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Satellite Motion in the Gravitational Field of a Viscoelastic Planet with a Core

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Abstract—The motion of a "planet–satellite" system in a gravitational field of mutual attraction forces is investigated. The planet is modeled by a body consisting of a solid core and a viscoelastic shell made of Kelvin–Voigt material. The satellite is modeled by a material point. A system of integro-differential equations of motion of a mechanical system is derived from the variational d'Alembert–Lagrange principle within the linear model of the theory of elasticity. Using the asymptotic method of separation of motions, an approximate system of equations of motion is constructed in vector form. This system describes the dynamics of a system with allowance for disturbances caused by elasticity and dissipation. The solution of the quasistatic problem of elasticity theory for the deformable shell of a planet is obtained in the explicit form. An averaged system of differential equations describing the evolution of satellite's orbital parameters is derived. For partial cases phase trajectories are constructed, stationary solutions are found, and their stability is investigated. As examples, some planets of the Solar system and their satellites are considered. This problem is a model for studying the tidal theory of planetary motion.

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The tidal evolution of the "planet–satellite" system has been investigated by many authors [1–3]. This work uses the methods of the analytical mechanics of systems with an infinite number of degrees of freedom [4]. This approach has been applied earlier, in particular, to a series of problems on the translational–rotational motion of a viscoelastic sphere [5–7].

1. STATEMENT OF THE PROBLEM. EQUATIONS OF MOTION

Consider the problem of the translational-rotational motion of a "planet–satellite system" in a gravitational field of mutual attraction forces. The satellite will be modeled by the material point P with mass μ. The planet will be modeled by a body consisting of a solid core and a viscoelastic shell, which occupies the region $V = V_0 \cup V_1$ in three-dimensional Euclidean space in the absence of deformations. Here $V_0 = \left\{\mathbf{r} \in E^3 : r_0 < |\mathbf{r}| \le r_1\right\}$. Let ρ_0, ρ_1 be densities of a core and viscoelastic shell, respectively, and m_0, m_1 be their masses. The material of planet's shell is assumed to be homogeneous and isotropic.

We introduce the inertial coordinate system $OXYZ$ with the origin at the center of masses of the "planet–satellite" system. To describe the rotational motion of a planet we introduce a movable coordinate system $Cx_1x_2x_3$, rigidly connected with the core and the Koenig system of axes $C\xi_1\xi_2\xi_3$, where C is the center of mass of a planet in its natural undeformed state (Fig. 1).

The position of point M of a planet in the inertial coordinate system is determined by the vector field

$$\mathbf{R}_M(\mathbf{r},t) = \mathbf{OC} + \Gamma(\mathbf{r} + \mathbf{u}(\mathbf{r},t)), \tag{1}$$

where Γ is the operator of transition from the movable coordinate system $Cx_1x_2x_3$ to the Koenig system of axes $C\xi_1\xi_2\xi_3$; $\mathbf{u}(\mathbf{r}, t)$ is the elastic displacement vector, which is zero for the points of the solid core V_0. Since O is the center of mass of the considered mechanical system, then

$$\int_V \mathbf{R}_M(\mathbf{r}, t)\rho dv + \mu \cdot \mathbf{OP} = 0. \tag{2}$$

where $\rho = \begin{cases} \rho_0, & \text{если} \quad \mathbf{r} \in V_0, \\ \rho_1, & \text{если} \quad \mathbf{r} \in V_1 \end{cases}.$

Here we introduce into consideration vector $\mathbf{R} = \mathbf{CP}$. Then from (1) and (2) we obtain

$$\begin{aligned} \mathbf{OP} &= \frac{m}{m+\mu}\mathbf{R} - \frac{1}{m+\mu}\int_{V_1}\Gamma\mathbf{u}\rho_1 dv_1, \\ \mathbf{OC} &= -\frac{\mu}{m+\mu}\mathbf{R} - \frac{1}{m+\mu}\int_{V_1}\Gamma\mathbf{u}\rho_1 dv_1. \end{aligned} \tag{3}$$

Here m is the mass of a planet, $m = m_0 + m_1$.

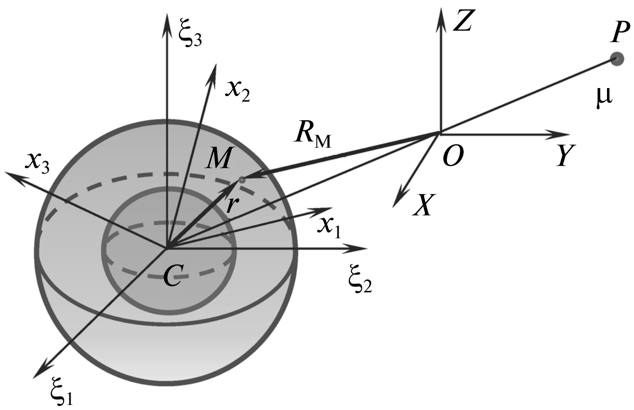

Fig. 1

The potential energy of the gravitational field is determined by the functional

$$\Pi = -f\mu\int_V \frac{\rho dv}{\left|-\mathbf{R} + \Gamma(\mathbf{r}+\mathbf{u})\right|}, \tag{4}$$

where f is the universal gravitational constant.

We will specify the functional of the potential energy of elastic deformation in accordance with the linear model of the elasticity theory:

$$\mathscr{E} = \int_{V_1} \mathscr{E}[\mathbf{u}]dv_1, \quad \mathscr{E}[\mathbf{u}] = \alpha_1\left(I_E^2 - \alpha_2 II_E\right),$$

$$\alpha_1 = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}, \quad \alpha_2 = \frac{2(1-2\nu)}{1-\nu},$$

$$\alpha_1 > 0, \quad 0 < \alpha_2 < 3, \tag{5}$$

$$I_E = \sum_{j=1}^{3} e_{jj}, \quad II_E = \sum_{k<l}\left(e_{kk}e_{ll} - e_{kl}^2\right),$$

$$e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \quad \mathbf{u} = (u_1, u_2, u_3),$$

where E is the Young modulus of elasticity, ν is the Poisson coefficient of a viscoelastic shell of the planet, and I_E and II_E are invariants of the tensor of small deformations.

The dissipative properties of a viscoelastic shell will be described by the dissipative functional $\mathscr{D} = \int_{V_1} \mathscr{D}[\dot{\mathbf{u}}]dv_1$, $\mathscr{D}[\dot{\mathbf{u}}] = \chi\mathscr{E}[\dot{\mathbf{u}}]$, which corresponds to the Kelvin–Voigt model (here $\chi > 0$ is the coefficient of internal viscous friction).

We let $\mathbf{R}_P = \mathbf{OP}$. The equations of motion of the "planet–satellite" system are obtained from the variational d'Alembert–Lagrange principle:

$$\int_V \left(\ddot{\mathbf{R}}_M, \delta\mathbf{R}_M\right)\rho dv + \mu\left(\ddot{\mathbf{R}}_P, \delta\mathbf{R}_P\right) + \delta\Pi + \int_{V_1}\left(\nabla_{\mathbf{u}}\mathscr{E}[\mathbf{u}] + \nabla_{\dot{\mathbf{u}}}\mathscr{D}[\dot{\mathbf{u}}], \delta\mathbf{u}\right)dv_1 = 0. \tag{6}$$

According to equalities (1) and (3), we have:

$$\begin{aligned}
\ddot{\mathbf{R}}_M &= -\frac{\mu}{m+\mu}\ddot{\mathbf{R}} - \frac{1}{m+\mu}\Gamma\int_{V_1}\left[\boldsymbol{\omega}\times[\boldsymbol{\omega}\times\mathbf{u}] + 2\boldsymbol{\omega}\times\dot{\mathbf{u}}\right.\\
&\quad \left. + \dot{\boldsymbol{\omega}}\times\mathbf{u} + \ddot{\mathbf{u}}\right]\rho_1 dv_1 + \Gamma\left[\boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u})] + 2\boldsymbol{\omega}\times\dot{\mathbf{u}}\right.\\
&\quad \left. + \dot{\boldsymbol{\omega}}\times(\mathbf{r}+\mathbf{u}) + \ddot{\mathbf{u}}\right], \quad \delta\mathbf{R}_M = -\frac{\mu}{m+\mu}\delta\mathbf{R}\\
&\quad - \frac{1}{m+\mu}\Gamma\int_{V_1}[\delta\boldsymbol{\alpha}\times\mathbf{u} + \delta\mathbf{u}]\rho_1 dv_1 + \Gamma[\delta\boldsymbol{\alpha}\times(\mathbf{r}+\mathbf{u}) + \delta\mathbf{u}],\\
\ddot{\mathbf{R}}_P &= \frac{m}{m+\mu}\ddot{\mathbf{R}} - \frac{1}{m+\mu}\Gamma\int_{V_1}\left[\boldsymbol{\omega}\times[\boldsymbol{\omega}\times\mathbf{u}] + 2\boldsymbol{\omega}\times\dot{\mathbf{u}}\right.\\
&\quad \left. + \dot{\boldsymbol{\omega}}\times\mathbf{u} + \ddot{\mathbf{u}}\right]\rho_1 dv_1,\\
\delta\mathbf{R}_P &= \frac{m}{m+\mu}\delta\mathbf{R} - \frac{1}{m+\mu}\Gamma\int_{V_1}[\delta\boldsymbol{\alpha}\times\mathbf{u} + \delta\mathbf{u}]\rho_1 dv_1.
\end{aligned} \tag{7}$$

Here $\boldsymbol{\omega}$ is the angular velocity vector of the planet, and $\delta\boldsymbol{\alpha}$ is the vector arising in varying the orthogonal operator Γ:

$$\boldsymbol{\omega}\times(\cdot) = \Gamma^{-1}\dot{\Gamma}(\cdot), \quad \delta\Gamma(\cdot) = \Gamma[\delta\boldsymbol{\alpha}\times(\cdot)].$$

Substituting expressions (7) for $\ddot{\mathbf{R}}_M$, $\delta\mathbf{R}_M$, $\ddot{\mathbf{R}}_P$, $\delta\mathbf{R}_P$ into equality (6) and equating the coefficients for independent variations $\delta\mathbf{R}$, $\delta\boldsymbol{\alpha}$, $\delta\mathbf{u}$, we obtain the equations of motion of the "planet–satellite" system in the form

$$-\frac{\mu}{m+\mu}\int_V \ddot{\mathbf{R}}_M\rho dv + \frac{\mu m}{m+\mu}\ddot{\mathbf{R}}_P + f\mu\int_V \frac{\mathbf{R} - \Gamma(\mathbf{r}+\mathbf{u})}{\left|\mathbf{R} - \Gamma(\mathbf{r}+\mathbf{u})\right|^3}\rho dv = 0, \tag{8}$$

$$\begin{aligned}
&\int_V \left[\mathbf{r} + \mathbf{u} - \frac{1}{m+\mu}\int_{V_1}\mathbf{u}\rho_1 dv_1\right]\times\Gamma^{-1}\ddot{\mathbf{R}}_M\rho dv\\
&- \frac{\mu}{m+\mu}\int_{V_1}\mathbf{u}\rho_1 dv_1 \times \Gamma^{-1}\ddot{\mathbf{R}}_P\\
&- f\mu\int_V \frac{(\mathbf{r}+\mathbf{u})\times\left[\Gamma^{-1}\mathbf{R} - (\mathbf{r}+\mathbf{u})\right]}{\left|\mathbf{R} - \Gamma(\mathbf{r}+\mathbf{u})\right|^3}\rho dv = 0,
\end{aligned} \tag{9}$$

$$\rho_1\left\{\Gamma^{-1}\ddot{\mathbf{R}}_M - \frac{1}{m+\mu}\int_V \Gamma^{-1}\ddot{\mathbf{R}}_M \rho dv - \frac{\mu}{m+\mu}\Gamma^{-1}\ddot{\mathbf{R}}_P - \frac{f\mu\left(\Gamma^{-1}\mathbf{R} - (\mathbf{r}+\mathbf{u})\right)}{\left|\mathbf{R} - \Gamma(\mathbf{r}+\mathbf{u})\right|^3}\right\} + \nabla_{\mathbf{u}}\mathcal{E}[\mathbf{u} + \chi\dot{\mathbf{u}}] = 0. \quad (10)$$

2. DISTURBED SYSTEM OF THE EQUATIONS OF MOTION. DEFORMATIONS OF THE VISCOELASTIC SHELL OF THE PLANET

We assume that the rigidity of the viscoelastic shell of the planet is great, i.e., the dimensionless parameter $\tilde{\varepsilon} = \rho\omega_0^2 r_1^2 E^{-1}$ is small, where ω_0 is the value of the magnitude of the initial angular velocity of a planet. Having chosen the scales of dimensional units in the appropriate manner, we can introduce the small parameter $\varepsilon = E^{-1}$. For $\varepsilon = 0$, the elastic displacement vector $\mathbf{u}$ is assumed to be zero. In this case we will obtain the problem of the motion of a mechanical system consisting of an absolutely rigid body of spherical shape and of a material point in the field of forces of mutual attraction. The undisturbed system of the equations of motion has the form:

$$\ddot{\mathbf{R}} + \frac{f(m+\mu)}{R^3}\mathbf{R} = \mathbf{0}, \quad A\dot{\boldsymbol{\omega}} = \mathbf{0},$$
$$A = \frac{8\pi}{15}\left[\rho_0 r_0^5 + \rho_1\left(r_1^5 - r_0^5\right)\right], \quad (11)$$

where A is the moment of inertia of a planet in an undeformed state with respect to the diameter.

For $\varepsilon \neq 0$, according to the method of separation of motions [4], we determine from equation (10) the viscoelastic shell deformations caused by the field of external forces and forces of inertia of the translational motion. We will search for the solution of equation (10) in the form of expansion in powers of the small parameter ε: $\mathbf{u} = \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots$. The boundary-value problem for determining the function $\mathbf{u}_1$ of the first approximation has the form [8]:

$$\varepsilon\nabla_{\mathbf{u}}\mathcal{E}[\mathbf{u}_1 + \chi\dot{\mathbf{u}}_1] = \frac{2}{3}\rho_1\omega^2\mathbf{r} + \nabla_{\mathbf{r}}U + \nabla_{\mathbf{r}}V, \quad (12)$$

$$\mathbf{u}_1\big|_{r=r_0} = \mathbf{0}, \quad \boldsymbol{\sigma}_n\big|_{r=r_1} = \mathbf{0}. \quad (13)$$

Here

$$\varepsilon\nabla_{\mathbf{u}}\mathcal{E}[\mathbf{u}_1] = -\frac{1}{2(1+\nu)}\left[\frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u}_1 + \Delta\mathbf{u}_1\right],$$

$$\boldsymbol{\sigma}_n = (\sigma_1, \sigma_2, \sigma_3), \quad \mathbf{n} = (\gamma_1, \gamma_2, \gamma_3),$$

$$\sigma_i = \frac{E\nu\gamma_i}{(1+\nu)(1-2\nu)}\operatorname{div}\mathbf{u}_1 + \frac{E}{2(1+\nu)} \times \left(\frac{\partial\mathbf{u}_1}{\partial x_i} + \operatorname{grad}u_i, \mathbf{n}\right), \quad i = 1, 2, 3,$$

$$U = \rho_1\left\{\frac{1}{6}\omega^2 r^2 - \frac{1}{2}(\boldsymbol{\omega}, \mathbf{r})^2\right\},$$
$$V = -\frac{3f\mu\rho_1}{R^3}\left\{\frac{1}{6}r^2 - \frac{1}{2}(\boldsymbol{\xi}, \mathbf{r})^2\right\},$$

$$\boldsymbol{\xi} = \Gamma^{-1}\mathbf{R}/R, \quad R = |\mathbf{R}|, \quad \omega = |\boldsymbol{\omega}|.$$

Boundary conditions (13) imply zeroing of the elastic displacement vector for the points of the internal surface of a spherical shell attached to a solid core, and zeroing of stresses on the external surface of a spherical shell.

The solution of the boundary-value problem (12)–(13) has the form [8]:

$$\mathbf{u}_1 = \mathbf{u}_{10} + \mathbf{u}_{11} + \mathbf{u}_{12}, \quad (14)$$

where

$$\mathbf{u}_{10} = \frac{2}{3}\rho_1\omega^2\left(a_1 r^2 + a_2 + \frac{a_3}{r^3}\right)\mathbf{r}, \quad \mathbf{u}_{11} = \left(b_1 r^2 + b_2 + \frac{b_3}{r^3} + \frac{b_4}{r^5}\right)\nabla_{\mathbf{r}}U + \left(b_5 + \frac{b_6}{r^5} + \frac{b_7}{r^7}\right)U\mathbf{r}, \quad \mathbf{u}_{12} \approx \mathbf{u}_{120} - \chi\dot{\mathbf{u}}_{120}, \quad \mathbf{u}_{120} = \left(b_1 r^2 + b_2 + \frac{b_3}{r^3} + \frac{b_4}{r^5}\right)\nabla_{\mathbf{r}}V + \left(b_5 + \frac{b_6}{r^5} + \frac{b_7}{r^7}\right)V\mathbf{r}, \quad a_1 = -\frac{1+\nu}{5(k+2)},$$

$$a_2 = -\frac{a_1 r_1^2\left(4x^5 + 5k + 6\right)}{4x^3 + 3k + 2},$$

$$a_3 = -\frac{a_1 r_1^5 x^3\left((3k+2)x^2 - 5k - 6\right)}{4x^3 + 3k + 2}, \quad b_1 = -\frac{(1+\nu)}{\Delta_0} \times \left\{8(9k+14)x^{10} + 80x^7 + 24(k+1)(5k+11)x^5 - 5(k+2)(15k+16)x^3 + 2(3k+8)(5k+4)\right\},$$

$$b_2 = \frac{(1+\nu)r_1^2}{\Delta_0}\left\{8(9k+14)x^{12} + 8\left(15k^2 + 46k + 51\right)x^7 - \left(63k^2 + 114k + 56\right)x^5 + 4(3k+8)(4k+3)\right\}, \quad (15)$$

$$b_3 = \frac{2(1+\nu)r_1^5 x^3}{\Delta_0}\left\{40x^9 - 16(k+6)x^7 + (21k+16)x^2\right.$$

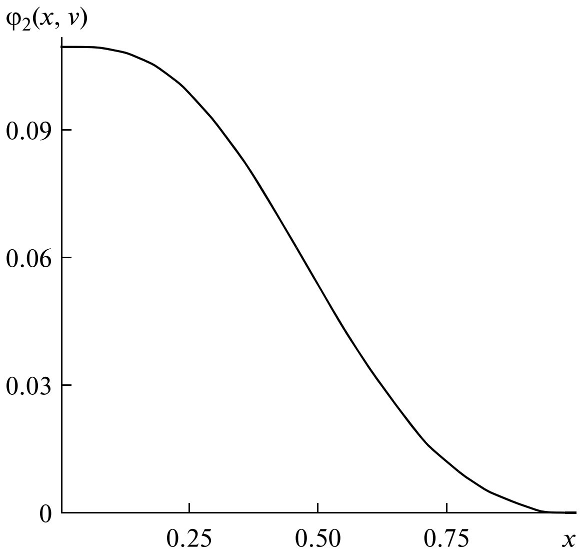

Fig. 2

$$-10(4k+3)\Big\},\quad b_4 = \frac{2(1+\nu)(k+1)r_1^7x^5}{\Delta_0}\Big\{24x^7$$

$$-2(3k+26)x^5 + (15k+16)x^2 - 6(4k+3)\Big\},$$

$$b_5 = -\frac{4(1+\nu)(k+1)}{\Delta_0}\Big\{60x^7 - 12(2k+17)x^5$$

$$+5(3k+26)x^3 - 2(3k+8)\Big\},\quad b_6 = (3k+1)b_3,$$

$$b_7 = -5b_4,\quad \Delta_0 = 8(2k+7)(9k+14)x^{10} + 200$$

$$\times\left(3k^2+8k+7\right)x^7 - 1008(k+1)^2x^5 + 25\left(27k^2\right.$$

$$\left.+56k+28\right)x^3 + 2(3k+8)(19k+14),$$

$$k = \frac{2\nu}{1-2\nu},\quad x = \frac{r_0}{r_1}.$$

Note that, according to the method of separation of motions [4], the time dependence of the vector-function $\mathbf{u}_{120}$ is accomplished in terms of quantities R and $\boldsymbol{\xi}$ in accordance with the undisturbed problem (11).

We introduce the movable coordinate system $Cx_1x_2x_3$, fixed with the planet. We place the origin C at the center of the solid core and direct the Cx_3 axis along the vector $\boldsymbol{\omega}$. Then the surface of a rotating deformed planet without allowance for tidal deformations can be described by the parametric vector equation:

$$\mathbf{S} = r_1\mathbf{e}_r + \varepsilon\mathbf{u}_{10}(r_1\mathbf{e}_r, t) + \varepsilon\mathbf{u}_{11}(r_1\mathbf{e}_r, t),$$
$$\mathbf{e}_r = (\cos\varphi\sin\theta; \sin\varphi\sin\theta; \cos\theta), \qquad (16)$$
$$0 \le \varphi \le 2\pi,\quad 0 \le \theta \le \pi.$$

Surface (16) is the surface of rotation. We will obtain a parametric equations of the curve appearing at intersection of this surface by the plane passing through the Cx_3 axis, for example, by the plane Cx_2x_3. We insert $\varphi = \pi/2$ into (16).

$$\mathbf{S} = (0; S_2, S_3),$$
$$S_2 = r_1\left[(1+d_1+d_2)\sin\theta + d_3\sin\theta\cos^2\theta\right],$$
$$S_3 = r_1\left[(1+d_1-2d_2)\cos\theta - d_3\cos\theta\sin^2\theta\right], \qquad (17)$$
$$0 \le \theta \le 2\pi.$$

Then $d_i = \varepsilon\rho_1^2\omega^2r_1^2\varphi_i(x,\nu)$, $i = 1, 2, 3$,

$$\varphi_1(x,\nu) = \frac{2(1+\nu)\left(3x^5-5x^3+2\right)}{15\left(4x^3+3k+2\right)},\quad \varphi_2(x,\nu) = \frac{(1+\nu)}{3\Delta_0}$$

$$\times\Big\{120(k+2)x^{12} - 2\left(15k^2+133k+218\right)x^{10}$$

$$+5\left(15k^2+31k+32\right)x^7 + 3(79k+74)x^5$$

$$-25\left(3k^2+14k+10\right)x^3 + 2(3k+8)(5k+4)\Big\}, \qquad (18)$$

$$\varphi_3(x,\nu) = \frac{(1+\nu)(k+1)}{\Delta_0}\Big\{2(9k+14)x^{10}$$

$$+25(3k+8)x^7 - 21(11k+26)x^5$$

$$+50(3k+7)x^3 - 4(3k+8)\Big\}.$$

Using formulas (17), we can obtain the values of the polar and equatorial radii for the chosen model of the planet with the core:

$r_{\rm eq} = r_1(1+d_1+d_2)$, $r_{\rm pol} = r_1(1+d_1-2d_2)$.

The value of the tidal bulge generated on a planetary surface by a satellite, is determined by the expression $|\varepsilon\mathbf{u}_{120}(r_1\boldsymbol{\xi},t)|$. Substituting $\mathbf{r} = r_1\boldsymbol{\xi}$ in (15), we obtain:

$$|\varepsilon\mathbf{u}_{120}(r_1\boldsymbol{\xi},t)| = \frac{6f\mu\rho_1r_1^3}{ER^3}\varphi_2(x,\nu), \qquad (19)$$

where function $\varphi_2(x,\nu)$ is defined by formula (18). Figure 2 shows a plot of function $\varphi_2(x,\nu)$ for $\nu = 0.2$.

As an example, we consider the system Earth–Moon. The amplitude of equilibrium lunar tide on the Earth is equal to 0.36 m [9]. Substituting into (19) the values of $|\varepsilon\mathbf{u}_{120}(r_1\boldsymbol{\xi},t)| = 0.36$ m, $r_1 = 6.378 \cdot 10^6$ m, $\mu = 7.349 \cdot 10^{22}$ kg, $R = 3.844 \cdot 10^8$ m, $E = 1.2 \times 10^{11}$ N/m^2, $f = 6.67 \cdot 10^{-11}$ N m^2/kg^2, we obtain $\rho_1\varphi_2(x,\nu) = 3.216 \cdot 10^2$ (kg/m^3). Taking into account that the density of upper layers of the Earth equals 2.6–3 (g/cm^3) [10], we obtain the range of variation of function $\varphi_2(x,\nu)$: $0.107 < \varphi_2(x,\nu) < 0.124$. In particular, if we let $\rho_1 = 3$ g/cm^3, $\nu = 0.2$, then for the considered two-layer model of the Earth we will have the following relation of internal and external radii of the viscoelastic layer: $x = 0.17$.

For constructing the disturbed system of the equations of motion of the mechanical "planet–satellite" system we will linearize equations (8)–(9) over the components of vector $\mathbf{u}$:

$$\ddot{\mathbf{R}} + \frac{f(m+\mu)}{R^3}\mathbf{R} + \frac{3f(m+\mu)}{mR^4}\Gamma\int_{V_1}[5\xi(\xi,\mathbf{r})(\xi,\mathbf{u}) - \xi(\mathbf{r},\mathbf{u}) - \mathbf{r}(\xi,\mathbf{u}) - \mathbf{u}(\xi,\mathbf{r})]\rho_1 dv_1 = 0, \quad (20)$$

$$\dot{\mathbf{L}} - \frac{3f\mu}{R^3}\Gamma\int_{V_1}[[\mathbf{u}\times\xi](\xi,\mathbf{r}) + [\mathbf{r}\times\xi](\xi,\mathbf{u})]\rho_1 dv_1 = 0. \quad (21)$$

Here $\mathbf{L}$ is the vector of the angular momentum of a planet relative to the center of mass:

$$\mathbf{L} = \int_V \Gamma(\mathbf{r}+\mathbf{u}) \times \frac{d}{dt}[\Gamma(\mathbf{r}+\mathbf{u})]\rho dv. \quad (22)$$

Substituting into equations (20) and (21) the quantity $\mathbf{u} = \varepsilon\mathbf{u}_1$, where $\mathbf{u}_1$ is defined by equality (14), and calculating triple integrals over the region V_1, we obtain the vector system of differential equations describing the translational-rotational motion of the "planet–satellite" system with allowance for disturbances caused by the elasticity and dissipation:

$$\ddot{\mathbf{R}} + \frac{f(m+\mu)}{R^3}\mathbf{R} + \frac{3f(m+\mu)\rho_1^2\varepsilon D}{mR^4}\Gamma\left\{-5\xi(\xi,\boldsymbol{\omega})^2 + \xi\boldsymbol{\omega}^2 + 2\boldsymbol{\omega}(\xi,\boldsymbol{\omega}) + \frac{6f\mu}{R^3}\left(1 + \frac{3\chi\dot{R}}{R}\right)\xi + \frac{6\chi f\mu}{R^3}\dot{\xi}\right\} = 0, \quad (23)$$

$$\dot{\mathbf{L}} + \frac{6f\mu\rho_1^2\varepsilon D}{R^3}\Gamma\left\{\frac{3\chi f\mu}{R^3}[\dot{\xi}\times\xi] + (\xi,\boldsymbol{\omega})[\boldsymbol{\omega}\times\xi]\right\} = 0,$$

$$D = \frac{4\pi r_1^7}{105}\varphi(x,\nu), \quad \varphi(x,\nu) = \frac{(1+\nu)}{\Delta_0}\{-16(9k + 14)x^{17} - 200(3k+8)x^{14} + 672(4k+9)x^{12} - (210k^2 + 3044k + 5824)x^{10} + (525k^2 + 1256k + 1576)x^7 + 84(17k+12)x^5 - 25(21k^2 + 92k + 56)x^3 + 210k^2 + 716k + 416\}. \quad (24)$$

The system of equations (23)–(24) has a first integral law of conservation of the angular momentum of the "planet–satellite" system relative to the common center of masses:

$$m_r\mathbf{R}\times\dot{\mathbf{R}} + \mathbf{L} = \mathbf{G}_0, \quad (25)$$

where $m_r = \mu m/(m+\mu)$, $\mathbf{G}_0$ is the constant vector.

3. EVOLUTION OF THE ORBITAL MOTION OF A SATELLITE

The angular momentum of a planet, defined by equality (22), can be represented in the form: $\mathbf{L} = A\Gamma\boldsymbol{\omega} + \varepsilon\mathbf{L}_1$. Retaining the terms of the first order in ε and taking into account the integral (25), we substitute into equation (23) the following expression for $\Gamma\boldsymbol{\omega}$, $\Gamma\xi$, $\Gamma\dot{\xi}$ **:**

$$\Gamma\boldsymbol{\omega} = A^{-1}(\mathbf{G}_0 - m_r\mathbf{R}\times\dot{\mathbf{R}}), \quad \Gamma\xi = \mathbf{R}/R,$$
$$\Gamma\dot{\xi} = \frac{\dot{\mathbf{R}}}{R} - \frac{\mathbf{R}}{R^2}\dot{R} - \frac{[\mathbf{G}_0 - m_r\mathbf{R}\times\dot{\mathbf{R}}]\times\mathbf{R}}{AR}.$$

Then we obtain the vector differential equation of the orbital motion of a satellite in the form of: $\mu\ddot{\mathbf{R}} = \mathbf{F}_0 + \varepsilon\mathbf{F}_1 + \varepsilon\chi\mathbf{F}_2$, where

$$\mathbf{F}_0 = -\frac{f_0\mu}{R^3}\mathbf{R}, \quad \mathbf{F}_1 = -C_1\left\{\frac{(\mathbf{G}_0 - m_r\mathbf{R}\times\dot{\mathbf{R}})^2}{A^2R^5}\mathbf{R} + \frac{2(\mathbf{R},\mathbf{G}_0)(\mathbf{G}_0 - m_r\mathbf{R}\times\dot{\mathbf{R}})}{A^2R^5} - \frac{5\mathbf{R}(\mathbf{R},\mathbf{G}_0)^2}{A^2R^7} + \frac{6f\mu}{R^8}\mathbf{R}\right\},$$

$$\mathbf{F}_2 = -C_2\left\{\frac{\dot{\mathbf{R}}}{R^8} + \frac{2\dot{R}}{R^9}\mathbf{R} - \frac{[\mathbf{G}_0 - m_r\mathbf{R}\times\dot{\mathbf{R}}]\times\mathbf{R}}{AR^8}\right\},$$

$$C_1 = 3f_0\mu\rho_1^2 D m^{-1}, \quad C_2 = 6f\mu C_1, \quad f_0 = f(m+\mu).$$

To obtain the evolution system of the equations of motion of a satellite we will transfer to the canonical Delaunay variables L, G, H, l, g, h [9, 11]. The undisturbed Hamiltonian of the problem of satellite motion under the effect of force $\mathbf{F}_0$ has the form: $H_0 = -f_0^2\mu^3/2L^2$.

We direct the OZ axis of the inertial coordinate system $OXYZ$ along the vector $\mathbf{G}_0$. Then $\mathbf{G}_0 = (0, 0, G_0)$. We write the components of vector $\mathbf{R}$ in the Delaunay variables:

$$\mathbf{R} = (R_X, R_Y, R_Z), R_X = R(\cos(g+\vartheta)\cos h - \sin(g+\vartheta)\cos i \sin h), \quad R_Y = R(\cos(g+\vartheta)\sin h + \sin(g+\vartheta)\cos i\cos h), \quad R_Z = R\sin(g+\vartheta)\sin i,$$

$$R = \frac{G^2}{f_0\mu^2(1+e\cos\vartheta)}, \quad e = \sqrt{1 - G^2/L^2},$$

$$\cos w = \frac{e + \cos\vartheta}{1 + e\cos\vartheta}, \quad l = w - e\sin w, \cos i = \frac{H}{G},$$

where e is the eccentricity of the orbit of the end of vector $\mathbf{R}$; ϑ, l, w are the true, mean and eccentric anomalies, respectively; g is the perihelion longitude from the ascending node; and i is the inclination of satellite's orbit.

The canonical equations of a disturbed motion of a satellite in the Delaunay variables have the form:

$$\dot{L} = \varepsilon Q_l,\quad \dot{G} = \varepsilon Q_g,\quad \dot{H} = \varepsilon Q_h,\quad \dot{l} = n - \varepsilon Q_L,$$
$$\dot{g} = -\varepsilon Q_G,\quad \dot{h} = -\varepsilon Q_H,\quad n = f_0^2\mu^3/L^3. \tag{26}$$

Here n is the mean motion of a satellite on its orbit, and the generalized forces $Q_l, \ldots, Q_H$ are determined from the expression for elementary work:

$$\delta A = (\varepsilon \mathbf{F}_1 + \varepsilon\chi \mathbf{F}_2, \delta \mathbf{R})$$
$$= \varepsilon\left(Q_l\delta l + Q_g\delta g + Q_h\delta h + Q_L\delta L + Q_G\delta G + Q_H\delta H\right).$$

Calculating the generalized forces and averaging the right-hand sides of the system of equations (26) over the fast angular variable l, we can obtain the closed system of ordinary differential equations with respect to the "action" variables L, G, H and slow angular variables g, h. We write the evolution system of equations of the orbital motion of a satellite in dimensionless variables n_0, e, i, g, h, where $n_0 = nAG_0^{-1}$. This system of equations has the form:

$$\dot{n}_0 = -\frac{3\Delta_1 n_0^{16/3}}{\left(1-e^2\right)^{15/2}}\left\{\left[\cos i - \frac{p}{n_0^{1/3}}\left(1-e^2\right)^{1/2}\right]\cdot F_2(e)\right.$$
$$\left.\times\left(1-e^2\right)^{3/2} - n_0\cdot F_3(e)\right\},\quad \dot{e} = \frac{\Delta_1 n_0^{13/3} e}{\left(1-e^2\right)^{13/2}}$$
$$\times\left\{\left[\cos i - \frac{p}{n_0^{1/3}}\left(1-e^2\right)^{1/2}\right]\cdot F_5(e)\cdot\left(1-e^2\right)^{3/2}\right.$$
$$\left. - n_0\cdot F_4(e)\right\},\quad \frac{di}{dt} = -\frac{\Delta_1 n_0^{13/3}\sin i}{\left(1-e^2\right)^5}$$
$$\times\left\{\frac{1}{2} + \left(\frac{9}{4} - \frac{3}{2}\sin^2 g\right)e^2 + \left(\frac{5}{16} - \frac{1}{4}\sin^2 g\right)e^4\right\},$$
$$\dot{g} = \frac{\Delta_1 n_0^{13/3}}{\left(1-e^2\right)^5}\cdot\cos i\cdot\sin 2g\cdot\left\{\frac{3e^2}{4} + \frac{e^4}{8}\right\} - \frac{\Delta_2 n_0^2}{\left(1-e^2\right)^{3/2}} \tag{27}$$
$$\times\left\{3\cos i - \frac{p}{n_0^{1/3}}\left(1-e^2\right)^{1/2}\right\} + \frac{\Delta_3 n_0^{7/3}\sin i}{\left(1-e^2\right)^2}$$
$$\times\left\{\frac{5}{2}\cos^2 i - \frac{1}{2} + \frac{15\mu n_0^2}{(m+\mu)\left(1-e^2\right)^3}\left(1 + \frac{3e^2}{2} + \frac{e^4}{8}\right)\right\},$$
$$\dot{h} = -\frac{\Delta_1 n_0^{13/3}}{\left(1-e^2\right)^5}$$
$$\times\sin 2g\cdot\left\{\frac{3e^2}{4} + \frac{e^4}{8}\right\} + \frac{\Delta_2 n_0^2}{\left(1-e^2\right)^{3/2}} - \frac{\Delta_3 n_0^{7/3}\cos i}{\left(1-e^2\right)^2}.$$

Here

$$\Delta_1 = \frac{18\varepsilon\chi\mu\rho_1^2 D}{m(m+\mu)f_0^{2/3}}\left(\frac{G_0}{A}\right)^{16/3},\quad \Delta_2 = \frac{3\varepsilon\mu\rho_1^2 D G_0^3}{(m+\mu)A^4},$$
$$\Delta_3 = \frac{3\varepsilon\rho_1^2 D}{m f_0^{2/3}}\left(\frac{G_0}{A}\right)^{13/3},\quad F_2(e) = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6,$$
$$F_3(e) = 1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \frac{25}{64}e^8,\quad F_4(e) = 9$$
$$+ \frac{135}{4}e^2 + \frac{135}{8}e^4 + \frac{45}{64}e^6,\quad F_5(e) = \frac{11}{2} + \frac{33}{4}e^2 + \frac{11}{16}e^4,$$
$$p = \frac{A^{1/3} f_0^{2/3} m_r}{G_0^{4/3}}.$$

We consider two partial cases of satellite motion.

Case 1: $i \equiv 0$. Then the first two equations of system (27) form a closed system of differential equations:

$$\dot{n}_0 = -\frac{3\Delta_1 n_0^{16/3}}{\left(1-e^2\right)^{15/2}}\left\{\left[1 - \frac{p}{n_0^{1/3}}\left(1-e^2\right)^{1/2}\right]\cdot F_2(e)\right.$$
$$\left.\times\left(1-e^2\right)^{3/2} - n_0\cdot F_3(e)\right\},\quad \dot{e} = \frac{\Delta_1 n_0^{13/3} e}{\left(1-e^2\right)^{13/2}} \tag{28}$$
$$\times\left\{\left[1 - \frac{p}{n_0^{1/3}}\left(1-e^2\right)^{1/2}\right]\cdot F_5(e)\cdot\left(1-e^2\right)^{3/2} - n_0\cdot F_4(e)\right\}.$$

The stationary solutions of system (28) are: $e = 0$, $n_0 = n_{0*}$, where the value of n_{0*} is a root of the equation

$$n_0 + \left(p/n_0^{1/3}\right) = 1. \tag{29}$$

If $p > 3\cdot 4^{-4/3}$, then equation (29) has no solutions. For $p = 3\cdot 4^{-4/3}$ equation (29) has one solution $n_0 = 1/4$. If $p < 3\cdot 4^{-4/3}$, then equation (29) has two solutions n_{01} and n_{02}: $n_{01} < 1/4 < n_{02}$.

In the case of existence of two stationary solutions we investigate their stability on the basis of the equations in variations. We let $n_0 = n_{0j} + \eta_1$, $e = \eta_2$ $(j = 1, 2)$. Then

$$\dot{\eta}_1 = -3\Delta_1 n_{0j}^{16/3}\left\{\frac{p}{3n_{0j}^{4/3}} - 1\right\}\cdot\eta_1,$$
$$\dot{\eta}_2 = \Delta_1 n_{0j}^{13/3}\left\{\frac{11}{2}\left[1 - \frac{p}{n_{0j}^{1/3}}\right] - 9n_{0j}\right\}\cdot\eta_2. \tag{30}$$

The solution of equations (30) will be sought in the form:

$$\eta_1 = B_1\exp\left\{\Delta_1 n_{0j}^{13/3}\lambda t\right\},\quad \eta_2 = B_2\exp\left\{\Delta_1 n_{0j}^{13/3}\lambda t\right\},$$

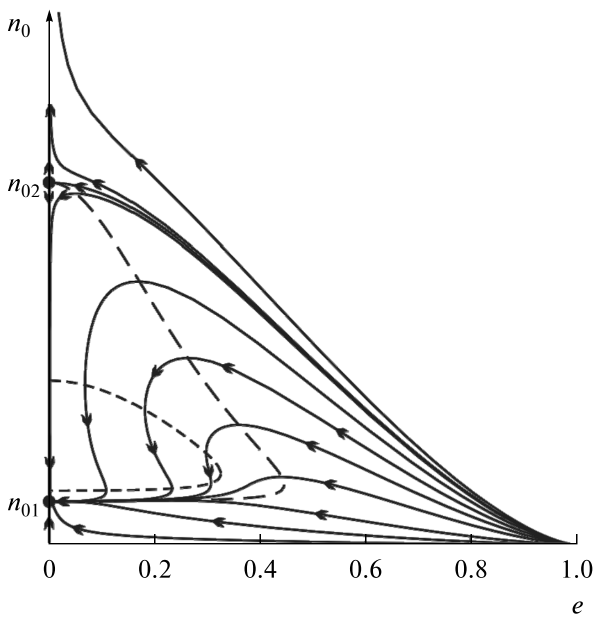

Fig. 3

where B_1, B_2 are arbitrary constants. The characteristic equation for the quantity λ has the form:

$$\left(\lambda + 3n_{0j}\left[\frac{p}{3n_{0j}^{4/3}} - 1\right]\right) \cdot \left(\lambda - \frac{11}{2}\left(1 - \frac{p}{n_{0j}^{4/3}}\right) + 9n_{0j}\right) = 0. \quad (31)$$

With allowance for equality (29) we obtain the following values of the roots of equation (31):

$$\lambda_1 = 4n_{0j} - 1, \quad \lambda_2 = -3.5n_{0j} < 0.$$

Because $\lambda_1 < 0$ for $j = 1$ and $\lambda_1 > 0$ for $j = 2$, the stationary solution $e = 0$, $n_0 = n_{01}$ is asymptotically stable, and the stationary solution $e = 0$, $n_0 = n_{02}$ is unstable. Figure 3 shows the phase portrait of the system of equations (28) for $p = 0.375$.

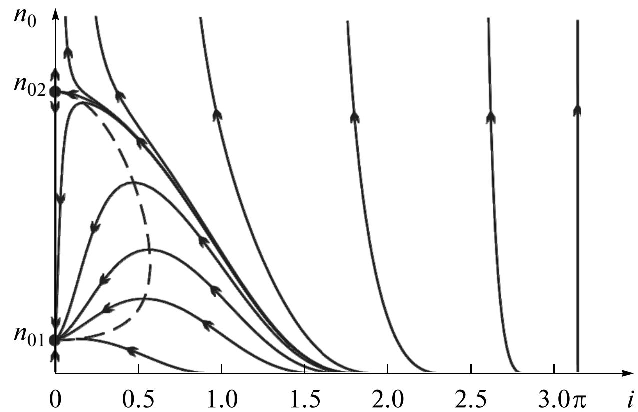

Fig. 4

Case 2: $e \equiv 0$. In this case we obtain the closed system of differential equations of the second order with respect to variables n_0, i:

$$\dot{n}_0 = -3\Delta_1 n_0^{16/3}\left\{\cos i - \frac{p}{n_0^{1/3}} - n_0\right\},$$
$$\frac{di}{dt} = -\frac{1}{2}\Delta_1 n_0^{13/3}\sin i. \quad (32)$$

The stationary solutions of system (32) are: $n_0 = n_{0*}$, $i = 0$, where n_{0*} is the root of equation (29). Depending on the value of parameter p, equation (29) either has no solutions, has one solution, or has two solutions n_{01} and n_{02}. As in the case 1, the stationary solution $n_0 = n_{01}$, $i = 0$ is asymptotically stable, and the stationary solution $n_0 = n_{02}$, $i = 0$ is unstable. The phase portrait of the system of equations (32) for the value of parameter $p = 0.375$ is shown in Fig. 4.

The table presents the numerical values of parameter p, the stationary values of n_{01}, n_{02} and the value of quantity $n_0 = n_0(0)$ at the present time for various "planet–satellite" systems. The dimensionless variable n_0 is proportional to the mean motion of a satellite

Table

Planet–satellite	p	$n_0(0)$	n_{01}	n_{02}
Earth–Moon	0.15535	7.2589×10^{-3}	3.7922×10^{-3}	0.83503
Mars–Phobos	1.5235×10^{-6}	3.2171	3.5358×10^{-18}	$1-1.5235\times10^{-6}$
Mars–Deimos	2.5391×10^{-7}	8.1268×10^{-1}	1.6369×10^{-20}	$1-2.5391\times10^{-7}$
Jupiter–Io	5.8980×10^{-4}	0.2335	2.0517×10^{-10}	0.9994
Jupiter–Europa	3.1685×10^{-4}	0.1163	3.1811×10^{-11}	0.9997
Jupiter–Ganymede	9.7643×10^{-4}	5.7650×10^{-2}	9.3093×10^{-10}	0.9990
Jupiter–Callisto	7.0902×10^{-4}	2.4717×10^{-2}	3.5644×10^{-10}	0.9993

on its orbit and is related with the semimajor axis a of satellite's orbit by the equation: $a = f_0^{1/3} \Big/ \left(G_0 A^{-1} n_0\right)^{2/3}$.

For all presented examples, except the Mars–Phobos system, a double inequality takes place: $n_{01} < n_0(0) < n_{02}$. According to the first equation of system (28), the value of variable n_0 decreases during the motion. This implies that the semimajor axes of satellite orbits increase, tending to asymptotically stable stationary values. For satellites of Jupiter and for the Martian satellite Deimos the current value of $n_0(0)$ is closer to the unstable stationary value n_{02}, and for the Earth–Moon system it is closer to the asymptotically stable value n_{01}. For the Mars–Phobos system $n_{01} < n_{02} < n_0(0)$. The value of variable n_0 increases. This implies that Phobos is approaching Mars.

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