar circular restricted problem is considered. As is known [4], singular points located at attracting centers M_1, M_2 can be eliminated by transforming the independent variable. In addition to this transformation, one can consider the transformation of the dependent variables coordinates and velocities. In the problems of eliminating the singularities one distinguishes local and global regularizations (beginning with the three-body problem). In global regularization all singularities are eliminated. The examples are: global Birkhoff regularization in the restricted three-body problem [4], the regularization of the problem of two motionless centers [6], the global Heggie regularization of the N-body problem [7]. As an example of local regularization, we mention the works [8, 9]. We will perform global regularization by transforming the time and applying two L-transformations of the second order, generated by generalized Levi–Civita matrices. Note that we performed a similar regularization in the motionless coordinate system [10]. Here we will consider regularization in the rotating coordinate system and present the numerical study of obtained regular equations. The detailed theory of L-matries is given in papers [10, 11]. In this paper we present necessary information on this issue.

We introduce the motionless coordinate system OX_1X_2 with its origin at the centers of mass of points M_1, M_2. Let $\mathbf{X} = (X_1,X_2)^T$ be the position vector of point M with mass m. This point is affected by the forces of attraction $\mathbf{F}_1$, $\mathbf{F}_2$ from the side of masses M_1, M_2, respectively (Fig. 1). On the basis of Newton's second law, the equation of motion of point M can be written as

$$m\frac{d^2\mathbf{X}}{dt^{*2}} = \gamma\frac{mm_1}{R_1^2}\frac{\overline{MM_1}}{R_1} + \gamma\frac{mm_2}{R_2^2}\frac{\overline{MM_2}}{R_2}, \qquad (1)$$

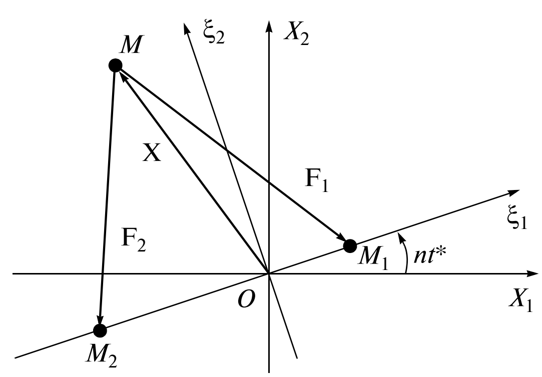

Fig. 1. Movable and motionless coordinate systems.

where γ is the gravitational constant, t^* is the physical time, and $R_1 = |\overline{MM_1}|$, $R_2 = |\overline{MM_2}|$ are the distances to attracting masses.

Points M_1, O, M_2 lie on the same straight line. Denote by n the angular velocity of points M_1, M_2. Then the angle nt^* is the longitude of point M_1. Let d_i is the distance from O to M_i ($i = 1, 2$). The coordinates of points M_1, M_2 equal

$$M_1(d_1\cos nt^*, d_1\sin nt^*), \quad M_2(-d_2\cos nt^*, d_2\sin nt^*)$$

We introduce the force function

$$\Phi^* = \Phi^*(t^*, X_1, X_2) = \gamma\left(\frac{m_1}{R_1} + \frac{m_2}{R_2}\right),$$

where

$$R_1 = \sqrt{(X_1 - d_1\cos nt^*)^2 + (X_2 - d_1\sin nt^*)^2},$$

$$R_2 = \sqrt{(X_1 + d_2\cos nt^*)^2 + (X_2 + d_2\sin nt^*)^2}.$$

Then equation (1), after reduction by m, can be written in the form

$$\frac{d^2\mathbf{X}}{dt^{*2}} = \frac{\partial\Phi^*}{\partial\mathbf{X}}, \quad \frac{\partial\Phi^*}{\partial\mathbf{X}} = \left(\frac{\partial\Phi^*}{\partial X_1}, \frac{\partial\Phi^*}{\partial X_2}\right)^T. \quad (2)$$

In equation (2) we will pass to the dimensionless quantities

$$t = nt^*, u_i = X_i/l, \ \rho_i = R_i/l, \ i = 1, 2,$$

where $l = d_1 + d_2$ is the distance between M_1, M_2. For the derivatives we obtain

$$\frac{dX_i}{dt^*} = l\cdot n\frac{du_i}{dt} = l\cdot n\dot{u}_i, \quad \frac{d^2X_i}{dt^{*2}} = l\cdot n^2\ddot{u}_i,$$

where the dot indicates the derivative with respect to the dimensionless time t.

Now we substitute the derivatives and dimensionless quantities into equation (2). Introducing the vector $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ and using Kepler's third law, we get the equation

$$\frac{d^2\mathbf{u}}{dt^2} = \frac{\partial\Phi}{\partial\mathbf{u}}, \quad \frac{\partial\Phi}{\partial\mathbf{u}} = \left(\frac{\partial\Phi}{\partial u_1}, \frac{\partial\Phi}{\partial u_2}\right)^T, \quad (3)$$

where $\Phi = \Phi(t, u_1, u_2) = \frac{\mu_1}{\rho_1} + \frac{\mu_2}{\rho_2}$ is the force function, μ_1, μ_2 are the ratios of masses:

$$\mu_1 = \frac{m_1}{m_1 + m_2} = \frac{d_2}{l}, \quad \mu_2 = \frac{m_2}{m_1 + m_2} = \frac{d_1}{l},$$

$$\rho_1 = \sqrt{(u_1 - \mu_2\cos t)^2 + (u_2 - \mu_2\sin t)^2},$$

$$\rho_2 = \sqrt{(u_1 + \mu_1\cos t)^2 + (u_2 + \mu_1\sin t)^2}$$

are the distances to attracting centers in the new variables.

2. CANONICAL EQUATIONS OF MOTION

For further transformations we write equation (3) in the canonical form

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathcal{H}}{\partial\mathbf{v}} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = -\frac{\partial\mathcal{H}}{\partial\mathbf{u}} = \frac{\partial\Phi}{\partial\mathbf{u}}, \quad (4)$$

where $\mathbf{v}^2/2 - \Phi$ is the Hamilton function.

We pass to the new coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2)^T$, $\boldsymbol{\eta} = (\eta_1, \eta_2)^T$, so that the new Hamilton function did not depend explicitly on time. We take the generating function of the form

$$W(t, \mathbf{v}, \xi) = -\mathbf{v}^T S\xi, \quad (5)$$

where S is the matrix of uniform rotation with unit angular velocity

$$S = S(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

According to the theory of canonical transformations, we have

$$\mathbf{u} = -\frac{\partial W}{\partial\mathbf{v}} = S\xi, \quad \boldsymbol{\eta} = -\frac{\partial W}{\partial\xi} = S^T\mathbf{v},$$

$$\mathcal{H} = \mathcal{K} + \frac{\partial W}{\partial t} = \mathcal{K} - \boldsymbol{\eta}^T I\xi,$$

where

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We express the Hamiltonian $\mathcal{K}$ in terms of new variables. We have $\mathbf{u}^2 = \xi^2$, $\mathbf{v}^2 = \boldsymbol{\eta}^2$,

$$\rho_1 = \sqrt{(\xi_1 - \mu_2)^2 + \xi_2^2} = |\xi - \xi_1|,$$

$$\rho_2 = \sqrt{(\xi_1 + \mu_2)^2 + \xi_2^2} = |\xi - \xi_2|,$$

where $\xi_1 = (\mu_2, 0)^T$, $\xi_2 = (-\mu_1, 0)^T$ are position vectors of points M_1 M_2, respectively, in a rotating coordinate system.

Now the explicit dependence on t in the force function disappears:

$$\Phi(u_1, u_2, t) = \frac{\mu_1}{|\xi - \xi_1|} + \frac{\mu_2}{|\xi - \xi_2|} = \tilde{\Phi}(\xi_1, \xi_2).$$

The new Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\eta}^2 - \tilde{\Phi}(\xi_1, \xi_2) - \boldsymbol{\eta}^T I\xi.$$

The equations of motion in the new variables $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ will be as follows:

$$\dot{\boldsymbol{\xi}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\eta}} = \boldsymbol{\eta} - I\boldsymbol{\xi}, \quad \dot{\boldsymbol{\eta}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\xi}} = \frac{\partial \tilde{\Phi}}{\partial \boldsymbol{\xi}} - I\boldsymbol{\eta} \tag{6}$$

or, in more details, in the coordinate form

$$\begin{gathered} \frac{d\xi_1}{dt} = \eta_1 + \xi_2, \quad \frac{d\xi_2}{dt} = \eta_2 - \xi_1, \\ \frac{d\eta_1}{dt} = -\frac{\mu_1(\xi_1 - \mu_2)}{\rho_1^3} - \frac{\mu_2(\xi_1 + \mu_1)}{\rho_2^3} + \eta_2, \\ \frac{d\eta_2}{dt} = -\frac{\mu_1\xi_2}{\rho_1^3} - \frac{\mu_2\xi_2}{\rho_2^3} - \eta_1. \end{gathered} \tag{7}$$

Since the Hamiltonian $\mathcal{H}$ does not depend explicitly on time t, then $\mathcal{H} = \text{const}$ is the first integral. This integral is identical to the Jacobi integral [4].

3. NEW VARIABLES

Denote by $\boldsymbol{\xi}^0$, $\boldsymbol{\eta}^0$ the initial values of variables $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ at time instant $t = 0$.

By analogy with paper [12], we consider the new variables, which are zeroed when point M collides with points M_1, M_2,

$$\mathbf{x}_i = \boldsymbol{\xi} - \boldsymbol{\xi}_i, \quad \mathbf{x}_i = (x_{i1}, x_{i2})^{\mathrm{T}}, \quad i = 1, 2. \tag{8}$$

Since we conserve the canonical form of the equations of motion, variables $\mathbf{x}_i$ should comply with the same number of conjugated momenta $\mathbf{y}_i$ we will introduce by the formula

$$\boldsymbol{\eta} = \mathbf{y}_1 + \mathbf{y}_2, \mathbf{y}_i = (y_{i1}, y_{i2})^{\mathrm{T}}, i = 1, 2. \tag{9}$$

With this approach, the number of degrees of freedom of the problem under consideration increases. Therefore, it is necessary to check the solutions of the new canonical system transfer, by means of transformations (8), (9), into the solutions of the original system (6).

We form the new Hamiltonian

$$\begin{gathered} H = H(\mathbf{x}, \mathbf{y}) = \mathcal{H}(\boldsymbol{\xi}(\mathbf{x}), \boldsymbol{\eta}(\mathbf{y})) \\ = \sum_{i=1}^{2} \frac{|\mathbf{y}_i|^2}{2} + \mathbf{y}_1^{\mathrm{T}}\mathbf{y}_2 - \sum_{i=1}^{2} \frac{\mu_i}{|\mathbf{x}_i|} - (\mathbf{y}_1 + \mathbf{y}_2)^T I(\mathbf{x}_1 + \boldsymbol{\xi}_1), \end{gathered}$$

where the arguments $\mathbf{x}$, $\mathbf{y}$ of the Hamiltonian H represent the four-dimensional vectors

$$\mathbf{x} = (x_1, \ldots, x_4)^{\mathrm{T}} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}, \quad \mathbf{y} = (y_1, \ldots y_4)^{\mathrm{T}} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}.$$

The order of the corresponding canonical system

$$\frac{d\mathbf{x}_i}{dt} = \frac{\partial H}{\partial \mathbf{y}_i}, \quad \frac{d\mathbf{y}_i}{dt} = -\frac{\partial H}{\partial \mathbf{x}_i}, \quad i = 1, 2 \tag{10}$$

is equal to eight.

In an expanded form, the equations of system (10) are as follows:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{y}_1 + \mathbf{y}_2 - I(\mathbf{x}_1 + \boldsymbol{\xi}_1), \quad i = 1, 2, \tag{11}$$

$$\begin{gathered} \frac{d\mathbf{y}_1}{dt} = -\frac{\mu_1}{|\mathbf{x}_1|^3}\mathbf{x}_1 - I(\mathbf{y}_1 + \mathbf{y}_2), \\ \frac{d\mathbf{y}_2}{dt} = -\frac{\mu_2}{|\mathbf{x}_2|^3}\mathbf{x}_2. \end{gathered} \tag{12}$$

For $\mathbf{x}_i$, $\mathbf{y}_i$ we will take the following initial values:

$$\mathbf{x}_i^0 = \boldsymbol{\xi}^0 - \boldsymbol{\xi}_i, \quad \mathbf{y}_i^0 = \frac{1}{2}\boldsymbol{\eta}^0, \quad i = 1, 2. \tag{13}$$

Then, obviously, $\mathbf{x}_i^0$, $\mathbf{y}_i^0$ are transferred, by means of (8), (9), into $\boldsymbol{\xi}^0$, $\boldsymbol{\eta}^0$.

With regard to the accepted initial conditions (13), we find from (11) the first integral of system (10) in the vector form: $\mathbf{x}_1 + \boldsymbol{\xi}_1 = \mathbf{x}_2 + \boldsymbol{\xi}_2$. These quantities represent two scalar integrals.

Now we will show that the solutions of system (6) can be obtained from the solutions of system (10), or, what is the same, from (11) and (12). Differentiating (8), we find

$$\dot{\boldsymbol{\xi}} = \dot{\mathbf{x}}_i + \dot{\boldsymbol{\xi}}_i \overset{(11)}{=} \mathbf{y}_1 + \mathbf{y}_2 - I(\mathbf{x}_1 + \boldsymbol{\xi}_1) \overset{(8.9)}{=} \boldsymbol{\eta} - I\boldsymbol{\xi} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\eta}}.$$

Similarly, from (9) we have

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \dot{\mathbf{y}}_1 + \dot{\mathbf{y}}_2 \overset{(12)}{=} -\sum_{i=1}^{2} \frac{\mu_i}{|\mathbf{x}_i|^3}\mathbf{x}_i - I(\mathbf{y}_1 + \mathbf{y}_2)^{(8.9)} \\ &= -\sum_{i=1}^{2} \frac{\mu_i}{|\boldsymbol{\xi} - \boldsymbol{\xi}_i|^3}(\boldsymbol{\xi} - \boldsymbol{\xi}_i) - I\boldsymbol{\eta} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\xi}}. \end{aligned}$$

Which was to be shown.

4. REGULARIZATION PROCEDURE

The right-hand part of equations (12) contains singularities generated by attracting points M_1, M_2. To eliminate these singularities, we will perform regularization. At first, we apply the uniform formalism and introduce the new time. Then we will make use of L-transformations [10].

In accordance with this sequence of operations, we will first consider the system

$$\frac{dx_0}{dt} = \frac{\partial H_a}{\partial y_0}, \ \frac{dy_0}{dt} = -\frac{\partial H_a}{\partial x_0}, \quad \frac{d\mathbf{x}_i}{dt} = \frac{\partial H_a}{\partial \mathbf{y}_i}, \ \frac{d\mathbf{y}_i}{dt} = -\frac{\partial H_a}{\partial \mathbf{x}_i}, \ i = 1, 2, \tag{14}$$

where $H_a = H(\mathbf{x}, \mathbf{y}) + y_0$. Because

$$\frac{dH_a}{dt} = \frac{\partial H_a}{\partial x_0}\dot{x}_0 + \frac{\partial H_a}{\partial y_0}\dot{y}_0 + \sum_{i=1}^{2}\left(\dot{\mathbf{x}}_1\frac{\partial H_a}{\partial \mathbf{x}_i} + \dot{\mathbf{y}}_i^{\mathrm{T}}\frac{\partial H_a}{\partial \mathbf{y}_i}\right) \overset{(14)}{=} 0,$$

we obtain that H_a is the integral of system (14). When choosing, at the initial time instant, for x_0, y_0 the values $x_0(0) = 0$, $y_0(0) = -H(\mathbf{x}(0), \mathbf{y}(0))$, the variable x_0 coincides with t, and the integral H_a assumes a zero value, regardless of the initial values for the other variables $\mathbf{x}$, $\mathbf{y}$. In addition, the last four equations of system (14) coincide with the equations of system (10).

Now we consider the time transformation $dt = \nu d\tau$ and the system

$$\frac{dx_0}{d\tau} = \frac{\partial H_b}{\partial y_0}, \ \frac{dy_0}{d\tau} = -\frac{\partial H_b}{\partial x_0}, \quad \frac{dx_i}{d\tau} = \frac{\partial H_b}{\partial \mathbf{y}_i}, \ \frac{d\mathbf{y}_i}{d\tau} = -\frac{\partial H_b}{\partial \mathbf{x}_i}, \ i = 1, 2, \tag{15}$$

where $\nu = \nu(\mathbf{x})$, $H_b = H_b(\mathbf{x}, y_0, \mathbf{y}) = \nu H_a = \nu(H + y_0)$. The initial values for variables x_0, $\mathbf{x}$, y_0, $\mathbf{y}$ of systems (14) and (15) are assumed to be identical

$$\mathbf{x}_i(0) = \mathbf{x}_i^0, \ \mathbf{y}_i(0) = \mathbf{y}_i^0, \ x_0(0) = 0, \quad y_0(0) = -H(\mathbf{x}^0, \mathbf{y}^0), \tag{16}$$

where $i = 1, 2$. Then the solutions of system (14) are obtained from the corresponding solutions of system (15). Now we introduce the vectors $\mathbf{q}_i = (q_{i1}, q_{i2})^T$, $i = 1, 2$, and consider the coordinate transformation

$$x_0 = q_0, \mathbf{x}_1 = L_1(\mathbf{q}_1)\mathbf{q}_1, \mathbf{x}_2 = L_2(\mathbf{q}_2)\mathbf{q}_2, \tag{17}$$

defined by two L-matrices of the second order

$$L_1(\mathbf{q}_1) = \begin{pmatrix} \mathbf{q}_1^T A_1 \\ \mathbf{q}_1^T A_2 \end{pmatrix}, \quad L_2(\mathbf{q}_2) = \begin{pmatrix} \mathbf{q}_2^T B_1 \\ \mathbf{q}_2^T B_2 \end{pmatrix}.$$

Here $\{A_1, A_2\}$, $\{B_1, B_2\}$ are two sets of generating matrices L_1 and L_2, respectively. Matrices A_1, A_2, B_1, B_2 are symmetric and orthogonal. These matrices are related by relationships $A_1A_2 + A_2A_1 = 0$, $B_1B_2 + B_2B_1 = 0$. From where it obviously follows that A_1, A_2, B_1, B_2 differ from a unit matrix. As shown in [10], if the matrix A_1 is specified, then the matrix A_2 is calculated by the formula

$$A_2 = \pm I A_1, \ I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A similar formula is obtained for the second set of generating matrices $\{B_1, B_2\}$. We note two more properties of L-matrices of the second order

$$L^T(\mathbf{q})L(\mathbf{q}) = |\mathbf{q}|^2 E, \ L(\mathbf{q})\mathbf{p} = L(\mathbf{p})\mathbf{q}. \tag{18}$$

We supplement the transformation (17) to a canonical one, generated by the generating function

$$\begin{aligned} W &= -y_0q_0 - \mathbf{y}_1^T L_1(\mathbf{q}_1)\mathbf{q}_1 - \mathbf{y}_2^T L_2(\mathbf{q}_2)\mathbf{q}_2 \\ &= -y_0q_0 - y_{11}\mathbf{q}_1^T A_1\mathbf{q}_1 - y_{12}\mathbf{q}_1^T A_2\mathbf{q}_1 \\ &\quad - y_{21}\mathbf{q}_2^T B_1\mathbf{q}_2 - y_{22}\mathbf{q}_2^T B_2\mathbf{q}_2. \end{aligned}$$

We find new momenta by the formulas

$$p_0 = -\frac{\partial W}{\partial q_0}, \ p_{ij} = \frac{\partial W}{\partial q_{ij}}, \ i, j = 1, 2.$$

We write these equations in vector form. Let $\mathbf{p}_i = (p_{i1}, p_{i2})^T$, $i = 1, 2$ are the conjugated momenta corresponding to vectors $\mathbf{q}_1$, $\mathbf{q}_2$. We have

$$\begin{aligned} \mathbf{p}_1 &= -\frac{\partial W}{\partial \mathbf{q}_1} = 2y_{11}A_1\mathbf{q}_1 + 2y_{12}A_2\mathbf{q}_1 \\ &= 2(A_1\mathbf{q}_1, A_2\mathbf{q}_1)\mathbf{y}_1 = 2\begin{pmatrix} \mathbf{q}_1^T A_1 \\ \mathbf{q}_1^T A_2 \end{pmatrix}^T \mathbf{y}_1. \end{aligned}$$

That is,

$$\mathbf{p}_1 = 2L_1^T(\mathbf{q}_1)\mathbf{y}_1. \tag{19}$$

We multiply (19) on the left side by $L_1(\mathbf{q}_1)$. Applying the first relation of (18), we find $\mathbf{y}_1 = \frac{1}{2|\mathbf{q}_1|^2}L_1(\mathbf{q}_1)\mathbf{p}_1$. In the same manner from $\mathbf{p}_2 = -\frac{\partial W}{\partial \mathbf{q}_2} = 2L_2^T(\mathbf{q}_2)\mathbf{y}_2$ we get $\mathbf{y}_2 = \frac{1}{2|\mathbf{q}_2|^2}L_2(\mathbf{q}_2)\mathbf{p}_2$. Thus, the transformation

$$x_0 = q_0, y_0 = p_0, \quad \mathbf{x}_i = L_i(\mathbf{q}_i)\mathbf{q}_i, \ \mathbf{y}_i = \frac{1}{2|\mathbf{q}_i|^2}L_i(\mathbf{q}_i)\mathbf{p}_i, \ i = 1, 2 \tag{20}$$

is canonical.

Since the generating function of this transformation does not depend explicitly on time τ, the new Hamiltonian H_c is obtained by substitution of variables (20) into the Hamiltonian H_b. We find the expression for the new

Hamiltonian. From the properties of the L-matrices we obtain the relationships

$$|\mathbf{x}_i|^2 = \mathbf{x}_i^T\mathbf{x}_i = \mathbf{q}_i^T L_i^T(\mathbf{q}_i)L_i(\boldsymbol{q}_i)\mathbf{q}_i = |q_i|^4 \Rightarrow |\mathbf{x}_i| = |\mathbf{q}_i|^2.$$

$$|\mathbf{y}_i|^2 = \mathbf{y}_i^T\mathbf{y}_i = \frac{1}{4|\mathbf{q}_i|^4}\mathbf{p}_i^T L_i^T(\mathbf{q}_i)L_i(\mathbf{q}_i)\mathbf{p}_i = \frac{|\mathbf{p}_i|^2}{4|\mathbf{q}_i|^2},\ i = 1, 2.$$

$$(\mathbf{y}_1+\mathbf{y}_2)^T I(\mathbf{x}_1+\boldsymbol{\xi}_1) = (x_{11}+\mu_2)(y_{12}+y_{22}) - x_{12}(y_{11}+y_{21}) = \frac{\mathbf{q}_1^T A_1\mathbf{q}_1+\mu_2}{2|\mathbf{q}_1|^2|\mathbf{q}_2|^2} \times (|\mathbf{q}_2|^2\mathbf{q}_1^T A_2\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_2\mathbf{p}_2) - \frac{\mathbf{q}_1^T A_2\mathbf{q}_1}{2|\mathbf{q}_1|^2|\mathbf{q}_2|^2}(|\mathbf{q}_2|^2\mathbf{q}_1^T A_1\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_1\mathbf{p}_2).$$

Therefore,

$$H_c = H_c(\mathbf{q}, p_0, \mathbf{p}) = H_b(\mathbf{x}(\mathbf{q}), p_0, \mathbf{y}(\mathbf{q}, \mathbf{p})) = \nu\Bigg(\sum_{i=1}^{2}\frac{|\mathbf{p}_i|^2}{8|\mathbf{q}_i|^2} + \frac{1}{4|\mathbf{q}_1|^2|\mathbf{q}_2|^2}\mathbf{p}_1^T L_1^T(\mathbf{q}_1)L_2(\mathbf{q}_2)\mathbf{p}_2 - \sum_{i=1}^{2}\frac{\mu_i}{|\mathbf{q}_i|^2} - \frac{1}{2|\mathbf{q}_1|^2|\mathbf{q}_2|^2}((\mathbf{q}_1^T A_1\mathbf{q}_1+\mu_2)(|\mathbf{q}_2|^2\mathbf{q}_1^T A_2\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_2\mathbf{p}_2) - \mathbf{q}_1^T A_2\mathbf{q}_1 \times (|\mathbf{q}_2|^2\mathbf{q}_1^T A_1\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_1\mathbf{p}_2)) + p_0\Bigg),$$

where the arguments $\mathbf{q}$, $\mathbf{p}$ of the Hamiltonian H_c represent four-dimensional vectors $\mathbf{q} = (\mathbf{q}_1^T, \mathbf{q}_2^T)^T$, $\mathbf{p} = (\mathbf{p}_1^T, \mathbf{p}_2^T)^T$.

To eliminate in the equations of motion the singularities generated by point M collisions with attracting points M_1, M_2, we take for function ν of time transformation the expression

$$\nu = |\mathbf{x}_1||\mathbf{x}_2| = |\mathbf{q}_1|^2|\mathbf{q}_2|^2.$$

Then

$$H_c = \frac{1}{8}(|\mathbf{p}_1|^2|\mathbf{q}_2|^2 + |\mathbf{p}_2|^2|\mathbf{q}_1|^2) + \frac{1}{4}\mathbf{p}_1^T L_1^T(\mathbf{q}_1)L_2(\mathbf{q}_2)\mathbf{p}_2 - \mu_1|\mathbf{q}_2|^2 - \mu_2|\mathbf{q}_1|^2 - \frac{1}{2}(\mathbf{q}_1^T A_1\mathbf{q}_1+\mu_2) \times (|\mathbf{q}_2|^2\mathbf{q}_1^T A_2\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_2\mathbf{p}_2) + \frac{1}{2}\mathbf{q}_1^T A_2 q_1 \times (|\mathbf{q}_2|^2\mathbf{q}_1^T A_1\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_1\mathbf{p}_2) + p_0|\mathbf{q}_1|^2|\mathbf{q}_2|^2.$$

In terms of the new variables the equations of motion take the form

$$\frac{dq_0}{d\tau} = \frac{\partial H_c}{\partial p_0},\ \frac{dp_0}{d\tau} = -\frac{\partial H_c}{\partial q_0}, \quad \frac{d\mathbf{q}_i}{d\tau} = \frac{\partial H_c}{\partial \mathbf{p}_i},\ \frac{d\mathbf{p}_i}{d\tau} = -\frac{\partial H_c}{\partial \mathbf{q}_i},\ i = 1, 2. \tag{21}$$

Now we write in more details the equations of system (21). The first pair of equations of this system contains the transformation of time and the law of change of quantity p_0:

$$dt/d\tau = |\mathbf{q}_1|^2|\mathbf{q}_2|^2,\ \frac{dp_0}{d\tau} = 0.$$

It follows from the last equation that $p_0 = \text{const}$. Equations for $\mathbf{q}_i$, $\mathbf{p}_i$ take the form

$$\begin{aligned}
\frac{d\mathbf{q}_1}{d\tau} &= \frac{|\mathbf{q}_2|^2}{4}\mathbf{p}_1 + \frac{1}{4}L_1^T(\mathbf{q}_1)L_2(\mathbf{q}_2)\mathbf{p}_2 - \frac{1}{2}(\mathbf{q}_1^T A_1\mathbf{q}_1+\mu_2)|\mathbf{q}_2|^2 A_2\mathbf{q}_1 + \frac{1}{2}\mathbf{q}_1^T A_2\mathbf{q}_1|\mathbf{q}_2|^2 A_1\mathbf{q}_1,\\
\frac{d\mathbf{q}_2}{d\tau} &= \frac{|\mathbf{q}_1|^2}{4}\mathbf{p}_2 + \frac{1}{4}L_2^T(\mathbf{q}_2)L_1(\mathbf{q}_1)\mathbf{p}_1 - \frac{1}{2}(\mathbf{q}_1^T A_1\mathbf{q}_1+\mu_2)|\mathbf{q}_1|^2 B_2\mathbf{q}_2 + \frac{1}{2}\mathbf{q}_1^T A_2\mathbf{q}_1|\mathbf{q}_1|^2 B_1\mathbf{q}_2,\\
\frac{d\mathbf{p}_1}{d\tau} &= -\frac{|\mathbf{p}_2|^2}{4}\mathbf{q}_1 - \frac{1}{4}L_1^T(\mathbf{p}_1)L_2(\mathbf{q}_2)\mathbf{p}_2 + 2\mu_2\mathbf{q}_1 - 2p_0|\mathbf{q}_2|^2\mathbf{q}_1 + (|\mathbf{q}_2|^2\mathbf{q}_1^T A_2\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_2\mathbf{p}_2)A_1 q_1\\
&\quad + \frac{1}{2}(\mathbf{q}_1^T A_1\mathbf{q}_1+\mu_2)(|\mathbf{q}_2|^2 A_2\mathbf{p}_1 + 2\mathbf{q}_2^T B_2\mathbf{p}_2\mathbf{q}_1) - (|\mathbf{q}_2|^2\mathbf{q}_1^T A_1\mathbf{p}_1 + |\mathbf{q}_1|^2\mathbf{q}_2^T B_1\mathbf{p}_2)A_2\mathbf{q}_1\\
&\quad - \frac{1}{2}\mathbf{q}_1^T A_2\mathbf{q}_1(|\mathbf{q}_2|^2 A_1\mathbf{p}_1 + 2\mathbf{q}_2^T B_1\mathbf{p}_2\mathbf{q}_1),\\
\frac{d\mathbf{p}_2}{d\tau} &= -\frac{|\mathbf{p}_1|^2}{4}\mathbf{q}_2 - \frac{1}{4}L_2^T(\mathbf{p}_2)L_1(\mathbf{q}_1)\mathbf{p}_1 + 2\mu_1\mathbf{q}_2 - 2p_0|\mathbf{q}_1|^2\mathbf{q}_2 + \frac{1}{2}(\mathbf{q}_1^T A_1\mathbf{q}_1+\mu_2)\\
&\quad \times (2\mathbf{q}_1^T A_2\mathbf{p}_1\mathbf{q}_2 + |\mathbf{q}_1|^2 B_2\mathbf{p}_2) - \frac{1}{2}\mathbf{q}_1^T A_2\mathbf{q}_1 \times (2\mathbf{q}_1^T A_1\mathbf{p}_1\mathbf{q}_2 + |\mathbf{q}_1|^2 B_1\mathbf{p}_2).
\end{aligned} \tag{22}$$

These equations, obviously, do not contain singularities arising from the collisions of a passively gravitating point M with points M_1, M_2. The right-hand sides of these equations represent polynomials with respect to all variables; that is, these quantities represent infinitely differentiable and unlimited functions on the whole space of variables $\mathbf{q}_i$, $\mathbf{p}_i$. As continuous functions, they are limited in any closed limited set. Therefore, for each such set the theorem of existence and uniqueness of the solution of the Cauchy problem is valid [13]. Consequently, for any initial values one can construct converging Pikar's approximations, which can be extended or newly reconstructed at any point of any compact set. In this sense, the problem can be considered an integrated one. Unfortunately, these solutions do not provide much information for qualitative analysis.

Note that the right-hand part of the equation for $\mathbf{q}_1$ is linear with respect to variables $\mathbf{p}_1$, $\mathbf{p}_2$; for $\mathbf{q}_2$ it is linear with respect to $\mathbf{q}_2$, $\mathbf{p}_1$, $\mathbf{p}_2$; for $\mathbf{p}_1$ it is linear with respect to $\mathbf{p}_1$; and for $\mathbf{p}_2$ it is linear with respect to $\mathbf{q}_2$, $\mathbf{p}_2$.

Now we substitute into regular equations (22) the specific L-matrices. We introduce the notations and the ten-dimensional vector

$$z = (z_1, z_2, \ldots, z_{10})^T = (\mathbf{q}_1^T, \mathbf{q}_2^T, \mathbf{p}_1^T, \mathbf{p}_2^T, t, p_0)^T.$$

Suppose that L-matrices are applied that have a positive determinant. Then one can take as generating L-matrices the matrices of the form

$$A_1 = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ -\lambda_2 & -\lambda_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda_2 & \lambda_1 \\ \lambda_1 & -\lambda_2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \tilde{\lambda}_1 & -\tilde{\lambda}_2 \\ -\tilde{\lambda}_2 & -\tilde{\lambda}_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \tilde{\lambda}_2 & \tilde{\lambda}_1 \\ \tilde{\lambda}_1 & -\tilde{\lambda}_2 \end{pmatrix},$$

where parameters $\lambda_1, \lambda_2, \tilde{\lambda}_1, \tilde{\lambda}_2$ are related by relationships $\lambda_1^2 + \lambda_2^2 = 1$, $\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 = 1$. This means that, actually, equations (22) contain two arbitrary parameters.

The L-matrices applied in (22) take the form

$$L_1(\mathbf{q}_1) = \begin{pmatrix} \lambda_1 z_1 - \lambda_2 z_2 & -\lambda_1 z_1 - \lambda_1 z_2 \\ \lambda_2 z_1 + \lambda_1 z_2 & \lambda_1 z_1 - \lambda_2 z_2 \end{pmatrix},$$

$$L_2(\mathbf{q}_2) = \begin{pmatrix} \tilde{\lambda}_1 z_3 - \lambda_2 z_4 & (-\tilde{\lambda})_2 z_3 - \tilde{\lambda}_1 z_4 \\ \tilde{\lambda}_2 z_3 + \tilde{\lambda}_1 z_4 & \tilde{\lambda}_1 z_3 - \tilde{\lambda}_2 z_4 \end{pmatrix},$$

$$L_1(\mathbf{p}_1) = \begin{pmatrix} \lambda_1 z_5 - \lambda_2 z_6 & (-\lambda)_2 z_5 - \lambda_1 z_6 \\ \lambda_2 z_5 + \lambda_1 z_6 & \lambda_1 z_5 - \lambda_2 z_6 \end{pmatrix},$$

$$L_2(\mathbf{p}_2) = \begin{pmatrix} \tilde{\lambda}_1 z_7 - \tilde{\lambda}_2 z_8 & -\tilde{\lambda}_2 z_7 - \tilde{\lambda}_1 z_8 \\ \tilde{\lambda}_2 z_7 + \tilde{\lambda}_1 z_8 & \tilde{\lambda}_1 z_7 - \tilde{\lambda}_2 z_8 \end{pmatrix}.$$

After some simplifications, the system (22) can be written in the form

$$\begin{aligned}
\frac{dz_1}{d\tau} &= a_1(\lambda_2 z_7 - \lambda_1 z_8) + a_2(\lambda_1 z_7 + \lambda_2 z_8) + \frac{1}{4} r_2 z_5 \\
&\quad - \frac{1}{2}\mu_2 r_2(\lambda_1 z_2 + \lambda_2 z_1) + \frac{1}{2} r_1 r_2 z_2, \\
\frac{dz_2}{d\tau} &= a_1(\lambda_1 z_7 + \lambda_2 z_8) - a_2(\lambda_2 z_7 - \lambda_1 z_8) + \frac{1}{4} r_2 z_6 \\
&\quad - \frac{1}{2}\mu_2 r_2(\lambda_1 z_1 - \lambda_2 z_2) - \frac{1}{2} r_1 r_2 z_1, \\
\frac{dz_3}{d\tau} &= -\frac{1}{2}\mu_2 r_1 A + a_5\lambda_1 + a_6\lambda_2 + \frac{1}{4} r_1 z_7 \\
&\quad + a_3(\lambda_1 z_5 - \lambda_2 z_6) + a_4(\lambda_1 z_6 + \lambda_2 z_5), \\
\frac{dz_4}{d\tau} &= \frac{1}{2}\mu_2 r_1 B - a_6\lambda_1 + a_5\lambda_2 + \frac{1}{4} r_1 z_8 \\
&\quad - a_4(\lambda_1 z_5 - \lambda_2 z_6) + a_3(\lambda_1 z_6 + \lambda_2 z_5), \\
\frac{dz_5}{d\tau} &= z_1 a_9 + (a_{10}A + a_{11}B)\lambda_1 + (a_{10}B - a_{11}A)\lambda_2 \\
&\quad + \frac{1}{2}(2z_1^2 + r_1) r_2 z_6 - r_2 z_1 z_2 z_5 + a_8(\lambda_1 z_6 + \lambda_2 z_5) \\
&\quad + a_7(\lambda_1 z_5 - \lambda_2 z_6), \\
\frac{dz_6}{d\tau} &= z_2 a_9 + (a_{12}A + a_{13}B)\lambda_1 + (a_{12}B - a_{13}A) \\
&\quad \times \lambda_2 - \frac{1}{2}(2z_2^2 + r_1) r_2 z_5 + r_2 z_1 z_2 z_6 \\
&\quad + a_8(\lambda_1 z_5 - \lambda_2 z_6) - a_7(\lambda_1 z_6 + \lambda_2 z_5), \\
\frac{dz_7}{d\tau} &= a_{14} z_3 + \frac{1}{2}\mu_2 r_1(\tilde{\lambda}_2 z_7 + \tilde{\lambda}_1 z_8) + a_{17} z_3 \\
&\quad + a_{15}(\lambda_1 z_8 - \lambda_2 z_7) + a_{16}(\lambda_1 z_7 + \lambda_2 z_8), \\
\frac{dz_8}{d\tau} &= a_{14} z_4 + \frac{1}{2}\mu_2 r_1(\tilde{\lambda}_1 z_7 - \tilde{\lambda}_2 z_8) + a_{17} z_4 \\
&\quad + a_{15}(\lambda_1 z_7 + \lambda_2 z_8) - a_{16}(\lambda_1 z_8 - \lambda_2 z_7), \quad \frac{dz_9}{d\tau} = r_1 r_2,
\end{aligned} \tag{23}$$

where

$$r_1 = z_1^2 + z_2^2, \quad r_2 = z_3^2 + z_4^2,$$

$$p_1 = z_5^2 + z_6^2, \quad p_2 = z_7^2 + z_8^2,$$

$$A = \tilde{\lambda}_1 z_4 + \tilde{\lambda}_2 z_3, \quad B = \tilde{\lambda}_2 z_4 - \tilde{\lambda}_1 z_3, \quad C = z_1^2 - z_2^2,$$

$$a_1 = \frac{1}{4}(\tilde{\lambda}_1(z_1 z_4 - z_2 z_3) + \tilde{\lambda}_2(z_2 z_4 + z_1 z_3)),$$

$$a_2 = \frac{1}{4}(\tilde{\lambda}_1(z_2 z_4 + z_1 z_3) - \tilde{\lambda}_2(z_1 z_4 - z_2 z_3)),$$

$$a_3 = \frac{1}{4}(z_2A - z_1B),\ a_4 = \frac{1}{4}(z_2B + z_1A),$$

$$a_5 = \left(-\frac{1}{2}AC - z_1z_2B\right)r_1,\ a_6 = \left(-\frac{1}{2}BC + z_1z_2A\right)r_1,$$

$$a_7 = \frac{1}{4}(z_7B + z_8A),\ a_8 = \frac{1}{2}(2\mu_2r_2 - z_7A + z_8B),$$

$$a_9 = \mu_2(2 + z_7A - z_8B) - \frac{1}{4}p_2 - 2z_{10}r_2,$$

$$a_{10} = r_1z_2z_8 + 2z_1^2(z_2z_8 + z_1z_7),$$
$$a_{11} = r_1z_2z_7 + 2z_1^2(z_2z_7 - z_1z_8),$$
$$a_{12} = r_1z_1z_8 - 2z_2^2(z_2z_7 - z_1z_8),$$
$$a_{13} = r_1z_1z_7 + 2z_2^2(z_2z_8 + z_1z_7),$$
$$a_{14} = 2\mu_1 - \frac{1}{4}p_1 - 2z_{10}r_1,$$

$$a_{15} = \frac{1}{2}\tilde{\lambda}_1r_1C + z_1z_2\tilde{\lambda}_2r_1 - \frac{1}{4}z_5(\tilde{\lambda}_1z_2 - \tilde{\lambda}_2z_1) - \frac{1}{4}z_6(\tilde{\lambda}_1z_1 + \tilde{\lambda}_2z_2),$$

$$a_{16} = \frac{1}{2}\tilde{\lambda}_2r_1C - z_1z_2\tilde{\lambda}_1r_1 - \frac{1}{4}z_5(\tilde{\lambda}_1z_1 + \tilde{\lambda}_2z_2) + \frac{1}{4}z_6(\tilde{\lambda}_1z_2 - \tilde{\lambda}_2z_2),$$

$$a_{17} = \mu_2((z_5z_2 + z_6z_1)\lambda_1 + (z_5z_1 - z_6z_2)\lambda_2) + (z_1z_6 - z_5z_2)r_1.$$

The equation for the variable z_{10} is not written here, since it is constant. Thus, the order of this system is nine.

Now we will obtain formulas for the conversion of the second-order L-transformation. Our calculations involve two L-matrices. The conversion formulas for them are identical. So, for brevity, we will write the L-matrix and pairs of vectors $\mathbf{q}$, $\mathbf{p}$ and $\mathbf{x}$, $\mathbf{y}$, corresponding to this matrix, without index. Instead of parameters λ_1, λ_2 we introduce the angular parameter $\psi \in [0, 2\pi)$: $\lambda_1 = \cos\psi$, $\lambda_2 = \sin\psi$. Then the matrix $L(\mathbf{q})$ can be represented in the form

$$L(\mathbf{q}) = S(\psi)L_+^0(\mathbf{q}),\ L_+^0(\mathbf{q}) = \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}, \qquad (24)$$

where $S(\psi)$ is the matrix of turning counterclockwise by angle ψ

$$S(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}.$$

In the complex form the transformation $\mathbf{x} = L(\mathbf{q})\mathbf{q}$ has the form

$$x_1 + ix_2 = (\lambda_1 + i\lambda_2)(q_1 + iq_2)^2,\ i = \sqrt{-1}.$$

We will write it in the trigonometric form

$$|\mathbf{x}|(\cos\varphi + i\sin\varphi) = |\mathbf{q}|^2(\cos(2\chi + \psi) + i\sin(2\chi + \psi)),$$

where $\tan\varphi = x^2/x^1$, $\tan\chi = q_2/q_1$, $\tan\psi = \lambda_2/\lambda_1$. Therefore,

$$|\mathbf{q}|^2 = |\mathbf{x}|,\ \chi = \frac{\varphi - \psi + 2\pi k}{2},\ k = 0, 1.$$

Thus, if x_1, x_2 are known initial values, then the initial values of regular coordinates q_1, q_2 are calculated by the formulas (for $k = 0$)

$$q_1 = |\mathbf{q}|\cos\chi = \sqrt{|\mathbf{x}|}\cos\frac{\varphi - \psi}{2},$$

$$q_2 = |\mathbf{q}|\sin\chi = \sqrt{|\mathbf{x}|}\sin\frac{\varphi - \psi}{2}.$$

After finding the vector $\mathbf{q} = (q_1, q_2)^T$ we calculate the initial value of the regular momentum vector $\mathbf{p}$ by the formula: $\mathbf{p} = 2L^T(\mathbf{q})\mathbf{y}$, where $\mathbf{y}$ is the momentum vector conjugated to $\mathbf{x}$.

Note that in the system (23) (or (22)) one can introduce, instead of parameters λ_1, λ_2, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, two angular parameters ψ, $\tilde{\psi}$. Specification of different pairs ψ, $\tilde{\psi}$ implies the transition from regular variables $\mathbf{q}$, $\mathbf{p}$ to the other variables $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}$, which is performed by means of two orthogonal transformations (for matrix L_1 and for L_2). In so doing, the form of equations of systems (22), (23) does not change. This implies the invariance of these equations with respect to orthogonal transformations of regular coordinates.

5. NUMERICAL STUDY

Systems (6) and (23), with appropriate choice of the initial values, describe the motion of the same problem. Unlike (6), system (23) does not contain singularities. Thus it has an advantage when considering the orbits undergoing close encounters with attracting centers.

We denote the ratio of masses m_1 to m_2 as k. Then we have

$$\frac{m_1}{m_2} = \frac{\mu_1}{\mu_2} = k,\quad \mu_2k + \mu_2 = 1 \Rightarrow \mu_2 = \frac{1}{k+1},$$

$$\mu_1 = \frac{k}{k+1}.$$

Table 1. Comparative analysis

n	t (days)	$\delta\mathcal{H}\cdot 10^{-14}$	$\delta\mathcal{H}_*\cdot 10^{-14}$	N_6	N_{23}
1	101.38	9.79	0.52	37358	18054
2	230.33	37.74	0.22	72397	36864
3	404.56	39.00	11.55	101112	56901

From Kepler's third law we find the angular velocity

$$n = \sqrt{\frac{\gamma m_1(1+k)}{l^3 k}},$$

which is used at transition to dimensional quantities. We will perform numerical experiments for the example of the Earth–Moon system. In this case, $k = 81.3$, $l = 384400$ km, $\gamma m_1 = 398601.3$ km^3/s^2. We apply the Runge–Kutta–Felberg method of the eighth order with automatic choice of the integration step. The step of integration is controlled by the method of the seventh order. The corresponding program is called the pair *RKF*87 [14, 15]. As the relative local error of the method we will take the quantity $\varepsilon = 10^{-14}$.

As the control relation we will use the relative value of the integral of system (6)

$$\delta\mathcal{H} = \frac{|\mathcal{H}(0) - \mathcal{H}|}{|\mathcal{H}|}, \qquad (25)$$

where $\mathcal{H}(0)$ is the value of the integral at the initial time instant, $\mathcal{H}$ is the similar value at the current time instant. In system (23) the integral of motion is given by the relation $z_{10} = \text{const}$. From the tenth equation, which is not written here, it follows that z_{10} is the "ideal" constant. Therefore, for equitable comparison based on the found values of variables z_i, we will find the values of variables ξ_i, η_i and then calculate the relative value of the integral by formula (25). The corresponding value is denoted by $\delta\mathcal{H}_*$. Now we consider the hypothetical satellite with the initial values

$\xi_1 = .0122 = 4689.68$ km, $\xi_2 = .0171 = 6573.24$ km,

$\dot{\xi}_1 = -10.6455 = -10.90683$ km/s, $\dot{\xi}_2 = .0 = .0$ km/s.

The distance to the center of the Earth at the initial time instant is equal to $\rho_1 = 6573.27$ km.

The results of numerical integration of systems (6) and (23) are given in Table 1. All computations were performed with double precision (real*8). The second column presents approximate values of the length of intervals of physical time t (in days), over which the numerical integration of systems was carried out. The other columns present, for the end of the integration interval, the values of relative errors for the integral constant $\delta\mathcal{H}$, $\delta\mathcal{H}_*$ and the number of addresses to the subroutine of calculation of the right-hand side for both systems: N_6 for system (6), and N_{23} for system (23).

The satellite under consideration undergoes multiple close encounters with attracting centers. Figure 2, implemented in the Maple system, depicts its orbit in the interval $t \approx 101.38$ days.

It is seen from presented data that, with increasing integration interval, relative errors and the number of addresses to the subroutine for calculation of system's right-hand side increase. The computational expenses (numbers N_6, N_{23}) are almost twice as much as the classical equations (6). In addition, the relative error of the integral of motion $\delta\mathcal{H}_*$ in the regular case is less than that of the irregular one.

It was noted above that system (23) contains two arbitrary parameters ψ, $\tilde{\psi}$. In particular, in calculations of Table 1 for matrices L_1, L_2 we applied the values $\psi = 0$, $\tilde{\psi} = \pi/2$, respectively. Of interest is the comparison of the results of integration of system (23) with different parameters ψ, $\tilde{\psi}$ for the same orbit.

We consider the satellite with initial values

$\xi_1 = .0090 = 3459.60$ km, $\xi_2 = .0169 = 6496.36$ km,

$\dot{\xi}_1 = -10.6180 = -10.87865$ km/s, $\dot{\xi}_2 = .0 = .0$ km/s.

The distance to the Earth center at the initial time instant is equal to $\rho_1 = 6608.29$ km. The orbit of this

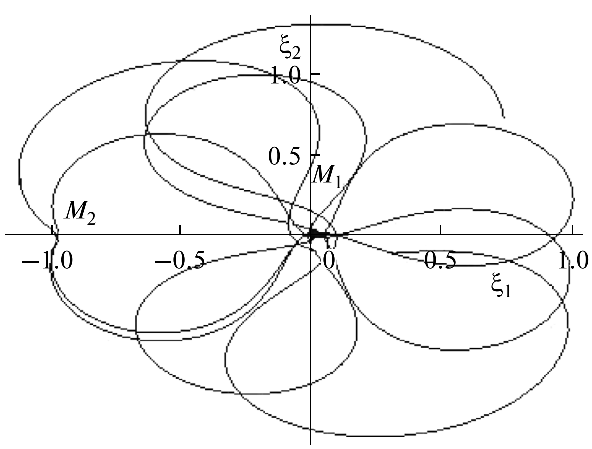

Fig. 2. Particle orbit on the interval $t \approx 101.38$ days.

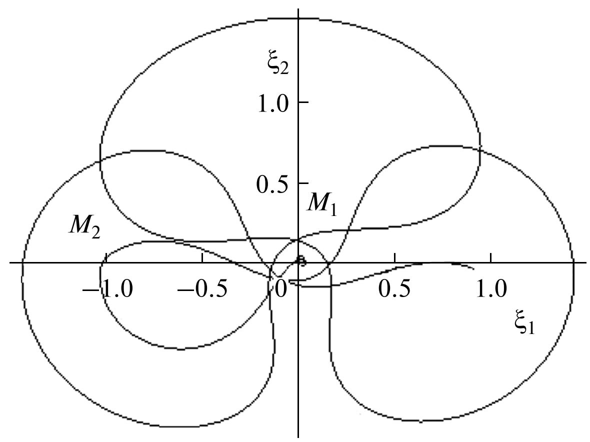

Fig. 3. Particle orbit on the interval $t \approx 70.25$ days.

Table 2. Integration with different parameters ψ, $\tilde{\psi}$

n	ψ	$\tilde{\psi}$	$\delta\mathcal{H}_* \cdot 10^{-14}$	N_{23}
1	0	0	1.15	14261
2	p/2	0	0.38	14569
3	0	p/2	0.08	14455
4	p/2	p/2	1.17	14312

satellite is presented in Fig. 3. Integration results (Table 2) show the nonequivalence of various systems of regular coordinates: the relative errors in the constant of the integral of motion and the volumes of calculations of the vector of accelerations are different for different parameters ψ, $\tilde{\psi}$.

CONCLUSION

The regular equations of the planar circular restricted three-body problem are obtained in the paper. In the regularization procedure we used two L-matrices of the second order. The equations contain two arbitrary parameters and are invariant with respect to orthogonal transformations of regular coordinates. Numerical calculations demonstrated the efficiency of the obtained equations for the orbits undergoing close encounters with the attracting centers.

REFERENCES

1. Subbotin, M.F., *Vvedenie v teoreticheskuyu astronomiyu* (Introduction into the Theoretical Astronomy), Moscow: Nauka, 1968.
2. Egorov, V.A., *Prostranstvennaya zadacha dostizheniya Luny* (Spatial Problem of Reaching the Moon), Moscow: Nauka, 1965.
3. Markeev, A.P., *Tochki libratsii v nebesnoi mekhanike i kosmodinamike* (Libration Points in the Celestial Mechanics and Cosmodynamics), Moscow: Nauka, 1978.
4. Szebehely, V., *Theory of Orbits in the Restricted Problem of Three Bodies*, New York: Academic Press, 1967.
5. Bruno, A.D., *Ogranichennaya zadacha trekh tel: Ploskie periodicheskie orbity* (Restricted Three-body Problem: Plane Periodic Orbits), Moscow: Nauka, 1990.
6. Demin, V.G., *Dvizhenie iskusstvennogo sputnika v netsentral'nom pole tyagoteniya* (The Motion of an Artificial Satellite in the Non-central Gravitational Field, Moscow: Nauka, 1968.
7. Heggie, D., A global regularization of the gravitational N-body problem, *Celestial Mech.*, 1974, vol. 10, pp. 217–241.
8. Aarseth, S. and Zare, K., A regularization of the Three-body problem, *Celestial Mech.*, 1974, vol. 10, pp. 185–205.
9. Mikkola, S. and Aarseth, S., A chain regularization method for the few-body problem, *Celestial Mech.*, 1990, vol. 47, pp. 375–390.
10. Poleshchikov, S.M. and Kholopov, A.A., *Teoriya L-matrits i regulyarizatsiya uravnenii dvizheniya v nebesnoi mekhanike* (Theory of L-matrices and Regularization of the Equations of Motion in the Celestial Mechanics), Syktyvkar: SLI, 1999.
11. Poleshchikov, S.M., Regularization of the equations of motion with L-transformation and numerical integration of the regular equations, *Celest. Mech. and Dyn. Astr.*, 2003, vol. 85, no. 4, pp. 341–393.
12. Broucke, R. and Lass, H., A note on relative motion in the general three-body problem, *Celestial Mech.*, 1973, vol. 8, pp. 5–10.
13. Stepanov, V.V., *Kurs differentsial'nykh uravnenii* (A Course of Differential Equations), Moscow: GITTL, 1953.
14. Poleshchikov, S.M., One integrable case of the perturbed two-body problem, *Cosmic Res.*, 2004, vol. 42, no. 4, pp. 398–407.
15. Poleshchikov, S.M., The motion of a particle in the perturbed field of an attracting center, *Cosmic Res.*, 2007, vol. 45, no. 6, pp. 493–505.

Translated by Yu. Preobrazhensky