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Construction of Optimum Controls and Trajectories of Motion of the Center of Masses of a Spacecraft Equipped with the Solar Sail and Low-Thrust Engine, Using Quaternions and Kustaanheimo–Stiefel Variables

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Abstract—The problem of optimum rendezvous of a controllable spacecraft (SC) with an uncontrollable spacecraft, moving over a Keplerian elliptic orbit in the gravitational field of the Sun, is considered. Control of the SC is performed using a solar sail and low-thrust engine. For solving the problem, the regular quaternion equations of the two-body problem with the Kustaanheimo–Stiefel variables and the Pontryagin maximum principle are used. The combined integral quality functional, which characterizes energy consumption for controllable SC transition from an initial to final state and the time spent for this transition, is used as a minimized functional. The differential boundary-value optimization problems are formulated, and their first integrals are found. Examples of numerical solution of problems are presented. The paper develops the application [1–6] of quaternion regular equations with the Kustaanheimo–Stiefel variables in the space flight mechanics.

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1. EQUATIONS OF MOTION

The spacecraft (SC) is considered as a material point B of variable mass $m = m(t)$. SC motion is considered in the coordinate system $OX_1X_2X_3$ (X) with the origin at the center O of attraction (at the center of masses of the Sun) and with the coordinate axes parallel to the axes of the inertial coordinate system. The controllable motion of a SC in a Newtonian central field of gravitation forces is described by the vector equation [7, 8]

$$\ddot{\mathbf{r}} + fMr^{-3}\mathbf{r} = \mathbf{p} + \mathbf{p}_{sol}, \quad r = |\mathbf{r}|. \qquad (1.1)$$

Here, $\mathbf{r}$ is the SC radius-vector drawn from the center of attraction, f is the gravitational constant, M is the mass of the attracting body, $\mathbf{p}$ is the SC acceleration vector produced by a low-thrust engine (the thrust vector of this engine related to the unit mass of SC), $\mathbf{p}_{sol}$ is the thrust vector of a solar sail related to the SC mass unit, defined by the relations [9] $\mathbf{p}_{sol} = dr^{-2}\cos^2\theta\mathbf{n} = dr^{-4}(\mathbf{r}\cdot\mathbf{n})^2\mathbf{n}$, $\mathbf{n}$ is the unit vector of the normal to sail's plane facing from the Sun, θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{n}$, d is the coefficient characterizing the sail area.

Now we introduce into consideration the coordinate system $\eta_1\eta_2\eta_3$ (η) with the origin at point B. Axis η_1 of this coordinate system is directed along the radius-vector $\mathbf{r}$. The angular position of the coordinate system η in the reference system X is specified by the normalized quaternion [10, 11]

$$\boldsymbol{\lambda} = \lambda_0 + \lambda_1\mathbf{i}_1 + \lambda_2\mathbf{i}_2 + \lambda_3\mathbf{i}_3,$$
$$\|\boldsymbol{\lambda}\|^2 = \lambda_0^2 + \lambda_1^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors (orts) of the hypercomplex space (imaginary Hamilton units); λ_j ($j = \overline{0.3}$) are components of the orientation quaternion $\boldsymbol{\lambda}$ (the Rodrigues–Hamilton (Euler) parameters), which are identical in the X and η bases. We assume that the nonholonomic relation

$$\omega_1 = 2(-\lambda_1\dot{\lambda}_0 + \lambda_0\dot{\lambda}_1 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3) = 0, \qquad (1.2)$$

is imposed on the motion of the trihedron η, where ω_1 is the projection of the vector of absolute angular velocity ω of the trihedron η on the direction of radius vector $\mathbf{r}$ (axis η_1); the upper dot means the derivative with respect to time t. We will write the equations of SC motion in a rotating coordinate system η using the

Kustaanheimo–Stiefel variables u_j [12] associated with parameters λ_j by relations [6, 13]

$$u_0 = r^{1/2}\lambda_0, \quad u_k = -r^{1/2}\lambda_k, \quad k = 1, 2, 3. \tag{1.3}$$

These equations are similar to the regular Kustaanheimo–Stiefel equations of the two-body problem [12] and have, in scalar recording, the form

$$\frac{d^2u_j}{d\tau^2} - \frac{h}{2}u_j = \frac{r}{2}q_j, \quad j = 0, 1, 2, 3,$$

$$\frac{dh}{d\tau} = 2\left(q_0\frac{du_0}{d\tau} + q_1\frac{du_1}{d\tau} + q_2\frac{du_2}{d\tau} + q_3\frac{du_3}{d\tau}\right), \tag{1.4}$$

$$\frac{dt}{d\tau} = r; \quad r = |\mathbf{r}| = u_0^2 + u_1^2 + u_2^2 + u_3^2,$$

$$\begin{aligned} q_0 &= u_0(p_1 + p_{sol1}) - u_3(p_2 + p_{sol2}) + u_2(p_3 + p_{sol3}),\\ q_1 &= u_1(p_1 + p_{sol1}) + u_2(p_2 + p_{sol2}) + u_3(p_3 + p_{sol3}),\\ q_2 &= -u_2(p_1 + p_{sol1}) + u_1(p_2 + p_{sol2}) + u_0(p_3 + p_{sol3}),\\ q_3 &= -u_3(p_1 + p_{sol1}) - u_0(p_2 + p_{sol2}) + u_1(p_3 + p_{sol3}). \end{aligned} \tag{1.5}$$

In these equations τ is a new independent variable, p_k and p_{solk} are projections of vectors $\mathbf{p}$ and $\mathbf{p}_{sol}$ on the axis X_k, and h is the total mechanical energy of SC's unit mass, defined by the equations

$$h = \frac{1}{2}v^2 - fM\frac{1}{r}, \quad v = |\mathbf{v}| = |\dot{\mathbf{r}}|, \tag{1.6}$$

$$\dot{h} = (\mathbf{p} + \mathbf{p}_{sol}) \cdot \mathbf{v}.$$

With variables u_j equation (1.6) takes the form

$$h = 2r\sum_{j=0}^{3}\dot{u}_j^2 - fM\frac{1}{r} = \frac{1}{r}\left(2\sum_{j=0}^{3}\left(\frac{du_j}{d\tau}\right)^2 - fM\right). \tag{1.7}$$

The equations of SC motion (1.4), (1.5) form a system of differential equations of the tenth order with respect to the variables u_j, $du_j/d\tau$ ($j = \overline{0,3}$), h and t.

Variables u_j and their derivatives $\dot{u}_j$ are associated with the Cartesian coordinates of SC x_k and their derivatives $\dot{x}_k$ by the nonlinear relations

$$\begin{aligned} x_1 &= r(\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2) = u_0^2 + u_1^2 - u_2^2 - u_3^2,\\ x_2 &= 2r(\lambda_1\lambda_2 + \lambda_0\lambda_3) = 2(u_1u_2 - u_0u_3),\\ x_3 &= 2r(\lambda_1\lambda_3 - \lambda_0\lambda_2) = 2(u_1u_3 + u_0u_2); \end{aligned} \tag{1.8}$$

$$\begin{aligned} v_1 &= \dot{x}_1 = 2(u_0\dot{u}_0 + u_1\dot{u}_1 - u_2\dot{u}_2 - u_3\dot{u}_3),\\ v_2 &= \dot{x}_2 = 2(u_2\dot{u}_1 + u_1\dot{u}_2 - u_3\dot{u}_0 - u_0\dot{u}_3),\\ v_3 &= \dot{x}_3 = 2(u_3\dot{u}_1 + u_1\dot{u}_3 + u_2\dot{u}_0 + u_0\dot{u}_2). \end{aligned} \tag{1.9}$$

Note that the projections p_{solk} of the acceleration from solar sail's thrust, appearing in (1.5), are determined by the relations

$$\begin{aligned} p_{solk} &= dr^{-4}(x_1n_1 + x_2n_2 + x_3n_3)^2n_k\\ &= dr^{-4}[(u_0^2 + u_1^2 - u_2^2 - u_3^2)n_1 + 2(u_1u_2 - u_0u_3)n_2\\ &\quad + 2(u_1u_3 + u_0u_2)n_3]^2n_k, \end{aligned}$$

where n_k is the projection of the unit vector of the normal $\mathbf{n}$ on the axis OX_k of the inertial coordinate system.

Equations (1.4), (1.5) possess the well-known advantages of regular Kustaanheimo–Stiefel equations [6, 12]: they (1) are regular for SC motions in the Newtonian gravitational field (not degenerated for $r = 0$), which is important in the study of SC motion over elongated orbits; (2) assume the form of linear differential equations with constant coefficients for the undisturbed Keplerian motions of SC; (3) are close to the linear form for small p_k and p_{solk}.

In the quaternion recording the equations and relations (1.3)–(1.5), (1.8), (1.9) assume the form [6, 14, 15]

$$\frac{d^2\mathbf{u}}{d\tau^2} - \frac{h}{2}\mathbf{u} = \frac{r}{2}\mathbf{q}, \tag{1.10}$$

$$\frac{dh}{d\tau} = 2scal\left(\frac{d\bar{\mathbf{u}}}{d\tau} \circ \mathbf{q}\right), \quad dt/d\tau = r; \tag{1.11}$$

$$\begin{aligned} \mathbf{q} &= -\mathbf{i}_1 \circ \mathbf{u} \circ (\mathbf{p}_x + \mathbf{p}_{solx}),\\ \mathbf{p}_{solx} &= dr^{-4}(\mathbf{r}_x \cdot \mathbf{n}_x)^2\mathbf{n}_x = dr^{-4}(scal(\mathbf{r}_x \circ \mathbf{n}_x))^2\mathbf{n}_x; \end{aligned} \tag{1.12}$$

$$r = \|\mathbf{u}\|^2 = \mathbf{u} \circ \bar{\mathbf{u}} = \bar{\mathbf{u}} \circ \mathbf{u}, \quad \mathbf{r}_x = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \mathbf{u}, \tag{1.13}$$

$$\begin{aligned} \mathbf{v}_x &= \frac{d\mathbf{r}_x}{dt} = 2\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \frac{d\mathbf{u}}{dt} = 2r^{-1}\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \frac{d\mathbf{u}}{d\tau},\\ \boldsymbol{\lambda} &= r^{-1/2}\bar{\mathbf{u}}, \quad \mathbf{u} = r^{1/2}\bar{\boldsymbol{\lambda}}. \end{aligned} \tag{1.14}$$

Here, $\mathbf{u} = u_0 + u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3$ is the quaternionic variable associated with the radius vector $\mathbf{r}$ and quaternion $\boldsymbol{\lambda}$ by relations (1.13), (1.14); $\mathbf{p}_x$, $\mathbf{p}_{solx}$, $\mathbf{r}_x$, $\mathbf{v}_x$ are mappings of vectors $\mathbf{p}$, $\mathbf{p}_{sol}$, $\mathbf{r}$, $\mathbf{v}$ on the basis X defined as the quaternions

$$\mathbf{p}_x = p_1\mathbf{i}_1 + p_2\mathbf{i}_2 + p_3\mathbf{i}_3,$$

$$\mathbf{p}_{solx} = p_{sol1}\mathbf{i}_1 + p_{sol2}\mathbf{i}_2 + p_{sol3}\mathbf{i}_3,$$

$$\mathbf{r}_x = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3,$$

$$\mathbf{v}_x = \upsilon_1\mathbf{i}_1 + \upsilon_2\mathbf{i}_2 + \upsilon_3\mathbf{i}_3 = \dot{x}_1\mathbf{i}_1 + \dot{x}_2\mathbf{i}_2 + \dot{x}_3\mathbf{i}_3;$$

symbol $\circ$ denotes quaternion multiplication, the upper dash means the conjugated quaternion, so that $\bar{\mathbf{u}} = u_0 - u_1\mathbf{i}_1 - u_2\mathbf{i}_2 - u_3\mathbf{i}_3$, scal$(\cdot)$ is the scalar part of quaternion $(\cdot)$.

The equations of motion of SC (1.10)–(1.11) represent a closed system of differential equations with respect to variables $\mathbf{u}$, $d\mathbf{u}/d\tau$, h, t, which contains the oscillatory quaternion equation (1.10). We write the equations of SC motion in the normal Cauchy form. Designating

$$\mathbf{s} = d\mathbf{u}/d\tau = du_0/d\tau + \mathbf{i}_1 du_1/d\tau + \mathbf{i}_2 du_2/d\tau + \mathbf{i}_3 du_3/d\tau,$$

we get

$$d\mathbf{u}/d\tau = \mathbf{s}, \quad d\mathbf{s}/d\tau = (h/2)\mathbf{u} + (r/2)\mathbf{q}, \quad dh/d\tau = 2scal(\bar{\mathbf{s}} \circ \mathbf{q}), \quad dt/d\tau = r. \tag{1.15}$$

For the velocity vector mapping on the basis X we have: $\mathbf{v}_x = 2r^{-1}\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \mathbf{s}$. In the matrix-vector form the equations of SC motion are as follows:

$$d\mathbf{u}/d\tau = \mathbf{s}, \quad d\mathbf{s}/d\tau = (h/2)\mathbf{u} + (r/2)\mathbf{q},$$
$$d\mathbf{h}/d\tau = 2\mathbf{s}^T\mathbf{q}, \quad d\mathbf{t}/d\tau = \mathbf{u}^T\mathbf{u},$$
$$\mathbf{q} = -K(\mathbf{i}_1)K(\mathbf{u})(\mathbf{p}_x + \mathbf{p}_{solx}),$$
$$\mathbf{r}_x = K(\bar{\mathbf{u}})K(\mathbf{i}_1)\mathbf{u}, \quad \mathbf{v}_x = 2r^{-1}K(\bar{\mathbf{u}})K(\mathbf{i}_1)\mathbf{s},$$
$$\boldsymbol{\lambda} = r^{-1/2}\bar{\mathbf{u}}, \quad \mathbf{u} = r^{1/2}\bar{\boldsymbol{\lambda}},$$

where $\mathbf{u}$, $\mathbf{s}$, $\mathbf{q}$, $\boldsymbol{\lambda}$ are four-dimensional vector-columns with components u_j, s_j, q_j, λ_j $(j = \overline{0.3})$, respectively (so, $\mathbf{u} = (u_0, u_1, u_2, u_3)$); $\mathbf{r}_x = (0, x_1, x_2, x_3)$, $\mathbf{v}_x = (0, \dot{x}_1, \dot{x}_2, \dot{x}_3)$, $\mathbf{p}_x = (0, p_1, p_2, p_3)$; $\bar{\mathbf{u}} = (u_0, -u_1, -u_2, -u_3)$, $\bar{\boldsymbol{\lambda}} = (\lambda_0, -\lambda_1, -\lambda_2, -\lambda_3)$; $K(\mathbf{u})$, $K(\mathbf{i}_1)$ are quaternion matrices, which match the quaternions $\mathbf{u}$ and $\mathbf{i}_1$ and have the form [11]:

$$K(\mathbf{u}) = \begin{pmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & -u_3 & u_2 \\ u_2 & u_3 & u_0 & -u_1 \\ u_3 & -u_2 & u_1 & u_0 \end{pmatrix},$$

$$K(\mathbf{i}_1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The equations of SC motion in variables u_j (1.15) have the first integrals:

$$h\|\mathbf{u}\|^2 - 2\|\mathbf{s}\|^2 = \Psi_1 = \text{const},$$
$$\|\mathbf{u}\|^2 = \mathbf{u} \circ \bar{\mathbf{u}} = u_0^2 + u_1^2 + u_2^2 + u_3^2, \tag{1.16}$$
$$\|\mathbf{s}\|^2 = \mathbf{s} \circ \bar{\mathbf{s}} = s_0^2 + s_1^2 + s_2^2 + s_3^2,$$

$$\text{scal}(\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \mathbf{s}) = u_1 s_0 - u_0 s_1 + u_3 s_2 - u_2 s_3 = \boldsymbol{\Psi}_2 = \text{const}. \tag{1.17}$$

In order that integrals (1.16) and (1.17) respond to the problem of SC motion, it is necessary that constants $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ have strictly defined values. In accordance with expression (1.7) for the energy of motion, constant $\boldsymbol{\Psi}_1 = -fM$, and, in accordance with the nonholonomic constraint equation (1.2), constant $\Psi_2 = 0$ (relation (1.17)) for $\Psi_2 = 0$ coincides with the bilinear relation introduced in [12] when constructing the regular equations of the two-body problem; this bilinear equation is equivalent [6, 13] to the condition (1.2)).

2. STATEMENT OF THE PROBLEM

Let us state the following problem: one should construct controls $\mathbf{n}$ and $\mathbf{p}$ subject to the limitations

$$|\mathbf{n}| = 1, \quad |\mathbf{p}| \le p_{\max}, \tag{2.1}$$

which transfer the SC, whose motion is described by equations (1.10)–(1.13) or (1.15), (1.12):

$$d\mathbf{u}/d\tau = \mathbf{s}, \quad d\mathbf{s}/d\tau = (h/2)\mathbf{u} + (r/2)\mathbf{q}, \quad dh/d\tau = 2\text{scal}(\bar{\mathbf{s}} \circ \mathbf{q}); \tag{2.2}$$

and by relations (1.12), (1.13), from the initial state

$$\tau = 0, \quad \mathbf{u} = \mathbf{u}(0) = \mathbf{u}^0, \quad \mathbf{s} = \mathbf{s}(0) = \mathbf{s}^0, \quad h = h(0) = h_0, \tag{2.3}$$

that satisfies the relations

$$\bar{\mathbf{u}}(0) \circ \mathbf{i}_1 \circ \mathbf{u}(0) = \mathbf{r}_x(0), \quad 2\|\mathbf{u}(0)\|^{-2}\bar{\mathbf{u}}(0) \circ \mathbf{i}_1 \circ \mathbf{s}(0) = \mathbf{v}_x(0), \tag{2.4}$$

into the final state

$$\tau = \tau_k, \quad \mathbf{u} = \mathbf{u}(\tau_k), \quad \mathbf{s} = \mathbf{s}(\tau_k), \quad h = h(\tau_k) = h^*, \tag{2.5}$$

that satisfies the relations

$$\bar{\mathbf{u}}(\tau_k) \circ \mathbf{i}_1 \circ \mathbf{u}(\tau_k) = \bar{\mathbf{u}}^*(\tau_k^*) \circ \mathbf{i}_1 \circ \mathbf{u}^*(\tau_k^*),$$
$$\bar{\mathbf{u}}(\tau_k) \circ \mathbf{i}_1 \circ \mathbf{s}(\tau_k) = \bar{\mathbf{u}}^*(\tau_k^*) \circ \mathbf{i}_1 \circ \mathbf{s}^*(\tau_k^*), \tag{2.6}$$
$$\tau_k^* = \tau^*(\tau_k),$$

and minimizes the quality functional

$$J = \int_0^{t_k}(\alpha_1 + \alpha_2 p^2(t))dt = \int_0^{\tau_k}\|\mathbf{u}(\tau)\|^2(\alpha_1 + \alpha_2 p^2(\tau))d\tau, \tag{2.7}$$
$$\alpha_1, \alpha_2 - \text{const} \ge 0.$$

The final values t_k and τ_k of variables t and τ are not specified in advance and have to be found. Variables $\mathbf{u}^*$, $\mathbf{s}^*$ satisfy differential equations

$$d\mathbf{u}^*/d\tau^* = \mathbf{s}^*, \quad d\mathbf{s}^*/d\tau^* = (h^*/2)\mathbf{u}^*, \quad h^* = \mathrm{const}; \tag{2.8}$$

$$dt/d\tau^* = r^* = \|\mathbf{u}^*\|^2 = \mathbf{u}^* \circ \bar{\mathbf{u}}^* \tag{2.9}$$

and describe the motion of the phase point ($\mathbf{u}^*$, $\mathbf{s}^*$), which must coincide, at time instant t_k, with the phase point ($\mathbf{u}$, $\mathbf{s}$) characterizing the state of SC.

Equation (2.8), (2.9) can be interpreted as the differential equations of motion in the Newtonian gravitational field of the uncontrollable SC, which should meet the controllable SC, or as the differential equations of programmed (undisturbed) motion of the controllable SC.

We suppose that the initial position of the phase point ($\mathbf{u}^*$, $\mathbf{s}^*$) (for $t = 0$, $\tau^* = 0$) satisfy the relations:

$$\begin{aligned} \bar{\mathbf{u}}^*(0) \circ \mathbf{i}_1 \circ \mathbf{u}^*(0) &= \mathbf{r}_x^*(0) \\ &= x_1^*(0)\mathbf{i}_1 + x_2^*(0)\mathbf{i}_2 + x_3^*(0)\mathbf{i}_3, \\ 2\|\mathbf{u}^*(0)\|^{-2}\bar{\mathbf{u}}^*(0) \circ \mathbf{i}_1 \circ \mathbf{s}^*(0) = \mathbf{v}_x^*(0) &= (d\mathbf{r}_x^*/dt)_{t=0}, \end{aligned} \tag{2.10}$$

where $\mathbf{r}^*(0)$, $\mathbf{v}^*(0)$ are specified initial values of the radius vector and velocity vector of an uncontrollable SC or programmed motion of SC. In equations (2.8) h^* is the known constant Keplerian energy of an uncontrollable SC, which is determined by the relation

$$h^* = (1/2)(v^*(0))^2 - fM/r^*(0). \tag{2.11}$$

The general solution of the system (2.8), (2.9) can be represented by means of the Stumpf function and has the form [6, 12]:

$$\begin{aligned} \mathbf{u}^*(\tau^*) &= c_0(z)\mathbf{u}^*(0) + \tau^* c_1(z)\mathbf{s}^*(0), \\ \mathbf{s}^*(\tau^*) &= c_1(z)[(h^*/2)\tau^*\mathbf{u}^*(0) + \mathbf{s}^*(0)] \\ &\quad + z[c_3(z) - c_2(z)]\mathbf{s}^*(0); \\ t(\tau^*) &= \|\mathbf{u}^*(0)\|^2\tau^* c_1(4z) + fM\tau^{*3}c_3(4z) \\ &\quad + 2\tau^{*2}[\mathrm{scal}(\bar{\mathbf{s}}^*(0) \circ \mathbf{u}^*(0))]c_2(4z), \\ z &= (-h^*/2)\tau^{*2}, \end{aligned} \tag{2.12}$$

where $c_j(z)$ and $c_j(4z)$, $j = \overline{0.3}$ are the Stumpf functions of arguments z and $4z$.

In the case of motion of an uncontrollable SC over an elliptical orbit $h^* < 0$ and equations (2.8) assume the form of the equations of motion of a single-frequency four-dimensional harmonic oscillator. Variables $\mathbf{u}^*$ and $\mathbf{s}^*$ in this case repesent harmonic functions of variable τ^* determined by the relations [6, 12]

$$\begin{aligned} \mathbf{u}^* &= k^{-1}\mathbf{s}^*(0)\sin(k\tau^*) + \mathbf{u}^*(0)\cos(k\tau^*), \\ \mathbf{s}^* &= \mathbf{s}^*(0)\cos(k\tau^*) - k\mathbf{u}^*(0)\sin(k\tau^*); \\ t &= \int_0^{\tau_k}\|\mathbf{u}^*(\tau^*)\|^2 d\tau^* \\ &= -\frac{fM}{2h^*}\tau^* + \frac{1}{2k}\left[\|\mathbf{u}^*(0)\|^2 + \frac{fM}{2h^*}\right]\sin(2k\tau^*) \\ &\quad - \frac{1}{h^*}[\mathrm{scal}(\bar{\mathbf{s}}^*(0) \circ \mathbf{u}^*(0))](1 - \cos(2k\tau^*)), \\ k &= (-0.5h^*)^{1/2}, \quad h^* = \mathrm{const} < 0. \end{aligned} \tag{2.13}$$

Thus, the final state of the phase point ($\mathbf{u}^*$, $\mathbf{s}^*$) is determined by relations (2.12) or (2.13) for $\tau^* = \tau_k^* = \tau^*(\tau_k)$. Further we will consider the motion of an uncontrollable SC over an elliptical orbit. Therefore, the final state of the phase point ($\mathbf{u}^*$, $\mathbf{s}^*$) will be determined by relations (2.13). The relationship between the independent variables τ and τ^* can be presented on the basis of (1.11) and (2.9) in the differential form

$$\frac{d\tau^*}{d\tau} = \frac{r(\tau)}{r^*(\tau^*)} = \frac{\|\mathbf{u}(\tau)\|^2}{\|\mathbf{u}^*(\tau^*)\|^2}. \tag{2.14}$$

For this reason we will include equation (2.14) into the set of differential equations of controllable SC motion. Note that the fourth equality (2.5) is automatically met when meeting conditions (2.6), in virtue of integral (1.16) and equality $\Psi_1 = -fM$.

The quality functional (2.7) in the problem under consideration characterizes the energy consumption for SC transition from the initial to the final state and the time spent for this transition. In the case when $\alpha_1 = 1$, $\alpha_2 = 0$, the functional $J = t_k$, and in this case the stated problem represents the problem of optimum fast response.

The stated problem represents the problem of “soft” rendezvous (docking) of controllable and uncontrollable SCs. Further we will consider also another problem—the problem of a “hard” rendezvous of SCs, in which the final state of a controllable SC should satisfy not relations (2.5), (2.6), but relations

$$\begin{aligned} \tau &= \tau_k, \\ \bar{\mathbf{u}}(\tau_k) \circ \mathbf{i}_1 \circ \mathbf{u}(\tau_k) &= \bar{\mathbf{u}}^*(\tau_k^*(\tau_k)) \circ \mathbf{i}_1 \circ \mathbf{u}^*(\tau_k^*(\tau_k)). \end{aligned} \tag{2.15}$$

Note that the initial values $u(0)$, $s(0)$, and $u^*(0)$, $s^*(0)$ of variables u, s and u^*, s^*, satisfying relations (2.4) and (2.10), can be uniquely defined in terms of $\mathbf{r}(0)$,

$\mathbf{v}(0)$ and $\mathbf{r}^*(0)$, $\mathbf{v}^*(0)$ using the algorithm described in [6, 16].

3. NECESSARY OPTIMALITY CONDITIONS AND TRANSVERSALITY CONDITIONS

We will solve the stated problem by means of the Pontryagin maximum principle. We introduce the conjugated variables: quaternion conjugated variables μ, ν, corresponding to quaternion phase variables $\mathbf{u}$, $\mathbf{s}$, and the scalar conjugated variables η, ϑ^*, corresponding to the Keplerian energy h and "time" τ^*. We compose the Hamilton function

$$\begin{aligned} H &= -r(\alpha_1 + \alpha_2 p^2) + \sum_{j=0}^{3} \mu_j s_j \\ &+ \sum_{j=0}^{3} \nu_j[(h/2)u_j + (r/2)q_j] \\ &+ 2\eta \sum_{j=0}^{3} s_j q_j + \vartheta^* r(r^*(\tau^*))^{-1} \qquad (3.1) \\ &= \sum_{j=0}^{3} \mu_j s_j + (h/2)\sum_{j=0}^{3} \nu_j u_j + (\vartheta^*(r^*(\tau^*))^{-1} - \alpha_1)r \\ &- \alpha_2 r p^2 + \sum_{j=0}^{3} \chi_j q_j, \end{aligned}$$

where

$$r = \sum_{j=0}^{3} u_j^2, \quad r^* = \sum_{j=0}^{3} u_j^{*2}, \quad \chi_j = (r/2)\nu_j + 2\eta s_j,$$

and quantities q_j are defined by (1.5) and (1.12).

The system of equations for the conjugated variables has the form:

$$\begin{aligned} d\boldsymbol{\mu}/d\tau &= -(h/2)\mathbf{v} + 2(\alpha_1 + \alpha_2 p^2)\mathbf{u} \\ &- 2\vartheta(r^*(\tau^*))^{-1}\mathbf{u} - (\mathbf{v}, \mathbf{q}_p)\mathbf{u} + \mathbf{i}_1 \circ \boldsymbol{\chi} \circ \mathbf{p}_x \\ &+ dr^{-4}(\mathbf{r}, \mathbf{n})\{(\mathbf{r}, \mathbf{n})[(-8r^{-1}(\boldsymbol{\chi}, \mathbf{q}_n) + (\mathbf{v}, \mathbf{q}_n))\mathbf{u} \\ &- \mathbf{i}_1 \circ \boldsymbol{\chi} \circ \mathrm{n}_x] + 4(\boldsymbol{\chi}, \mathbf{q}_n)\mathbf{q}_n\}, \qquad (3.2) \\ d\mathbf{v}/d\tau &= -\boldsymbol{\mu} - 2\eta\mathbf{q}, \\ d\eta/d\tau &= -(1/2)(\mathbf{v}, \mathbf{u}), \\ d\vartheta^*/d\tau &= 2\vartheta^* r r^{*-2}(\mathbf{u}^*, \mathbf{s}^*); \end{aligned}$$

$$\begin{aligned} \mathbf{q} &= -\mathbf{i}_1 \circ \mathbf{u} \circ (\mathbf{p}_x\ \ \mathbf{p}_{solx}), \\ \mathbf{p}_{solx} &= dr^{-4}(\mathbf{r}, \mathbf{n})^2\mathbf{n}_x = dr^{-4}(\mathrm{scal}(\mathbf{r}_x \circ \mathbf{n}_x))^2\mathbf{n}_x, \\ \mathbf{q}_p &= -\mathbf{i}_1 \circ \mathbf{u} \circ \mathbf{p}_x, \quad \mathbf{q}_n = -\mathbf{i}_1 \circ \mathbf{u} \circ \mathbf{n}_x, \\ (\mathbf{r} \cdot \mathbf{n}) &= -\mathrm{scal}(\mathbf{r}_x \circ \mathbf{n}_x), \quad \mathbf{r}_x = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \mathbf{u}, \qquad (3.3) \\ r &= \|\mathbf{u}\|^2 = \mathbf{u} \circ \bar{\mathbf{u}}, \quad r^* = \|\mathbf{u}^*\|^2 = \mathbf{u}^* \circ \bar{\mathbf{u}}^*, \\ (\mathbf{v}, \mathbf{q}_p) &= \mathrm{scal}(\mathbf{v} \circ \bar{\mathbf{q}}_p), \quad (\boldsymbol{\chi}, \mathbf{q}_n) = \mathrm{scal}(\boldsymbol{\chi} \circ \bar{\mathbf{q}}_n), \\ (\mathbf{v}, \mathbf{u}) &= \mathrm{scal}(\mathbf{v} \circ \bar{\mathbf{u}}), \quad (\mathbf{u}^*, \mathbf{s}^*) = \mathrm{scal}(\mathbf{u}^* \circ \bar{\mathbf{s}}^*), \\ \boldsymbol{\chi} &= 2\eta\mathbf{s} + (r/2)\mathbf{v}, \end{aligned}$$

where $(\mathbf{a}, \mathbf{b})$ is the scalar product of four-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ that is equal to the scalar part of the quaternion product $\mathbf{a} \circ \bar{\mathbf{b}}$. It follows from equation (3.2) that

$$\vartheta^* = Qr^*(\tau^*), \qquad (3.4)$$

where Q is the arbitrary constant.

According to the condition of maximum for the Hamilton–Pontryagin function, the optimum control for the unit vector of normal $\mathbf{n}$ to solar sail's plane is determined by the relation

$$\begin{aligned} \mathbf{n} &= (2/(3zr))\mathbf{r} + (1/(br))(z - 2/(3z))\boldsymbol{\kappa}_v, \\ b &= r^{-2}(\boldsymbol{\kappa}_v, \mathbf{r}), \\ z &= [(1/6)(4 - a^2 + a(8 + a^2)^{1/2})]^{1/2}, \qquad (3.5) \\ a &= br|\boldsymbol{\kappa}_v|^{-1}, \\ \boldsymbol{\kappa}_v &= \mathrm{vect}\boldsymbol{\kappa}, \quad \boldsymbol{\kappa} = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \boldsymbol{\chi}, \quad \boldsymbol{\chi} = 2\eta\mathbf{s} + (r/2)\mathbf{v}, \\ r &= \mathbf{u} \circ \bar{\mathbf{u}}, \end{aligned}$$

and for low thrust the optimum control is determined by the formula

$$\mathbf{p}_x = \begin{cases} (2\alpha_2 r)^{-1}\boldsymbol{\kappa}_v, & \text{if } |\boldsymbol{\kappa}_v| \le 2\alpha_2 r p_{\max}, \\ p_{\max}|\boldsymbol{\kappa}_v|^{-1}\boldsymbol{\kappa}_v, & \text{if } |\boldsymbol{\kappa}_v| > 2\alpha_2 r p_{\max}, \end{cases} \qquad (3.6)$$

$$\boldsymbol{\kappa}_v = \mathrm{vect}\boldsymbol{\kappa}, \quad \boldsymbol{\kappa} = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \boldsymbol{\chi},$$
$$\boldsymbol{\chi} = 2\eta\mathbf{s} + (r/2)\mathbf{v}, \quad r = \mathbf{u} \circ \bar{\mathbf{u}},$$

where $\boldsymbol{\kappa}_v = \mathrm{vect}\boldsymbol{\kappa}$ is the vector part of the quaternion $\boldsymbol{\kappa} = \kappa_0 + \boldsymbol{\kappa}_v = \kappa_0 + \kappa_1\mathbf{i}_1 + \kappa_2\mathbf{i}_2 + \kappa_3\mathbf{i}_3$; $(\boldsymbol{\kappa}_v, \mathbf{r})$ is the scalar product of three-dimensional vectors $\boldsymbol{\kappa}_v$ and $\mathbf{r}$.

Note that formula (3.5) follows from vector relationship $\mathbf{n} = (1/3)[2(\mathbf{r} \cdot \mathbf{n})^{-1}\mathbf{r} + (\boldsymbol{\kappa}_v \cdot \mathbf{n})^{-1}\boldsymbol{\kappa}_v]$, which was obtained from solution of the problem on the conventional extremum for the Hamilton–Pontryagin function, with regard to the fact that $|\mathbf{n}| = 1$, by the method of undefined Lagrange's multipliers. In the case of $\alpha_1 = 1$, $\alpha_2 = 0$ (the problem of fast response) the optimum control for the low thrust is determined by the relation

$$\mathbf{p}_x = p_{\max}|\boldsymbol{\kappa}_v|^{-1}\boldsymbol{\kappa}_v,$$
$$\boldsymbol{\kappa}_v = \mathrm{vect}\boldsymbol{\kappa}, \quad \boldsymbol{\kappa} = \bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \boldsymbol{\chi}, \quad \boldsymbol{\chi} = 2\eta\mathbf{s} + (r/2)\mathbf{v}, \tag{3.7}$$
$$r = \mathbf{u} \circ \bar{\mathbf{u}}.$$

So, the problem of construction of optimum controls and trajectories of SC motion in the Newtonian gravitational field is reduced to the integration of differential equations (2.2), (3.2), (3.3), (3.5), (3.6) in the case of $\alpha_2 > 0$, or (3.7)—in the case of $\alpha_2 = 0$. The equations form a closed system of differential equations of the twentieth order with respect to the variables u_j, s_j, h, τ^*, μ_j, ν_j, η, $\vartheta^*, j = \overline{0.3}$. The boundary conditions, which are necessary for solving the problem, are determined by the conditions indicated in Section 2 and by the transversality conditions.

The transversality conditions for the "soft" rendezvous have the form

$$\mathrm{scal}(\bar{\mathbf{u}}(\tau_k) \circ \mathbf{i}_1 \circ \mathbf{v}(\tau_k)) = 0,$$
$$\mathrm{scal}(\bar{\mathbf{u}}(\tau_k) \circ \mathbf{i}_1 \circ \boldsymbol{\mu}(\tau_k)) + \mathrm{scal}(\bar{\mathbf{s}}(\tau_k) \circ \mathbf{i}_1 \circ \mathbf{v}(\tau_k)) = 0,$$
$$\vartheta^*(\tau_k) = -\mathrm{scal}(\bar{\boldsymbol{\mu}}(\tau_k) \circ \mathbf{s}(\tau_k)) \tag{3.8}$$
$$-(h/2)\mathrm{scal}(\bar{\mathbf{v}}(\tau_k) \circ \mathbf{u}(\tau_k)),$$
$$\eta(\tau_k) = 0,$$

as those for the "hard" rendezvous have the form

$$\mathrm{scal}(\bar{\mathbf{u}}(\tau_k) \circ \mathbf{i}_1 \circ \boldsymbol{\mu}(\tau_k)) = 0,$$
$$\vartheta^*(\tau_k)$$
$$= -r^{-1}\mathrm{scal}[(\bar{\mathbf{u}}(\tau_k) \circ \mathbf{i}_1 \circ \boldsymbol{\mu}(\tau_k)) \circ \tag{3.9}$$
$$\circ (\bar{\mathbf{u}}^*(\tau_k^*) \circ \mathbf{i}_1 \circ \mathbf{s}^*(\tau_k^*))],$$
$$\mathbf{v}(\tau_k) = 0, \quad \eta(\tau_k) = 0.$$

The transversality conditions (3.8), (3.9) are obtained, respectively, from (2.6) and (2.15) by the method of undefined Lagrange multipliers with their subsequent elimination.

In addition, because τ_k is not specified in advance, for the optimum process at $\tau = \tau_k$, the condition for the Hamilton–Pontryagin function should be met:

$$H^{opt}\big|_{\tau_k} = 0. \tag{3.10}$$

In the case of solution of the problem on SC insertion into the specified orbit, the initial value of τ^* is not specified in advance. By this reason, the left end of the trajectory for the controllable system (1.15) occurs to be movable, and, on it, for $\tau = 0$ the transversality condition $\vartheta^* = 0$ should be met. In this case, it follows from (3.4) that $\vartheta^* = 0$.

4. ANALYSIS OF THE PROBLEM

The problem of construction of optimum controls and trajectories of SC motion in the Newtonian gravitational field was reduced to the to the boundary value problem described by the differential equations (2.2), (3.2), (3.5), (3.6) in the case of $\alpha_2 > 0$, or (3.5), (3.7), in the case of $\alpha_2 = 0$. The boundary conditions, which are necessary for solving the problem, are determined in the case of "soft" rendezvous by the relations (2.3)–(2.6), (2.10), (2.11), (2.13) (for $\tau^* = \tau_k^* = \tau^*(\tau_k)$), (3.8), (3.10). In the case of a "hard" rendezvous in the aforementioned conditions, instead of (2.5), (2.6) and (3.8) it is necessary to take (2.15) and (3.9), respectively. The obtained equations form the system of differential equations of the twentieth order with respect to variables u_j, s_j, h, τ^*, μ_j, ν_j, η, $\vartheta^*, j = \overline{0.3}$. In their integration twenty arbitrary constants will appear; the twenty-first unknown constant will be τ_k. For determining the constants we have twenty one conditions, and in the case of a "soft" rendezvous, sixteen boundary conditions (2.3), (2.6), four transversality conditions (3.8), and condition (3.10); and in the case of a "hard" rendezvous, we have thirteen boundary conditions (2.3), (2.15) and eight transversality conditions (3.9). In the case of solving the problem of optimum SC insertion into the specified orbit is necessary to solve the boundary value problem for the same system of equations with initial conditions (2.3) with $\tau = 0$, and, here, the specified initial condition for τ^* should be replaced by the transversality condition $\vartheta^* = 0$ for $\tau = 0$ and by the boundary conditions (2.6), (3.8), (3.10), for $\tau = \tau_k$.

Equations (2.2), (3.2), in addition to integrals (1.16), (1.17), have the integrals

$$H^{opt}(\tau) = 0, \tag{4.1}$$

$$\mathrm{scal}(\bar{\mathbf{s}} \circ \mathbf{i}_1 \circ \mathbf{v}) + \mathrm{scal}(\bar{\mathbf{u}} \circ \mathbf{i}_1 \circ \boldsymbol{\mu}) = \boldsymbol{\Psi} = \mathrm{const}, \tag{4.2}$$

$$\vartheta^*/r^* = Q = \mathrm{const}. \tag{4.3}$$

Besides, these equations have three scalar first integral

$$u_0\mu_1 - u_1\mu_0 - u_2\mu_3 + u_3\mu_2 + s_0\nu_1 - s_1\nu_0 - s_2\nu_3$$
$$+ s_3\nu_2 = c_1 = \mathrm{const},$$
$$u_0\mu_2 + u_1\mu_3 - u_2\mu_0 - u_3\mu_1 + s_0\nu_2 + s_1\nu_3 - s_2\nu_0$$
$$- s_3\nu_1 = c_2 = \mathrm{const},$$
$$u_0\mu_3 - u_1\mu_2 + u_2\mu_1 - u_3\mu_0 + s_0\nu_3 - s_1\nu_2$$
$$+ s_2\nu_1 - s_3\nu_0 = c_3 = \mathrm{const},$$

which in the quaternion form are as follows:

$$\mathrm{vect}(\bar{\mathbf{u}} \circ \boldsymbol{\mu} + \bar{\mathbf{s}} \circ \mathbf{v}) = \mathbf{c} = \mathrm{const},$$
$$\bar{\mathbf{u}} = u_0 - \mathbf{u}_v, \quad \boldsymbol{\mu} = \mu_0 + \mu_v, \quad \bar{\mathbf{s}} = s_0 - \mathbf{s}_v, \tag{4.4}$$
$$\mathbf{v} = \nu_0 + \nu_v,$$

and in the vector form they are:

$$u_0\boldsymbol{\mu}_v + s_0\mathbf{v}_v - \mu_0\mathbf{u}_v - \nu_0\mathbf{s}_v - \mathbf{u}_v \times \boldsymbol{\mu}_v - \mathbf{s}_v \times \mathbf{v}_v = \mathbf{c} = \text{const}.$$

The integral (4.1) takes place in virtue of stationarity (with respect to the independent variable τ) of the system of equations for the phase and conjugated variables and in virtue of (3.10). One can be convinced in the validity of integrals (4.2), (4.4) by direct checking. In virtue of the second of transversality conditions (3.8), in the integral (4.2) in the case of "soft" rendezvous the constant $\boldsymbol{\Psi} = 0$.

An analytical solution of the system of equations (2.2), (3.2), (3.5), (3.6) or (3.5), (3.7) is hardly possible in the general case. Therefore, in constructing the optimum controls and trajectories of SC motion, in the general case one has to rely upon the numerical solution of mentioned equations only, with using well-known techniques of solution of boundary value problems and integrals (1.16), (1.17), (4.1)–(4.4). The integrals (4.1)–(4.4), in which the conjugated variables are present, can be used for transferring the boundary conditions from the right end to the left one, which is important for numerical solution of the boundary-value problem. Note that the use of regular equations of the two-body problem (2.2) in the Kustaanheimo–Stiefel variables for numerical construction of optimum controls and trajectories of SC motion allows one to use the numerical integration techniques, which are superior to classical methods both in the accuracy, and in the volume of calculations on computers [12, 17].

5. EXAMPLES OF NUMERICAL SOLUTION OF THE PROBLEM

For numerical solution of the boundary-value problem, to which the Pontryagin maximum principle reduces the solution of the stated optimum control problem, one uses the combination of Newton's and gradient descent methods. The system of differential equations for determining the phase and conjugated variables is integrated by the Runge–Kutta method of the fourth order of accuracy. The use of KS-variables results in improving the convergence of the iterative process when solving the boundary-value optimum control problem. This is due to the fact that, in case of low thrust, the nonlinear terms in the system of differential equations (2.2), (3.2) occur to be small quantities.

For numerical solution of the problem, the transition to dimensionless variables was accomplished in the equations in accordance with the relations

$$\mathbf{u} = R^{1/2}\mathbf{u}^b, \quad \mathbf{s} = (fM)^{1/2}\mathbf{s}^b,$$
$$h = (fM/R)h^b, \quad \tau = (R/(fM))^{1/2}\tau^b,$$
$$t = R(R/(fM))^{1/2}t^b, \quad d = fMd^b, \quad \mathbf{r} = R\mathbf{r}^b,$$
$$\mathbf{v} = (fM/R)^{1/2}\mathbf{v}^b,$$
$$\mathbf{p} = (fM/R^2)\mathbf{p}^b, \quad p_{\max} = (fM/R^2)p^b_{\max}.$$

In these relations the dimensionless quantities are marked by the upper symbol "b"; R denotes the characteristic scale of length, such as the radius of the Earth's orbit, on which the controllable spacecraft is located at the initial time instant.

Note that the form of equations in dimensional and dimensionless variables is the same.

Below we present the results of numerical solution of two optimum control problems on a soft rendezvous of controllable and uncontrollable spacecrafts. For each problem three versions are considered, which differ in the values of phase coordinates (position and velocity) of an uncontrollable spacecraft at the initial time instant.

In the first problem the case is considered, where the controllable spacecraft does not undergo a low thrust effect ($\mathbf{p} = 0$), i.e., the spacecraft is controllable by means of the solar sail only, and the quality of control process is determined by the spent time (the problem of fast response in flight with a solar sail). In solving the first problem, in the Hamilton–Pontryagin function (3.1) and in the equations for conjugated variables (3.2) it is supposed that

$$\alpha_1 = 1, \alpha_2 = 0, \mathbf{p} = 0. \tag{5.1}$$

At the initial time instant the controllable spacecraft is located on the Earth orbit. Its position and velocity in the Cartesian coordinate system $OX_1X_2X_3$, whose origin lies at the center of the Sun, and the OX_1X_2 plane coincides with the plane of Earth's orbit, in the dimensionless variables is determined by the quantities: $x_1 = 1.0, x_2 = 0.0, x_3 = 0.0, \nu_1 = 0.0, \nu_2 = 1.0, \nu_3 = 0.0$.

The uncontrollable spacecraft is located near the orbit of Mars. The plane of orbit of an uncontrollable spacecraft is inclined to the OX_1X_2 plane. At the initial time instant the elongation between uncontrollable and controllable spacecrafts is determined by the angle φ_0. Table 1 gives the initial state of an uncontrollable spacecraft for three values of elongation.

The dimensionless quantity that characterizes the size of a sail and the mass of a spacecraft, is $d = 0.1$.

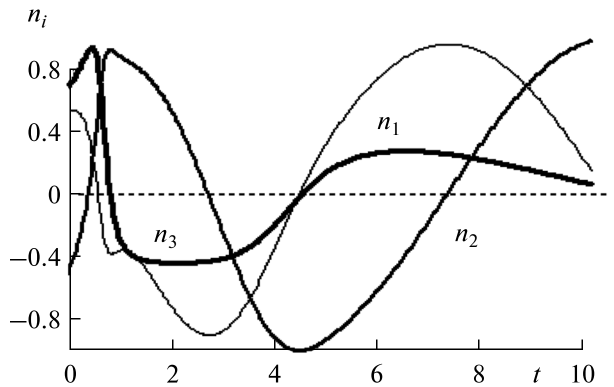

Fig. 1

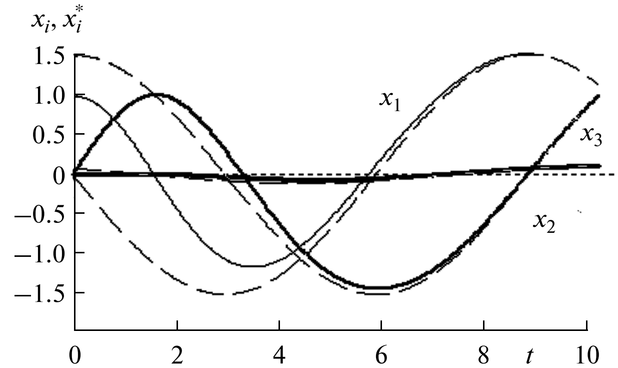

Fig. 2

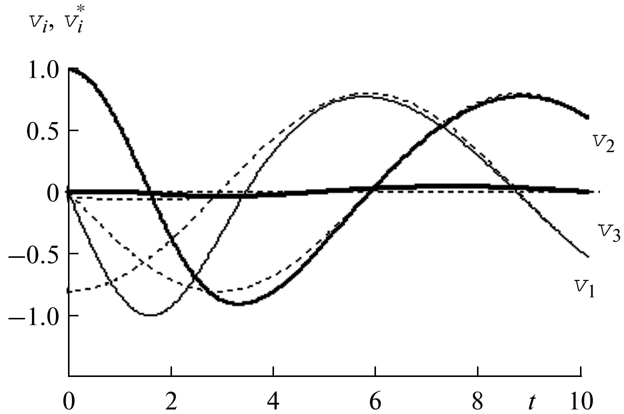

Fig. 3

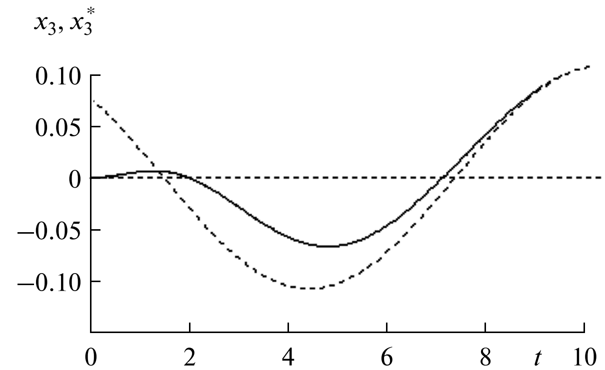

Fig. 4

Table 2, for the solution of the problem of fast response for three values of angle φ_0, indicates the intervals of dimensionless time of motion till the soft rendezvous and the time in earth years, as well as the dimensionless coordinates of the place of rendezvous.

Figures 2–5, for the solution of the first problem, in the case when $\varphi_0 = 90°$, presents the plots of variation of dimensionless Cartesian coordinates and projections of the velocity vector of a controllable spacecraft (solid lines) and uncontrollable one (dotted lines). Figure 1, for the same case, presents the plots of variation of projections of a unit vector of the normal to the plane of a solar sail.

In the second problem the controllable spacecraft is moving under the effect of a solar sail and low-thrust engine. In this case it is supposed that $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $d = 0.1$, $p_{max} = 0.1$.

Table 1

φ_0^0	x_1	x_2	x_3	v_1	v_2	v_3
75	0.3934	1.4682	0.0760	−0.7820	0.2095	−0.0405
90	0.0000	1.5200	0.0760	−0.8096	0.0000	−0.0405
120	−0.7600	1.3164	0.0760	−0.7011	−0.4048	−0.0405

Table 2

φ_0^0	Flight time	Flight time in Earth years	x_1	x_2	x_3
75	9.6764	1.5400	1.4972	0.2732	0.1013
90	10.1976	1.6230	1.1441	1.0022	0.1073
120	11.1576	1.7726	−0.2770	1.4946	0.0970

Table 3

φ_0^0	Flight time	Flight time in Earth years	x_1	x_2	x_3
75	4.6695	0.7431	−1.1952	−0.9325	−0.1061
90	5.0228	0.7994	−0.6698	−1.3609	−0.1015
120	5.5563	0.8843	0.5225	−1.4260	−0.0878

The initial states of controllable and uncontrollable spacecrafts, as in the first problem, are determined by relations (5.1) and by the data of Table 1, respectively. Table 3, for the solution of the second problem with the quality functional (2.7), indicates, for three values of angle φ_0, the intervals of time of motion till the soft rendezvous in dimensionless coordinates and in earth years, as well as the dimensionless coordinates of the place of rendezvous.

The time of motion in solution of the second problem is almost two times less than in the first one, because in the second problem the controllable spacecraft is moving not only under an effect of solar sail, but under an effect of low thrust as well.

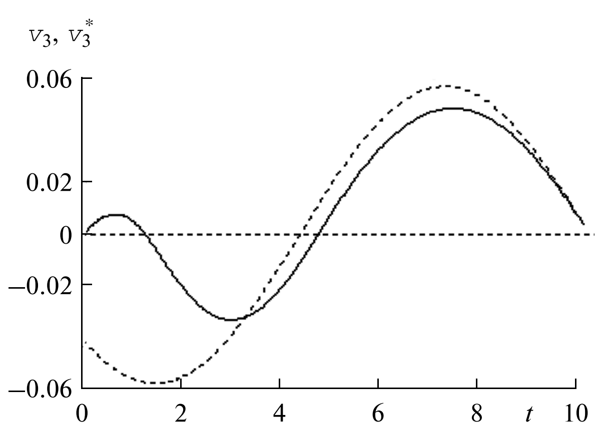

Fig. 5

Figures 8–11, for the solution of the second problem, in the case when $\varphi_0 = 90°$, presents the plots of variation of dimensionless Cartesian coordinates and projections of the velocity vector of a controllable spacecraft (solid lines) and uncontrollable one (dotted lines). Figures 6 and 7, for the same case, present the plots of variation of projections of a unit vector of the normal to solar sail's plane and the projections of the low-thrust vector in dimensionless variables. Figure 12 shows the trajectories of motion of a controllable (solid line) and uncontrollable (dotted line) spacecrafts.

Since the orbit of an uncontrollable SC has small angle of inclination to the plane of Earth's orbit, the dimensionless projections x_3 and v_3 of the radius-vector **r** and velocity vector **v** for controllable and uncontrollable SCs have small values. For this reason, in Figs. 4, 5, and 10, 11 the plots of variation of these quantities have been presented at a smaller scale.

It is seen from the plots for projections of a unit vector of the normal to solar sail's plane (the control vector **n**) (Figs. 1 and 6), that the projection n_3 of the control varies within a wide range, whereas the projection x_3 and v_3 of the radius-vector and velocity vector of SC assume small values as compared to the other projections of these vectors.

For checking the accuracy of solution of the boundary-value optimum control problem, the first integrals (1.16), (1.17), (4.1)–(4.4) of the system of

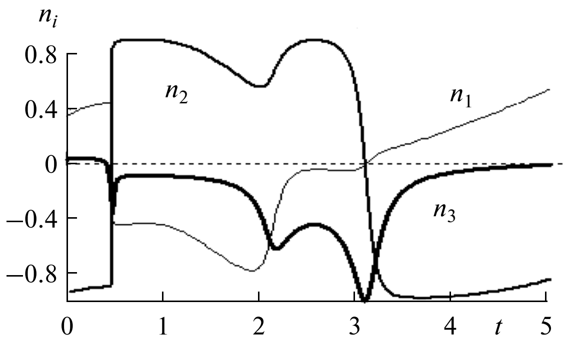

Fig. 6

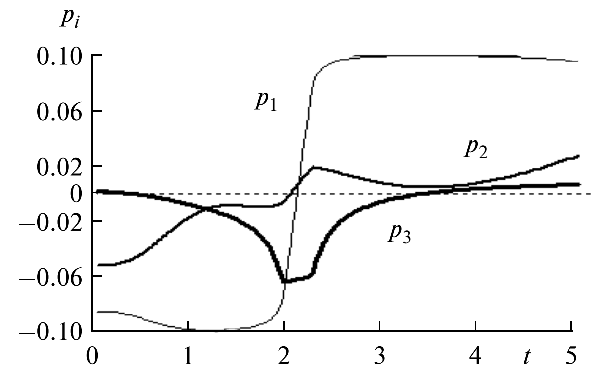

Fig. 7

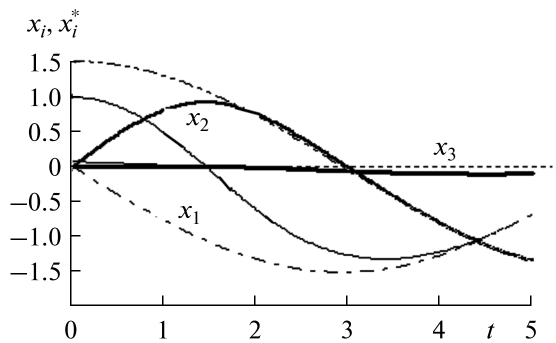

Fig. 8

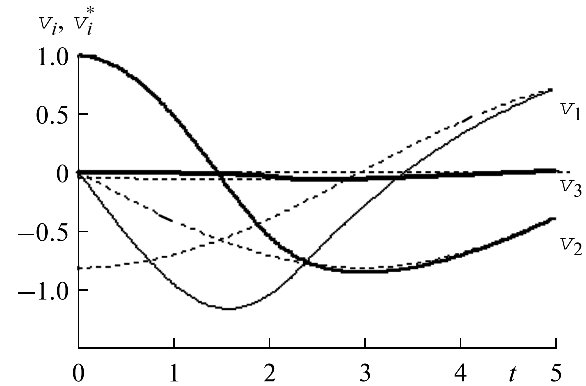

Fig. 9

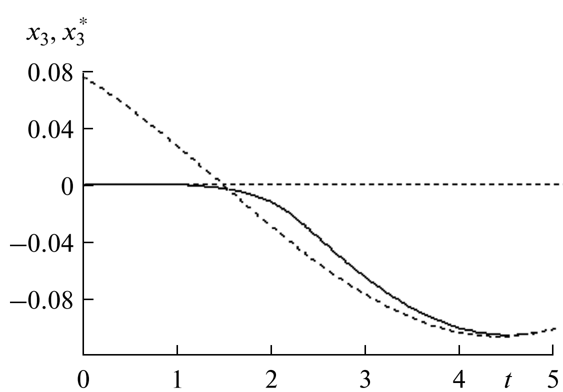

Fig. 10

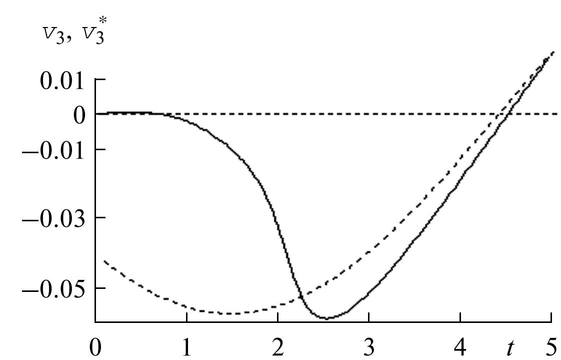

Fig. 11

differential equations (2.2), (3.2) have been calculated. Calculations have shown that the first twelve digits after the comma keep constant values in the integrals, which indicates the high accuracy of performed calculations.

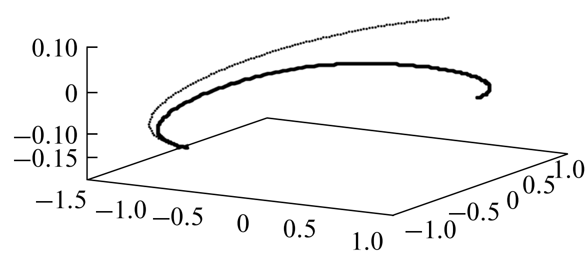

Fig. 12

ACKNOWLEDGMENTS

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