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Guaranteed Deterministic Approach to Superhedging: the Semicontinuity and Continuity Properties of Solutions of the Bellman–Isaacs Equations

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Abstract—We consider a guaranteed deterministic statement of the problem of discrete-time superreplication: the aim of hedging a contingent claim is to ensure the coverage of possible payout under the option contract for all feasible scenarios. These scenarios are given by a priori given compact sets that depend on the price history: the price increments at each time must lie in the corresponding compact sets. The lack of transaction costs is assumed; the market with trading constraints is considered. The game-theoretic interpretation implies that the corresponding Bellman–Isaacs equations hold. In the present paper, we propose several conditions for the solutions of these equations to be semicontinuous or continuous.

Keywords: guaranteed estimate, deterministic price dynamics, superreplication, option, Bellman–Isaacs equations, multivalued mapping, semicontinuity, continuity, robust no arbitrage condition

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1. INTRODUCTION

This paper continues a series of publications by the present author [3, 4]. These papers develop a model of the financial market and an uncertain deterministic evolution of prices with discrete time: asset prices evolve in a deterministic manner under uncertainty described using a priori information about possible price increments; namely, it is assumed that they lie in given compact sets depending on the price history (such a model is an alternative to the traditional probabilistic market model[1]).

Within the framework of the above-described market model, we study the problem of pricing options, by which we mean nondeliverable[2] OTC contracts the payments for which depend on the evolution of the prices of the underlying assets up to the moment of expiration. The seller of the option assumes a contingent liability that, unlike contingent insurance policy liabilities, can be protected against market risk through hedging in the markets.[3] One of the most important ways to hedge contingent liabilities on a sold option is the superreplication,[4] in other words, the superhedging (we prefer to use the second of the two equivalent terms).

In the aforementioned series of publications, we focus on the problem of superhedging options in the presence of trading constraints (within the framework of a financial market model with uncertain

[1] In the deterministic approach that we propose, a reference probability measure is not initially specified as it is assumed in the probabilistic approach; see, e.g., [8].

[2] For the purposes of risk management, derivative financial instruments are used, which, as a rule, are nondeliverable contracts.

[3] This includes transactions with underlying assets and a risk-free asset.

[4] This term originated from the fact that it is impossible to replicate contingent liabilities in incomplete markets (which is possible only in complete markets).

deterministic price evolution). The problem of pricing an option with superhedging is to determine the minimum level of funds at the initial moment[5] that are necessary for the seller when choosing an appropriate hedging strategy to guarantee coverage of the contingent obligation on the sold option (the payments for which, recall, depend on the price history).

We consider options of the American style (American options), when the counterparty of the seller (the owner of the option) can exercise the option (i.e., demand payment in accordance with the rules established by this contract) at any time up to the expiration of the option. Options of the European and Bermuda styles can be considered as a common case of American options under certain conditions of regularity, including the "no arbitrage" market in a certain sense.

Let us formalize the construction described above for the superhedging problem. The main premise in the proposed approach is the problem of "uncertain" price dynamics by the assumption of a priori information about price movements[6] at time t, namely, the assumption that the increments ΔX_t of discounted prices[7] lie in a priori given compact sets[8] $K_t(\cdot) \subseteq \mathbb{R}^n$, where the dot designates the price history up to time $t-1$ inclusive, $t = 1, \ldots, N$. By $v_t^*(\cdot)$ we denote the greatest lower bound of the portfolio value at time t, with a known history that guarantees, with a certain choice of an admissible hedging strategy, the validity of current and future obligations arising in relation to possible payments on an American option. The corresponding Bellman–Isaacs equations at discounted prices arise directly from the economic meaning by choosing the "best" admissible hedging strategy[9] at step t, $h \in D_t(\cdot) \subseteq \mathbb{R}^n$ for the worst-case scenario $y \in K_t(\cdot)$ of (discounted) price increments for given functions $g_t(\cdot)$ describing the potential payout of the option. Thus, we obtain the recurrence relations[10]

$$\begin{gathered} v_N^*(\bar{x}_N) = g_N(\bar{x}_N), \\ v_{t-1}^*(\bar{x}_{t-1}) = g_{t-1}(\bar{x}_{t-1}) \bigvee \inf_{h \in D_t(\bar{x}_{t-1})} \sup_{y \in K_t(\bar{x}_{t-1})} \left[v_t^*(\bar{x}_{t-1}, x_{t-1} + y) - hy \right], \\ t = N, \ldots, 1, \end{gathered} \tag{BA}$$

where $\bar{x}_{t-1} = (x_0, \ldots, x_{t-1})$ describes the back story with respect to the current time t. Conditions for (BA) to hold are stated in Theorem 3.1 in [3].

In this case, it is convenient (formally) to consider that $g_0 = -\infty$ (the lack of obligations for payments at the initial time); $g_t \geq 0$ for $t = 1, \ldots, N$ in the case of an American option. The set $D_t(\cdot)$ is assumed to be convex with $0 \in D_t(\cdot)$.

The multivalued mappings $x \mapsto K_t(x)$ and $x \mapsto D_t(x)$, as well as the functions $x \mapsto g_t(x)$, are assumed to be given for all $x \in (\mathbb{R}^n)^t$, $t = 1, \ldots, N$. Therefore, the functions $x \mapsto v_t^*(x)$ are given by Eqs. (BA) for all $x \in (\mathbb{R}^n)^t$.

In Eqs. (BA), the functions v_t^*, as well as the corresponding least upper bounds and greatest lower bounds, take values in the extended set of real numbers $\mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$—the two-point compactification[11] of $\mathbb{R}$.

[5] In other words, it is the premium charged to the option buyer if the option seller uses pricing that is consistent with superhedging.

[6] The increments are taken "backward," i.e. $\Delta X_t = X_t - X_{t-1}$, where X_t is the vector of discounted prices at time t; the ith component of this vector is the unit price of the ith asset.

[7] We assume that the risk-free asset has a constant price equal to one.

[8] The dot designates variables describing price evolution. More precisely, this is the back story, $\bar{x}_{t-1} = (x_0, \ldots, x_{t-1}) \in (\mathbb{R}^n)^t$ for K_t, while for the functions v_t^* and g_t introduced below, this is the history, $\bar{x}_t = (x_0, \ldots, x_t) \in (\mathbb{R}^n)^{t+1}$.

[9] The vector h describes the size of the positions taken in assets, i.e., the ith component of this vector is the number of units of the ith asset bought or sold.

[10] The $\bigvee$ sign denotes the maximum, and $hy = \langle h, y \rangle$ is the inner product of a vector h by a vector y.

[11] The neighborhoods of points $-\infty$ and $+\infty$ have the form $[-\infty, a)$, $a \in \mathbb{R}$ and $(b, +\infty]$, $b \in \mathbb{R}$, respectively.

The derivation of the Bellman–Isaacs equations (BA) is readily obtained by a reasoning of an "engineering" nature. In an informal economic language, this can be explained as follows, assuming, for simplicity, that the least upper bounds and greatest lower bounds in (BA) are attained. Let $t \le N$; by the current (present) time $t-1$ the history of (discounted) prices $x_1, \dots, x_{t-1}$ is known. The value V_{t-1} of the portfolio hedging the contingent obligation on a sold American option for guaranteed performance of obligations must be, first, not less than current obligations equal to the potential payments $g_t(x_1, \dots, x_{t-1})$. Second, the portfolio value $V_t = V_{t-1} + H_t \Delta X_t$ at the next time (here the strategy H_t is formed at time $t-1$ and can only depend on the price history $x_1, \dots, x_{t-1}$) must be guaranteed, for any scenario $\Delta X_t = y \in K_t(x_1, \dots, x_{t-1})$ of price movement at step t, not less than $v_t^*(x_1, \dots, x_{t-1}, x_{t-1} + y)$. Thus, to cover future obligations, the value of the portfolio V_{t-1} when choosing the strategy $H_t = h \in D_t(x_1, ..., x_{t-1})$ must be at least $v_t^*(x_1, \dots, x_{t-1}, x_{t-1} + y) - hy$ in the worst-case scenario $y \in K_t(x_1, ..., x_{t-1})$ of price movement at step t, i.e., with $y \in K_t(x_1, ..., x_{t-1})$ maximizing the expression $v_t^*(x_1, \dots, x_{t-1}, x_{t-1} + y) - hy$. The resulting value is minimized by choosing the strategy $h \in D_t(x_1, ..., x_{t-1})$ for estimating the required reserves to cover potential future payments. It remains to set $v_t^*(x_1, \dots, x_{t-1})$ equal to the maximum of two values: the current liabilities and the amount of reserves to cover potential future payments.

We will call a trajectory of asset prices $(x_0, \dots, x_t) = \bar{x}_t$ on the time interval $[0, t] = \{0, \dots, t\}$ possible if $x_0 \in K_0,\ \Delta x_1 \in K_1(x_0),\ \dots,\ \Delta x_t \in K_t(x_0, \dots, x_{t-1});\ t = 0, 1, \dots, N$. By B_t we denote the set of possible trajectories of asset prices on the time interval $[0, t]$; hence

$$B_t = \big\{(x_0, \dots, x_t) : x_0 \in K_0, \Delta x_1 \in K_1(x_0), \dots, \Delta x_t \in K_t(x_0, \dots, x_{t-1})\big\}. \tag{1.1}$$

Note that (1.1) is equivalent to the recurrence relations[12]

$$\begin{aligned} B_t &= \big\{(\bar{x}_{t-1}, x_t) :\ \bar{x}_{t-1} \in B_{t-1}, \Delta x_t \in K_t(\bar{x}_{t-1})\big\} \\ &= \big\{(\bar{x}_{t-1}, x_t) : \bar{x}_{t-1} \in B_{t-1}, x_t \in x_{t-1} + K_t(\bar{x}_{t-1})\big\}, \quad t = 1, \dots, N. \end{aligned} \tag{1.2}$$

One of the conditions for Eqs. (BA) to hold is the assumption, stated in Theorem 3.1 in [3], on the boundedness of the payout functions g_t, owing to which the functions v_t^* are bounded above. The assumption is as follows:

$$\begin{gathered} \text{there exist constants } C_t \ge 0 \text{ such that for each } t = 1, \dots, N \\ \text{and all possible trajectories } \bar{x}_t = (x_0, \dots, x_t) \in B_t \\ \text{one has } g_t(x_0, \dots, x_t) \le C_t. \end{gathered} \tag{B}$$

We will assume the constants C_t to be chosen minimal; i.e.,

$$C_t = \sup_{x\ \in B_t} g_t(x);$$

and we will denote

$$C = \bigvee_{t=1}^{N} C_t. \tag{1.3}$$

For convenience of notation, in the last variable we make an "additive" replacement of the functions v_t^*, assuming

$$w_t(\bar{x}_{t-1}, y) = w_t(x_1, \dots, x_{t-1}, y) = v_t^*(x_1, \dots, x_{t-1}, x_{t-1} + y), \tag{T}$$

[12] Here $x + A = \{z :\ z - x \in A\}$.

and will use w_t here and in what follows on the right-hand sides of the Bellman–Isaacs equations, $t = N, \ldots, 0$, i.e., in the form

$$v^*_{t-1}(\cdot) = g_{t-1}(\cdot) \bigvee \inf_{h \in D_t(\cdot)} \sup_{y \in K_t(\cdot)} \left[w_t(\cdot, y) - hy \right], \quad t = N, \ldots, 1.$$

Recall that here and in what follows, the dot denotes the "current" variables; in the last formula, for example, the argument is $\bar{x}_{t-1}$.

In what follows, we adopt the assumptions listed in Theorem 3.1 in [3] and the assumptions listed in item 1 of Remark 3.1 in [3] to be true.

The proposed approach allows one, to a certain extent, to simplify the mathematical technique and make the formulation of statements more understandable for economists; the advantages of the approach include game-theoretic interpretation.[13] Owing to the constructiveness of the approach, the question of the "smoothness" of solutions suggests itself immediately, directly from the form of Eqs. (BA), which is the main topic of this paper. These "smoothness" properties, as we will see from subsequent publications, will be required to establish one of the conditions of the game equilibrium (the upper semicontinuity of solutions of (BA)) as well as to establish conditions for the coincidence of solutions of the superhedging problem in the probabilistic and deterministic approaches (the continuity of solutions of Eqs. (BA)). Of particular interest is the circumstance established in this work that it is a coarse[14] (robust) condition of no guaranteed arbitrage with unlimited profit *NDSAUP* introduced in [4] that is generally required to obtain the continuity of solutions of Eqs. (BA).

2. GENERAL CONDITIONS FOR SEMICONTINUITY AND CONTINUITY OF SOLUTIONS OF THE BELLMAN–ISAACS EQUATIONS

The "smoothness" properties of solutions of the Bellman-Isaacs equations (BA) are determined by the corresponding "smoothness" properties of the payout functions $g_t(\cdot)$ as well as of the multivalued mappings $K_t(\cdot)$ and $D_t(\cdot)$ specifying the a priori information about price increments and trading constraints, respectively. Actually, it is the guaranteed deterministic approach that creates the incentive to study this "smoothness."

Note that the requirements for certain properties of "smoothness" of the description of market dynamics are obtained from considerations of the model of a financial market to be realistic with an uncertain deterministic evolution of prices, a fact that we have already noted in [3].

As realistic stochastic scenarios of market behavior we consider the (probability) distributions of a discrete-time stochastic process describing the evolution of prices for which the conditional distributions of the current price depend continuously (in the weak topology) on the price history. In other words, for the stochastic model of price dynamics, the transition kernels Q_t corresponding to the conditional probabilities of price $X_t \in \mathbb{R}^n$ at time t with a known history $\bar{X}_{t-1} = \bar{x}_{t-1} \in (\mathbb{R}^n)^t$ have the Feller property. If for the distribution of the vector $\bar{X}_t = (\bar{X}_{t-1}, X_t)$ there exists a (regular) version of the conditional distribution $P(X \in \cdot | \bar{X}_{t-1} = x) = Q_t(x, \cdot)$ that has the Feller property, then this option is unique for $x \in \text{supp}(P_{\bar{X}_{t-1}})$, where $P_{\bar{X}_{t-1}}$ is the distribution of the random vector $\bar{X}_{t-1}$ and $\text{supp}(\pi)$ denotes the (topological) support of the measure π. In this case, it is natural to choose the Feller version of the regular conditional distribution, and the deterministic

[13] With no trading constraints, this interpretation allows one to give an economically important explanation of the origin of risk-neutral probabilities as one of the properties of the most unfavorable mixed market strategies.

[14] One of our fundamental considerations is that since the description of uncertainty in the market cannot be accurate in practice, fundamental properties such as the "no arbitrage" market (in one sense or another), should not change under small disturbances of the market model. In this regard, in [4] we introduced a new concept of structural stability for the market to be "no arbitrage" and established criteria of a geometric nature for this property.

and stochastic approaches lead to the same notions of "no arbitrage," when the reference probability measure is specified using Feller transition kernels (by the Ionescu–Tulcea theorem).

We have proposed the following formalization of the realism of the model[15] in the context of the approach we are considering.

Definition 2.1. We say that a financial market model with an uncertain evolution of prices is realistic if there exist mixed market strategies for which there are Feller transition kernels $Q_t(x,\cdot)$—conditional distributions X_t with a known history $\bar{X}_{t-1} = x$ such that[16] $\operatorname{supp} Q_t(x,\cdot) = x + K_t(x)$ for $t = 1,\ldots,N$.

The results[17] obtained in [5] imply the following criterion for a realistic model.

Theorem 2.1. *For a realistic financial market model with uncertain deterministic price evolution*[18] *it is necessary and sufficient that the multivalued mappings $K_t(\cdot)$, $t = 1,\ldots,N$, be lower semicontinuous.*

Obviously, the semicontinuity properties (from above or from below) for the multivalued mappings $\bar{x}_{t-1} \mapsto K_t(\bar{x}_{t-1})$ and $\bar{x}_{t-1} \mapsto x_{t-1} + K_t(\bar{x}_{t-1})$, $t = 1,\ldots,N$, are equivalent, as is the case for the functions v_t^* and w_t, $t = 0,\ldots,N$.

Here and in what follows, by $\mathcal{N}(Y)$ we denote the class of all nonempty subsets of Y, and by $\mathcal{K}(Y)$ we denote the class of all nonempty compact subsets of the topological space Y.

Now let us establish sufficient conditions for the "regular behavior" of the set of possible trajectories for price dynamics that are given by the compact set K_0 of initial price states and compact-valued[19] mappings $\bar{x}_{t-1} \mapsto K_t(\bar{x}_{t_1})$, $t = 1,\ldots,N$. To this end, we make the following additional assumption:[20]

$$\begin{gathered}\text{the multivalued mapping } (x_0,\ldots,x_{t-1}) \mapsto K_t(x_0,\ldots,x_{t-1}) \\ \text{of } (\mathbb{R}^n)^t \text{ into } \mathcal{K}(\mathbb{R}^n) \text{ is upper } h\text{-semicontinuous.}\end{gathered} \tag{USC – PH}$$

Remark 2.1.

1. In the general case of a multivalued mapping, the upper semicontinuity for a metric space Y implies the upper h-semicontinuity,[21] see Proposition 2.61 in [9].
2. The lower semicontinuity and upper semicontinuity are equivalent for compact-valued mappings (Theorem 2.68 in [9]).

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[15] It is hardly possible to give economic reasons for stochastic price dynamics to be specified by transition kernels (conditional probabilities for a given price history) that do not satisfy the Feller property.

[16] The last condition reflects the realism of deterministic scenarios of price increments.

[17] In this paper, the general case of topological spaces is considered, and the necessary conditions for the existence of a Feller kernel with given supports are weaker than the sufficient conditions for its existence, but in the case of a finite-dimensional Euclidean space, the necessary and sufficient conditions coincide.

[18] In this case, the compactness of $K_t(\cdot)$, $t = 1,\ldots,N$ is not required—only closedness is assumed (as in potential topological supports of probability measures). Note that in the case of compact-valued mappings, the lower semicontinuity implies the lower h-semicontinuity.

[19] As per assumption (C) in [3].

[20] The upper (lower) semicontinuity in the sense of Pompeiu–Hausdorff, in other words, the upper (lower) h-semicontinuity of the multivalued mapping $F : X \mapsto \mathcal{N}(Y)$ at the point $x_0 \in X$, is defined for a topological space X and a metric space Y with a metric ρ as the continuity of the numerical function $x \mapsto e_\rho(F(x), F(x_0))$ (respectively, $x \mapsto e_\rho(F(x_0), F(x))$) at the point x_0, where $e_\rho(A,B)$ is the Pompeiu deviation of the set A from the set B, $e_\rho(A,B) = \sup\{\rho(x,B), x \in A\}$, $\rho(x,B) = \inf\{\rho(x,x'),\ x' \in B\}$. Note that the Pompeiu–Hausdorff distance is $h_\rho(A,B) = e_\rho(A,B) \bigvee e_\rho(B,A)$. A multivalued mapping is upper (lower) h-semicontinuous if it is upper (lower) semicontinuous at all points in the domain.

[21] Upper (lower) semicontinuity is defined as the openness of the set $\{x \in X :\ F(x) \subseteq G\}$ for any open $G \subseteq Y$ (respectively, as the openness of the sets $\{x \in X :\ F(x) \cap G \neq \emptyset\}$ for any open $G \subseteq Y$).

3. If a mapping $F : X \mapsto \mathcal{N}(Y)$ is an h-continuous multivalued mapping with closed values, then the graph $\{(x, y) \in X \times Y :\ y \in F(x)\}$ is closed (in the topology of the product of spaces); see Proposition 2.63 in [9].
4. The image[22] $F(x) = \bigcup\limits_{x\in K} F(x)$ of a compact set $K \subseteq X$ for a compact-valued semicontinuous mapping $F : X \mapsto \mathcal{K}(Y)$ is compact (Corollary 2.20 in [9]).

Proposition 2.1. *Let condition (USC-PH) be satisfied. Then*

1. *The sets B_t described by relations (1.2) are compact, $t = 0, \ldots, N$.*
2. *If, in addition, the potential payout functions g_t occurring in Eq. (BA) are upper semicontinuous, then the uniform boundedness condition (B) is satisfied.*

Proof. Assertion 1 is easy to verify by induction. Indeed, this holds for $t = 0$, because $B_0 = K_0$ is compact. If this is true for $t = 0, \ldots, s-1$, where $s \in \{1, \ldots, N-1\}$, then, with the use of (1.2), the set

$$B_s = \big\{(\bar{x}_{s-1}, x_s) : \bar{x}_{s-1} \in B_{s-1}, x_s = x_{s-1} + K_s(\bar{x}_{s-1})\big\}$$

is the graph of the multivalued mapping $F :\ B_{s-1} \mapsto \mathcal{K}(\mathbb{R})$, where $F_s(\bar{x}_{s-1}) = x_{s-1} + K_s(\bar{x}_{s-1})$, which is upper semicontinuous in the sense of Pompeiu–Hausdorff, and the set B_{s-1} is compact.

In accordance with item 3 in Remark 2.1, the set B_s is closed. By item 2 in Remark 2.1, the mapping F is upper semicontinuous, and by item 4 in Remark 2.1, the image $F(B_s)$ is compact. Since the closed set B_s is contained in the compact set $B_{s-1} \times F(B_s)$, it follows that the set B_s is compact.

Assertion 2 readily follows from Assertion 1, since the semicontinuous functions g_t are bounded above (and attain maximum) on the compact sets B_t for $t = 1, \ldots, N$. ∎

We need classical results—Berge's three theorems [6, 7], see also [9]. For the reader's convenience, we present their wording. Assume that X and Y are Hausdorff topological spaces.

Theorem 2.2 (Berge). *If a numerical function $g : X \times T \mapsto [-\infty, +\infty]$ is upper semicontinuous and a multivalued mapping $F : X \mapsto \mathcal{N}(Y)$ is lower semicontinuous, then the function $g_* : X \mapsto [-\infty, +\infty]$ defined by*

$$g_*(x) = \inf_{y\in F(x)} g(x, y) \tag{2.1}$$

is upper semicontinuous.

Theorem 2.3 (Berge). *If a numerical function $g : X \times T \mapsto [-\infty, +\infty]$ is upper semicontinuous and a compact-valued mapping $F : X \mapsto \mathcal{K}(Y)$ is upper semicontinuous,*[23] *then the function $g^* : X \mapsto [-\infty, +\infty]$ defined by*

$$g^*(x) = \sup_{y\in F(x)} g(x, y) \tag{2.2}$$

is upper semicontinuous.

Remark 2.2. Since

$$-\inf_{y\in F(x)} \big[-g(x, y)\big] = \sup_{y\in F(x)} g(x, y), \tag{2.3}$$

we see that Theorem 2.2 can be stated in an equivalent form.[24]

[22] Here image is understood in the sense of a multivalued mapping.
[23] Note that in the book [6], the compactness of F is included in the definition of upper semicontinuity, in addition to the fact that the set $\{x \in X :\ F(x) \subseteq G\}$ is open for any open $G \subseteq Y$.
[24] Theorem 2.2′ is Proposition 3.1 in [9], and Theorem 2.3 is Proposition 3.3 in [9].

Theorem 2.2′. *For a lower semicontinuous function $g : X \times Y \mapsto [-\infty, +\infty]$ and a lower semicontinuous multivalued mapping $F(x)$, the function $g^* : X \mapsto [-\infty, +\infty]$ defined by (2.2) is lower semicontinuous.*

Accordingly, Theorem 2.3 can also be stated in equivalent form.

Theorem 2.3′. *For a lower semicontinuous function $g : X \times Y \mapsto [-\infty, +\infty]$ and an upper semicontinuous compact-valued mapping $F : X \mapsto \mathcal{K}(Y)$, the function $g_* : X \mapsto [-\infty, +\infty]$ defined by (2.1) is lower semicontinuous.*

A statement similar to Theorem 2.2 can also be obtained by weakening the requirements for the function g but strengthening the requirements for the multivalued function F.

Proposition 2.2. *If for each $y \in Y$ the functions $x \mapsto g_y(x) = g(x, y) \in [-\infty, +\infty]$, $x \in X$, are upper semicontinuous and the sets $\{x \in X : \ y \in F(x)\}$ are open, then the function $g_* : X \mapsto [-\infty, +\infty]$ defined by (2.1) is upper semicontinuous.*

Proof. Set

$$\varphi_y(x) = \begin{cases} 0 & \text{if } y \in F(x) \\ +\infty & \text{if } y \notin F(x). \end{cases}$$

Then

$$g_*(x) = \inf_{y \in Y} \left[g_y(x) + \varphi_y(x) \right]$$

with the function $\varphi_y(\cdot)$ being upper semicontinuous and hence the function $g_y(\cdot) + \varphi_y(\cdot)$ being upper semicontinuous as well. Since the greatest lower bound of upper semicontinuous functions is upper semicontinuous, it follows that the function $g_*(\cdot)$ is upper semicontinuous. ■

By using identity (2.3), we can restate Proposition 2.2 in an equivalent manner.

Proposition 2.2′. *If for each $y \in Y$ the functions $x \mapsto g_y(x) = g(x, y) \in [-\infty, +\infty]$ are lower semicontinuous and the sets $\{x \in X : \ y \in F(x)\}$ are open, then the function g^* defined by (2.2) is lower semicontinuous.*

Remark 2.3.

1. If for each $y \in Y$ the sets $\{x \in X : \ y \in F(x)\}$ are open, then the multivalued function F is lower semicontinuous, since for an open $G \subseteq Y$ the sets $\{x \in X : \ F(x) \cap G \neq \emptyset\} = \bigcup_{y \in G} \{x : \ y \in F(x)\}$ will be open as unions of open sets.
2. The openness of the sets $\{x \in X : \ y \in F(x)\}$, generally speaking, does not follow from the lower (or upper) semicontinuity of the mapping F. In the case of a Y-metric space with a metric ρ, it is well known (see [9, Propositions 2.26, 2.61, and 2.64]) that for each $y \in Y$ the functions $x \mapsto \rho(y, F(x))$ are upper (or, respectively, lower) semicontinuous. If F takes closed values, i.e., the set $F(x)$ is closed for each $x \in X$, then about the set $\{x \in X : \ y \in F(x)\} = \{x \in X : \rho(y, F(x)) = 0\}$ one can only assert that it is closed (or, accordingly, is a set of the type[25] G_δ).
3. The joint upper (lower) semicontinuity of the numerical function $g(x, y)$ implies the upper (lower) semicontinuity of the functions $x \mapsto g_y(x) = g(x, y)$ of one variable: if g is upper semicontinuous, then considering the Cauchy net (x_α, y_α), where $x_\alpha \equiv x$ and $y_\alpha \to y^*$, we note that $(x_\alpha, y_\alpha) \to (x, y^*)$ and hence $\limsup_\alpha g_x(y_\alpha) = \limsup_\alpha g(x_\alpha, y_\alpha) \leq g(x, y^*) = g_x(y^*)$.
   In a similar way, we can also prove the assertion about the lower semicontinuity; however,

[25] That is, representable in the form of a countable intersection of open sets.

this also follows from what has already been proved by selecting $g = -f$, where f is lower semicontinuous.

4. In Theorems 2.3 and 2.3′, the maximum and minimum, respectively, are attained for each $x \in X$ at some $y^*(x) \in F(x)$.

Sufficient conditions for the continuity of the function g^* defined by (2.2) are given by the following theorem.[26]

Theorem 2.4 (Berge). *Let a function $g : X \times Y \mapsto \mathbb{R}$ be continuous, and let $F : X \mapsto \mathcal{K}(y)$ be a continuous[27] multivalued mapping. Then the function $g^* : Y \mapsto \mathbb{R}$ defined by (2.2) is continuous, and the multivalued mapping $M : X \mapsto \mathcal{K}(Y)$, where[28] $M(x) = \{y \in F(x) : \ g(x, y) = g^*(x)\}$, is upper semicontinuous.*

From the above results, one can readily obtain conditions of semicontinuity for the solution v_t^* of the main equations (BA). Using notation (T), we introduce the function

$$\bar{x}_{t-1} \mapsto \rho_t(\bar{x}_{t-1}) = \inf_{h \in D_t(\bar{x}_{t-1})} \sup_{y \in K_t(\bar{x}_{t-1})} \big[w_t(\bar{x}_{t-1}, y) - hy\big]. \tag{2.4}$$

Theorem 2.5. *Let condition (USC-PH) of the upper h-semicontinuity of multivalued mappings $\bar{x}_{t-1} \mapsto K_t(\bar{x}_{t-1})$, $t = 1, \dots, N$, be satisfied. Further, assume that the following conditions are also satisfied for $t = 1, \dots, N$:*

1. *The multivalued mappings $\bar{x}_{t-1} \mapsto D_t(\bar{x}_{t-1})$ are lower semicontinuous.*
2. *The numerical functions $\bar{x}_t \mapsto g_t(\bar{x}_t)$ are upper semicontinuous.*

Then the functions $\bar{x}_t \mapsto v_t^*(\bar{x}_t)$ defined by relations (BA) are upper semicontinuous, $t = 1, \dots, N$.

Proof. Let conditions 1 and 2 be satisfied. Let us prove the upper semicontinuity of the functions v_s by induction. For $s = N$ the assertions in the theorem obviously hold true, since $v_N^* = g_N$. Now let this be true for $s = N, \dots, t$. Let us show that this is also true for $s = t-1$, where $N \geq t > 1$. By the induction assumption, the function $(\bar{x}_{t-1}, y) \mapsto w_t(\bar{x}_{t-1}, y)$ is upper semicontinuous. Using (USC-PH) and Theorem 2.3, we conclude that the function

$$(\bar{x}_{t-1}, h) \mapsto \varphi(x, h) = \sup_{y \in K_t(\bar{x}_{t-1})} \big[w_t(\bar{x}_{t-1}, y) - hy\big], \tag{2.5}$$

is upper semicontinuous, because the function $((\bar{x}_{t-1}, h), y) \mapsto w_t(\bar{x}_{t-1}, y) - hy$ is jointly upper semicontinuous. Applying Theorem 2.2 to the function defined by (2.5), we conclude that the function ρ_t defined by (2.4) is upper semicontinuous, and consequently, so is the function

$$\bar{x}_{t-1} \mapsto v_{t-1}^*(\bar{x}_{t-1}) = g_{t-1}(\bar{x}_{t-1}) \bigvee \rho_t(\bar{x}_{t-1}). \quad \blacksquare \tag{2.6}$$

Assume that

cl(A) is the closure of A.

int(A) is the interior of A.

ri(A) is the relative interior of the convex set A.

Lemma 2.1. *Let X be a Hausdorff topological space, and let $F : X \mapsto 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ be a multivalued mapping taking convex values. Then*

[26] This result is often called "Berge's maximum theorem;" in [9], it is Theorem 3.4.

[27] Simultaneously upper and lower semicontinuous multivalued mapping.

[28] That is, $M(x)$ is the set of maximizers, those $y \in Y$ for which the maximum is attained in (2.2) for a given $x \in X$.

1. *The lower semicontinuity of $F(\cdot)$ is equivalent to the lower semicontinuity of the multivalued mapping $x \mapsto \mathrm{ri}(F(x))$.*
2. *If, in addition, $\mathrm{int}(F(x)) \neq \emptyset$ for all $x \in X$, then the lower semicontinuity $F(\cdot)$ is equivalent to the lower semicontinuity of $x \mapsto \mathrm{int}(F(x))$.*

Proof.

1. Since

$$\mathrm{ri}\left(F(x)\right) \subseteq F(x) \subseteq \mathrm{cl}\left(F(x)\right) \tag{2.7}$$

and, in view of the convexity of $F(x)$, by Theorem 6.3 in [11] we obtain

$$\mathrm{cl}\left(\mathrm{ri}\left(F(x)\right)\right) = \mathrm{cl}\left(F(x)\right). \tag{2.8}$$

By Proposition 2.38 in [9], the multivalued mapping $x \mapsto F(x)$ is lower semicontinuous if and only if so is the mapping $x \mapsto \mathrm{cl}(F(x))$, and so (2.7) and (2.8) imply the desired assertion.

2. In the case of $\mathrm{int}(F(x)) \neq \emptyset$, one has

$$\mathrm{ri}\left(F(x)\right) = \mathrm{int}\left(F(x)\right). \quad \blacksquare \tag{2.9}$$

Remark 2.4.

1. The nondegeneracy of the trading constraints, by which we mean the solidity of the convex set $D_t(\cdot)$, i.e., $\mathrm{int}(D_t(\cdot)) \neq \emptyset$, is quite a natural assumption for financial models; in this case, by virtue of the convexity of $D_t(\cdot)$, we can apply Lemma 2.1, so, without loss of generality, the sets $D_t(\cdot)$ can be considered to be open when we speak of the preservation of lower semicontinuity of $D_t(\cdot)$ with no other requirements.
2. On the other hand, the property of lower semicontinuity of the multivalued mapping $F : X \mapsto \mathcal{N}(Y)$ is equivalent to the lower semicontinuity of the multivalued mapping $\bar{F} : X \mapsto \mathcal{N}(Y)$, where $\bar{F}(x) = \mathrm{cl}(F(x))$, see Proposition 2.38 in [9]. In a number of cases, for a lower semicontinuous $D_t(\cdot)$ it is convenient to assume, without loss of generality, that the sets $D_t(\cdot)$ are closed (nondegeneracy is not required in this case) with no other requirements.
3. In Theorem 2.5 we can assume, without loss of generality, that the sets $D_t(\cdot)$ are closed, because the values $v_t^*(\cdot)$ do not change under closure of $D_t(\cdot)$. Indeed, denoting $\bar{D}_t(\cdot) = \mathrm{cl}(D_t(\cdot))$, we have the inequality

$$\inf_{h \in \bar{D}_t(x)} \varphi(x,h) \leq \inf_{h \in D_t(x)} \varphi(x,h), \tag{2.10}$$

where the function φ is given by formula (2.5). For each $h \in \bar{D}_t(x)$, there exists a sequence $h^n \in D_t(x)$ such that $h^n \to h$. By virtue of the upper semicontinuity of the function φ (established when proving Theorem 2.5)

$$\varphi(x,h) \geq \limsup_{n\to\infty} \varphi(x,h^n) \geq \inf_{h \in \bar{D}_t(x)} \varphi(x,h),$$

and so the equality takes place in (2.10).

On the other hand, applying relations (2.7) and (2.8) to $F(\cdot) = D_t(\cdot)$, we conclude that the value v_t^* does not change if $D_t(\cdot)$ is replaced by $D_t'(\cdot) = \mathrm{ri}(D_t(\cdot))$, and in the case of nondegeneracy, considering (2.9), we can consider, without loss of generality, that $D_t(\cdot)$ are open.

4. For a number of models, trading constraints are time constant and history independent, i.e., $D_t(\cdot) \equiv D$ (for example, when short positions on a risky asset are prohibited, i.e.,

when $D = [0,\infty)^n$). In this case, the set $\{x \in (\mathbb{R}^n)^{t-1} : \ h \in D_t(\cdot)\}$ is either empty (if $h \notin D$) or coincides with the entire space $(\mathbb{R}^n)^{t-1}$ (if $h \in D$), so the set is open (and hence Propositions 2.2 and 2.2′ are applicable) and closed at the same time, and the multivalued mappings $x \mapsto D_t(\cdot)$ are continuous.

5. By $\tilde{K}_t = K_t(B_{t-1})$ we denote the image B_{t-1} of the multivalued mapping K_t, where B_t is given by (1.1). Note that under condition (USC-PH) the function defined by (2.5) is bounded owing to the compactness of $\tilde{K}_t$, by Proposition 2.1, uniformly in $\bar{x}_{t-1} \in \tilde{K}_t$ for each h, because $C \geq w_t \geq 0:$ for each $y \in K_t(\bar{x}_{t-1}) \subseteq \tilde{K}_t$,

$$C - hy \geq w_t(\bar{x}_{t-1}, y) - hy \geq -hy,$$
$$\max_{y \in -\tilde{K}_t} hy \geq -hy \geq \min_{y \in -\tilde{K}_t} hy = -\max_{y \in \tilde{K}_t} hy,$$

based on which,[29] we conclude that

$$C + \sigma_{-\tilde{K}_t}(h) \geq \sup_{y \in K_t(\bar{x}_{t-1})} \big[w_t(\bar{x}_{t-1}, y) - hy\big] \geq -\sigma_{\tilde{K}_t}(h), \tag{2.11}$$

where $\sigma_A(h)$ is the support function of the set A.

6. By virtue of the upper semicontinuity of the function $y \mapsto w_t(\cdot, y) - hy$ in the conditions of Theorem 2.5 (i.e., under conditions 1 and 2), the least upper bound over y is attained at some value $y \in K_t(\bar{x}_{t-1})$.
7. Note that for $\bar{x}_{t-1} \in B_{t-1}$ the function (2.5) is convex in the variable h, takes finite values with the estimate (2.11), and, in particular, is continuous with respect to h (Corollary 10.1.1 in [11]).
8. The function $\rho_t(\cdot)$ defined by (2.4) can take the value $-\infty$. At the same time, since $0 \in D_t(x)$, the function $\rho_t(\cdot)$ is bounded above, $\rho_t(\cdot) \leq C$.

To ensure the property of lower semicontinuity of the Bellman–Isaacs functions defined by (BA), we need to make an additional assumption concerning the trading constraints; namely,

$$\text{the set } D_t(\bar{x}_{t-1}) \text{ is compact for each } \bar{x}_{t-1}, \quad t = 1, \dots, N. \tag{C − T}$$

Theorem 2.6. *Let condition (C-T) be satisfied, let complex-valued mappings $\bar{x}_{t-1} \mapsto D_t(\bar{x}_{t-1})$ be upper semicontinuous, let $\bar{x}_{t-1} \mapsto K_t(\bar{x}_{t-1})$ be lower semicontinuous, and let the numerical functions $\bar{x}_{t-1} \mapsto g_t(\bar{x}_{t-1})$ be lower semicontinuous for $t = 1, \dots, N$. Then the functions $\bar{x}_t \mapsto v_t^*(\bar{x}_t)$ defined by relations (BA) are lower semicontinuous, $t = 1, \dots, N$.*

Proof. We conduct the proof by induction. For $s = N$, the assertion holds since $v_N^* = g_N$. Let it hold for $s = N, \dots, t$. Let us show that it is also satisfied for $s = t-1$ (for $t > 1$). Theorem 2.2′ can be applied to the function φ defined by (2.5), because the function $((\bar{x}_{t-1}, h), y) \mapsto w_t(\bar{x}_{t-1}, y) - hy$ is jointly lower semicontinuous, and so φ is lower semicontinuous. Further, Theorem 2.3′ applies to the function $\rho_t(\cdot)$ defined by (2.4), and so this function is lower semicontinuous as well, and (2.6) implies the lower semicontinuity of the function $v_{t-1}^*(\cdot)$. ■

Theorem 2.7. *Let condition (C-T) be satisfied, let the compact-valued mappings $\bar{x}_{t-1} \mapsto D_t(\bar{x}_{t-1})$ and $\bar{x}_{t-1} \mapsto K_t(\bar{x}_{t-1})$ be continuous,[30] and let the numerical functions $\bar{x}_{t-1} \mapsto g_t(\bar{x}_{t-1})$ be continuous. Then*

1. *The functions $\bar{x}_t \mapsto v_t^*(\bar{x}_t)$ given by relations (BA) are continuous.*

[29] The support function of a compact set takes finite values.

[30] The continuity of a multivalued mapping means simultaneously upper and lower semicontinuity. For compact-valued mappings, this is equivalent to continuity in the Pompeiu–Hausdorff metric.

2. *The multivalued mappings* $(\bar{x}_{t-1}, h) \mapsto M_t(\bar{x}_{t-1}, h)$, *where* $M_t(\bar{x}_{t-1}, h)$ *is the set of maximizers* $y \in K_t(\bar{x}_{t-1})$ *for which the maximum of the function* (2.5) *is attained, as well as the multivalued mappings* $\bar{x}_{t-1} \mapsto N_t(\bar{x}_{t-1})$, *where* $N_t(\bar{x}_{t-1})$ *is the set of minimizers* $h \in D_t(\bar{x}_{t-1})$ *for which the minimum in* (2.4) *is attained, are lower semicontinuous,* $t = 1, \ldots, N$.

Proof. The assertion follows from Theorems 2.5 and 2.6 as well as Berge's theorem 2.4. ■

Remark 2.5.

1. If Y is a metric space,[31] then the upper semicontinuity property of the multivalued mapping is preserved under closure, see Proposition 2.40 in [9]. Therefore, if $D_t(\cdot)$ is replaced by $\bar{D}_t(\cdot) = \mathrm{cl}(D_t(\cdot))$, then Theorem 2.7 remains valid in view of item 2 in Remark 2.4 on the preservation of the value of the function $v_t^*(\cdot)$.
2. In fact, the above results also give the conditions of semicontinuity or continuity in a more general case, namely, for the model described in [3], formulas (3.6) and (3.7), if simultaneously with the trading constraints inherent in margin trading,[32] we impose constraints on borrowing a risk-free asset by specifying an h-convex function[33] $\alpha_t(h, \bar{x}_{t-1}) \leq 0$, with auxiliary constraints $h \in D_t(\cdot)$, the corresponding Bellman–Isaacs equations have the form

$$v_N^*(\bar{x}_N) = g_N(\bar{x}_N),$$
$$v_{t-1}^*(\bar{x}_{t-1}) = g_{t-1}(\bar{x}_{t-1}) \bigvee \inf_{h \in D_t(\bar{x}_{t-1})} \left[\sup_{y \in K_t(\bar{x}_{t-1})} (w_t(\bar{x}_{t-1}, y) - hy) \bigvee \mu h^{\oplus} \bar{x}_{t-1} \bigvee (h\bar{x}_{t-1} + \alpha_t(h, \bar{x}_{t-1})) \right],$$

   where $h^{\oplus} = ((h^1)^+, \ldots, (h^n)^+)$, $(a)^+ = 0 \bigvee a$, and $a \in \mathbb{R}$.

   For the applicability of Theorem 2.5, it is necessary to require the upper semicontinuity of the function $x \mapsto \alpha_t(h, x)$, $x \in B_{t-1}$; for the applicability of Theorem 2.6, the joint lower semicontinuity of α_t; and for the applicability of Theorem 2.7, the joint continuity of α_t.
3. Note that all the above results are of a general nature and are in no way connected with the assumptions of the no arbitrage type. To ensure the properties of lower semicontinuity and continuity of the Bellman–Isaacs functions w_t in the case when (C-T) fails, i.e., when D_t is unbounded, we need additional conditions relating the behavior of the multivalued mappings $K_t(\cdot)$ and $D_t(\cdot)$. Below we give the relevant conditions and proofs, which are more technical in nature than those presented above.

3. SMOOTHNESS CONDITIONS FOR SOLUTIONS OF THE BELLMAN–ISAACS EQUATIONS LINKING UNCERTAINTY OF PRICE MOVEMENTS AND TRADE RESTRICTIONS

We will use the notation

$K_t^*(\cdot) = \mathrm{conv}(K_t(\cdot))$ is the convex hull of $K_t(\cdot)$.

σ_A is the support function of the set A; i.e., $\sigma_A(y) = \sup_{h \in A} hy$.

$\mathrm{bar}(A) = \{y \in \mathbb{R}^n : \sigma_A(y) < +\infty\}$ is the barrier cone[34] of the set A.

[31] In fact, the normality of the topological space Y suffices.

[32] Margin trading in the financial market implies the presence of intermediaries (brokers) who allow (upon the conclusion of a general agreement with a market participant) borrowing in securities. At the same time, the regulator usually establishes requirements that the share of own funds in the portfolio of a participant in margin trading should not be lower than the established level; this leads to trading constraints.

[33] α_t is the maximum allowable debt (for example, bank credit limit) taken with negative sign.

[34] This cone is convex and contains the point 0.

In [4], the notion of structural stability of "no arbitrage" was introduced and it was proved (Theorem 2) that the structurally stable condition of no guaranteed arbitrage *RNDSAUP* is equivalent to the validity of the condition

$$0 \in \operatorname{int}\left\{z: \; z + K_t^*(\cdot) \bigcap \operatorname{bar}(D_t(\cdot)) \neq \emptyset\right\}. \tag{SR}$$

This condition and its strengthening play an important role in the proof of the results of this section, which are stated below. Note also that the proof of these results will essentially use the boundedness condition (B) for the payoff functions g_t, $t = 1, \ldots, N$.

Proposition 3.1. *Let the multivalued mappings $D_t(\cdot)$ take closed values for $t \in \{1, \ldots, N\}$, and let condition (B) be satisfied. Then*

$$D_t^a(\cdot) = \{h \in D_t(\cdot): \sup_{y \in K_t(\cdot)} [w_t(\cdot, y) - hy] \leq a\}$$

is compact[35] *for each $a \in \mathbb{R}$ if and only if condition RNDSAUP is satisfied.*

Proof. Denote

$$T_t(\cdot) = \Big\{z \in \mathbb{R}^n: \; \big(z + K_t^*(\cdot)\big) \cap \operatorname{bar}\big(D_t(\cdot)\big) \neq \emptyset\Big\}; \tag{3.1}$$

condition (SR), equivalent to *RNDSAUP*, is written as

$$0 \in \operatorname{int}\big(T_t(\cdot)\big). \tag{3.2}$$

Since $w_t \geq 0$, as well as owing to (2.1), $w_t \leq C$, we have the inequalities

$$\sup_{y \in K_t^*(\cdot)} [-hy] = \sup_{y \in K_t(\cdot)} [-hy] \leq \sup_{y \in K_t(\cdot)} \big[w_t(\cdot, y) - hy\big] \leq C + \sup_{y \in K_t(\cdot)} [-hy] = C + \sup_{y \in K_t^*(\cdot)} [-hy].$$

Denoting

$$\hat{D}_t^b(\cdot) = \left\{h \in D_t(\cdot): \sup_{y \in K_t^*(\cdot)} [-hy] \leq b\} = \{h \in D_t(\cdot): \; \sigma_{K_t^*(\cdot)}(-h) \leq b\right\}, \tag{3.3}$$

we obtain $\hat{D}_t^{a-C}(\cdot) \subseteq D_t^a(\cdot) \subseteq \hat{D}_t^a$, and so $D_t^a(\cdot)$ is compact for all $a \in \mathbb{R}$ if and only if the $\hat{D}_t^a(\cdot)$ are compact for all $a \in \mathbb{R}$.

Consider the function

$$h \mapsto f_{t,\cdot}(h) = \chi_{D_t(\cdot)}(h) + \sup_{y \in K_t^*(\cdot)} (-hy), \tag{3.4}$$

where

$$\chi_D(h) = \begin{cases} 0 & \text{if } h \in D \\ +\infty & \text{if } h \notin D. \end{cases}$$

The function defined by (3.4) is a closed[36] proper[37] convex function, because the same refers to the first term in (3.4), while the second term in (3.4) is an everywhere finite convex function and hence everywhere continuous (see Corollary 10.1.1 in [11]). In accordance with Corollary 14.2.2 in [11], for the set $\{h \in \mathbb{R}^n: \; f_{t,\cdot}(h) \leq a\} = \hat{D}_t^a(\cdot)$ to be bounded (and hence compact by virtue of the semicontinuity of the function defined by (3.4)) for each $a \in \mathbb{R}$ it is necessary and sufficient that 0

[35] The empty set is considered compact.
[36] Lower semicontinuous (the terminology of [11]).
[37] If a function takes finite values on a nonempty convex set and is $+\infty$ outside it.

be an inner point of the set $T'_t(\cdot) = \{z \in \mathbb{R}^n: \ f^*_{t,\cdot}(z) < \infty\}$, where $z \mapsto f^*_{t,\cdot}(z)$ is the function dual to the function $h \mapsto f_{t,\cdot}(h)$; i.e.,

$$T'_t(\cdot) = \left\{ z \in \mathbb{R}^n: \ \sup_{h\in\mathbb{R}^n} \left[hz - f_{t,\cdot}(h)\right] < \infty \right\}.$$

In this case, using the classical Kneser minimax theorem [10], we obtain

$$\sup_{h\in\mathbb{R}^n} \left\{ hz - \left[\chi_{D_t(\cdot)}(h) + \sup_{y\in K^*_t(\cdot)} (-hy) \right] \right\} = \sup_{h\in D_t(\cdot)} \left[hz + \inf_{y\in K^*_t(\cdot)} (hy) \right]$$
$$= \sup_{h\in D_t(\cdot)} \inf_{y\in z+K^*_t(\cdot)} (hy) = \inf_{y\in z+K^*_t(\cdot)} \sup_{h\in D_t(\cdot)} (hy) = \inf_{y\in z+K^*_t(\cdot)} \sigma_D(y).$$

If $\left(z + K^*_t(\cdot)\right) \cap \operatorname{bar}\left(D_t(\cdot)\right) = \emptyset$, then $\sigma_D(y) = \infty$ for all $y \in z + K^*_t(\cdot)$ and $\inf_{y\in z+K^*_t(\cdot)} \sigma_D(y) = +\infty$. If, however, $\left(z + K^*_t(\cdot)\right) \cap \operatorname{bar}\left(D_t(\cdot)\right) \neq \emptyset$, then $\inf_{y\in z+K^*_t(\cdot)} \sigma_D(y) < \infty$. Thus,

$$T'_t(\cdot) = \left\{ z \in \mathbb{R}^n: \ z + K^*_t(\cdot) \cap \operatorname{bar}\left(D_t(\cdot)\right) \neq \emptyset \right\};$$

i.e., $T'_t(\cdot) = T_t(\cdot)$; the latter implies the desired assertion. ■

Remark 3.1.

1. Note that, when proving Proposition 3.1, it was established that condition (SR) and the closedness of the sets $D_t(\cdot)$ are sufficient for the compactness of the sets $\hat{D}^a_t(\cdot)$.
2. It can readily be verified that the set $T_t(\cdot)$ defined by (3.1) is convex.
3. For the case of no trading constraints, i.e., when $D_t(\cdot) = \mathbb{R}^n$, the barrier cone $\operatorname{bar}\left(D_t(\cdot)\right)$ is $\{0\}$, and condition (3.2) is equivalent to the robust condition of no arbitrage opportunities *RNDAO*, which, in accordance with Proposition 1 in [4], is equivalent to the simultaneous validity of the condition of no arbitration opportunities *NDAO* and the full size of the compact sets $K_t(\cdot)$, $t = 1, \ldots, N$, i.e., the condition $0 \in \operatorname{int}(K^*_t(\cdot))$, $t = 1, \ldots, N$.

In the sequel, we assume that the $D_t(\cdot)$ are closed sets. (By virtue of item 3 in Remark 2.4 and item 1 in Remark 2.5, this is not a limitation for Theorems 2.5 and 2.7 to hold.)

Lemma 3.1. *Let $D_t(\cdot)$ be closed sets, and let condition RNDSAUP be satisfied; then for $a \geq C$ the function $\rho_t(\cdot)$ defined by (2.4) can be represented in the form*

$$\rho_t(\cdot) = \inf_{h\in\hat{D}^a_t(\cdot)} \sup_{y\in K_t(\cdot)} \left[w_t(\cdot, y) - hy\right]; \tag{3.5}$$

thus, $D_t(\cdot)$ can be replaced by a compact convex set $\hat{D}^a_t(\cdot)$ with $0 \in \hat{D}^a_t(\cdot)$. In particular, the lower semicontinuous (and convex) function $h \mapsto \sup_{y\in K_t(\cdot)} [w_t(\cdot, y) - hy]$ attains the minimum value $\rho_t(\cdot)$ at some point $h^*(\cdot) \in \hat{D}^C_t$.

Proof. If $h_0 \notin \hat{D}^a_t(\cdot)$, where $a \geq C$, then, since $0 \in D_t(\cdot)$, for such an h_0 we have

$$\sup_{y\in K^*_t(\cdot)} \left[w_t(\cdot, y) - h_0 y\right] \geq \sup_{y\in K^*_t(\cdot)} [-h_0 y] > a \geq C$$
$$\geq \sup_{y\in K_t(\cdot)} \left[w_t(\cdot, y) - hy\right]\Big|_{h=0} \geq \inf_{h\in D_t(\cdot)} \sup_{y\in K_t(\cdot)} \left[w_t(\cdot, y) - hy\right];$$

the latter implies (3.5). ■

We say that a multivalued mapping $F : X \mapsto \mathcal{N}(Y)$ is locally precompact if for any point $x_0 \in X$ there exists a neighborhood V_x of this point such that its image $F(V_x) = \bigcup_{x \in V_x} F(x)$ is precompact.[38] If, in addition, X is compact, then, obviously, the image of $F(x)$ is precompact. (It suffices to select a finite subcover from the cover V_x, $x \in X$, and note that a finite union of compact sets is compact.)

Lemma 3.2. *Let the following condition be satisfied: for each point $x_0 \in B_{t-1}$ there exists a neighborhood $V(x_0)$ of this point such that $\check{K}_t(x_0) = \bigcap_{x \in V(x_0)} K_t^*(x) \neq \emptyset$, and moreover,*[39]

$$0 \in \operatorname{int}\left(\left\{z : \ z + \check{K}_t(x_0) \cap \operatorname{bar}\left(D_t(\cdot)\right) \neq \emptyset\right\}\right). \tag{SSR}$$

Then the multivalued mapping $x \mapsto \hat{D}_t^b(x)$, $x \in B_{t-1}$, is locally precompact.

Proof. For $x \in V(x_0)$, in view of (3.3), we have

$$\hat{D}_t^b(x) \subseteq \check{D}_t^b(x_0) = \left\{h \in D_t(x_0) : \ \sigma_{\check{K}_t(x_0)}(-h) \leq b\right\};$$

reproducing the argument in Proposition 3.1 and replacing $K_t^*(x)$ by $\check{K}_t(x)$, we conclude that the sets $\check{D}_t^b(x_0)$, $x_0 \in B_{t-1}$, are compact for all $b \in \mathbb{R}$ if and only if (SSR) is satisfied; hence we obtain the local precompactness of the multivalued mapping $x \mapsto \hat{D}_t^b(x)$, $x \in B_{t-1}$, subject to (SSR). ■

Lemma 3.3. *Let multivalued mappings $x \mapsto D_t(x)$ be closed,*[40] *let multivalued mapping $x \mapsto K_t(x)$ be lower semicontinuous, and let B_t be precompact sets, $t = 1, \ldots, N$. Then the multivalued mappings $x \mapsto \hat{D}_t^a(x)$ are closed.*

Proof. Since $B_t = K_t(B_{t-1}) = \bigcup_{x \in B_{t-1}} K_t(x)$ is precompact by condition, it follows that the multivalued mapping $x \mapsto K_t(x)$ is uniformly bounded on B_{t-1}, and therefore, so are the convex hulls $K_t^*(x) = \operatorname{conv}(K_t(x))$ on B_{t-1},

$$\sup_{x \in B_{t-1}} \sup_{y \in K_t^*(x)} \|y\| = A < \infty. \tag{3.6}$$

The lower semicontinuity of $K_t(\cdot)$ implies the lower semicontinuity of $K_t^*(\cdot)$ by Proposition 2.24(a) in [9]. Consider sequences x^n and h^n, $n = 1, 2, \ldots$, such that $x^n \in B_{t-1}$, $x^n \to x^0$, $h^n \in \hat{D}_t^b(x^n)$, and $h^n \to h^0$. By Proposition[41] 9.10 in [2], in view of (3.6), we obtain

$$b \geq \sigma_{K_t^*(x^n)}(-h^n) \geq \sigma_{K_t^*(x^0)}(-h^0) - A\|h^n - h^0\|.$$

Based on this, we have

$$b \geq \liminf_{n \to \infty} \sigma_{K_t^*(x^n)}(-h^n) \geq \liminf_{n \to \infty} \sigma_{K_t^*(x^n)}(-h^0) \geq \sigma_{K_t^*(x^0)}(-h^0),$$

because the function $x \mapsto \sigma_{K_t^*(x)}(h)$ is lower semicontinuous for each h; see, e.g., Proposition 2.35 in [9]. Moreover, since the multivalued mapping $x \mapsto D_t(x)$ is closed by assumption, we have $h^0 \in D_t(x^0)$. Thus, $h^0 \in \{h \in D_t(x^0) : \ \sigma_{K_t^*(x^0)}(-h) \leq b\} = \hat{D}_t^b(x^0)$. ■

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[38] In other words, in the terminology of [2], this mapping is compactly bounded at all points from X; in [9], the mapping is also called "locally compact" (which, in our opinion, is unfortunate, since this term already refers to topological spaces).

[39] Of course, (SSR) implies $\check{K}_t(x_0) \neq \emptyset$. We can interpret $\check{K}_t(\cdot)$ as a new dynamics of the market with uncertainty diminished compared with $K_t^*(\cdot)$; in this case, the $\check{K}_t(\cdot)$ are convex compact sets.

[40] A multivalued mapping $F : X \mapsto \mathcal{N}(Y)$ is said to be closed if for Cauchy nets x_α and y_α such that $x_\alpha \to x$, $y_\alpha \in F(x_\alpha)$, and $y_\alpha \to y$ one has $y \in F(x)$. In other words, the graph of the mapping F is closed.

[41] This is the result in [2] concerning the Lipschitz property (with constant A) for support functions.

Theorem 3.1. *Let $t = 1, \ldots, N$, let numerical functions $\bar{x}_t \mapsto g_t(\bar{x}_t)$ be lower semicontinuous, let multivalued mappings $\bar{x}_{t-1} \mapsto D_t(\bar{x}_{t-1})$ be closed, let multivalued mappings $\bar{x}_{t-1} \mapsto K_t(\bar{x}_{t-1})$ be lower semicontinuous, let condition (SSR) be satisfied, and let sets B_t be precompact. Then the functions $(\bar{x}_{t-1}, y) \mapsto w_t(\bar{x}_{t-1}, y)$ defined by relations (BA) are lower semicontinuous.*

Proof. Using Lemmas 3.2 and 3.3, we prove by Proposition 2.23 in [9] that the locally precompact and closed mapping[42] $x \mapsto \hat{D}_t^b(x)$ is upper semicontinuous. Applying Lemma 3.1 and Theorem 2.6, where $D_t(\cdot)$ is replaced by $\hat{D}_t^a(\cdot)$ for some $a \geq C$, we obtain the desired assertion. ■

Remark 3.2. Note that the proof of Theorem 3.1 established the upper semicontinuity of the compact-valued map $x \mapsto \hat{D}_t^b(x)$ under the conditions of Theorem 3.1 concerning $K_t(\cdot)$, $D_t(\cdot)$, and B_t.

Proposition 3.2. *Suppose that multivalued mappings $x \mapsto K_t(x)$, $t = 1, \ldots, N$, are lower semicontinuous and that condition (SGNSAUP)[43] is satisfied for all x,*

$$\operatorname{int}\left(K_t^*(\cdot)\right) \bigcap \operatorname{bar}\left(D_t(\cdot)\right) \neq \emptyset. \tag{3.7}$$

Then

1. *There exist neighborhoods $V(x)$ of points x such that the following condition is satisfied for $\check{K}_t(x) = \bigcap_{x' \in V(x)} K_t^*(x')$:*[44]

$$\operatorname{int}\left(\check{K}_t(x)\right) \bigcap \operatorname{bar}\left(D_t(x)\right) \neq \emptyset. \tag{3.8}$$

2. *Condition (SSR) is satisfied.*

Proof.

1. Fix t and x. In accordance with (3.7), there exists an $r > 0$ and a $y \in \operatorname{bar}(D_t(x))$ such that $B_r(y) \subseteq K_t^*(x)$. In accordance with Proposition 2.42, (a) in [9], the lower semicontinuity of $K_t(\cdot)$ implies the similar property for $K_t^*(\cdot) = \operatorname{conv}(K_t(\cdot))$. By Lemma 2.51 in [9], for each given $\varepsilon \in (0, r)$ there exists a neighborhood $V(x)$ of the point x such that for each $x' \in V(x)$ one has $B_\varepsilon(y) \subseteq K_t^*(x')$. Consequently, by setting $\check{K}_t(x) = \bigcap_{x' \in V(x)} K_t^*(x')$, we obtain $\check{K}_t(x) \supseteq B_\varepsilon(y)$. Therefore, $\operatorname{int}(\check{K}_t(x)) \bigcap \operatorname{bar}(D_t(x)) \neq \emptyset$.
2. Consider the "new" dynamics of the market with price movement uncertainty described by the compact sets $\check{K}_t(\cdot)$, which are convex as intersections of convex compact sets. For such a market, condition (*SGNSAUP*) in Theorem 4.1 in [4] is satisfied. This is precisely (3.8); therefore, condition *RNDSAUP* is satisfied, and we can apply item 1 of Theorem 4.1 in [4], in accordance with which condition (SR) is satisfied, which for the "new" market dynamics is condition (SSR) for the "old" market dynamics, i.e., is described by $K_t(\cdot)$. ■

Fix $a \geq C \geq 0$, where the constant C is given by relation (1.3). Without loss of generality, we can assume[45] that $C > 0$, since otherwise $g_t(\cdot) \equiv 0$, which is not of interest from the viewpoint of economic interpretation.

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[42] A closed mapping takes closed values; see, e.g., Remark 2.12 in [9].
[43] This condition appears in Theorem 4.1 in [4] and implies the full size of compact sets $K_t(\cdot)$, i.e., $\operatorname{int}(K_t^*(\cdot)) \neq \emptyset$.
[44] Thus, the compact sets $\check{K}_t(\cdot)$ will also be full-size under the assumptions of Proposition 3.2.
[45] However, to ensure that $a > 0$ (which is required in the proof of Proposition 3.3), one can simply require $a > C$.

Proposition 3.3. *Let compact-valued mappings $K_t(\cdot)$ be continuous, let $D_t(\cdot)$ be lower semicontinuous and closed*[46] *on B_{t-1}, and let condition (SR) be satisfied. Then the multivalued mapping*

$$x \mapsto \hat{D}_t^a(x) = D_t(x) \bigcap E_t^a(x),\ x \in B_{t-1},$$

where

$$E_t^a(x) = \{h \in \mathbb{R}^n :\ \sigma_{K_t^*(x)}(h) \le a\},$$

is continuous.

Proof. Let us demonstrate the lower semicontinuity for $x \mapsto \hat{D}_t^a(x)$. We fix $t \in \{1, \ldots, N\}$ and start with establishing the lower semicontinuity for $E_t^a(\cdot)$. By Proposition 2.6, (f) in [9], it suffices to make sure that for a converging sequence $x_n \to x_0$ from B_{t-1} the Kuratowski lower limit $\liminf\limits_{n\to\infty} E_t^a(x_n)$ contains $E_t^a(x_0)$. Since $\liminf\limits_{n\to\infty} E_t^a(x_n) = \{x :\ \rho(x, E_t^a(x_n)) \to 0\}$ (see Remark 1.43 in [9]), for an arbitrary $h_0 \in E_t^a(x_0)$ it is necessary to show the existence of a sequence $h_n \in E_t^a(x_n)$ such that $h_n \to h_0$. Set

$$r_n = h_\rho(K_t^*(x_n), K_t^*(x_0)) \le h_\rho(K_t(x_n); K_t(x_0)) \to 0,$$

see inequality 5.12 in [2]. Here we use the fact that for compact-valued mappings ($K_t^*(\cdot)$ is compact-valued) the continuity coincides with h-continuity by Theorem 2.68 in [9]. Set $h_n = \alpha_n h_0$, where $\alpha_n = \frac{a}{a + r_n\|h_0\|} \to 1$ as $n \to \infty$. Using Proposition 9.11 in [2], we obtain

$$\sigma_{K_t^*(x_n)}(h_n) \le \sigma_{K_t^*(x_0)}(h_n) + r_n\|h_n\| = \alpha_n\sigma_{K_t^*(x_0)}(h_0) + r_n\alpha_n\|h_0\| \le \alpha_n a + \alpha_n r_n\|h_0\| = a;$$

i.e., $h_n \in E_t^a(x_n)$, $h_n \to h_0 \in E_t^a(x_0)$, and the lower semicontinuity of $E_t^a(\cdot)$ has been established.

Note that $a > 0$ by the above-made assumption and that the set $\frac{1}{a}E_t^a(x) = \{h : \sigma_{K_t^*(x)}(h) \le 1\}$ is the (Minkowski) polar for $K_t^*(\cdot)$; see [1, formula (70) and Theorem 12.2]. By Theorem 6.6, (a) in [1], the point 0 is an inner point of the set polar to $K_t^*(x)$, because it is bounded (by virtue of compactness, by Theorem 2.6 in [1]). Therefore, $0 \in \operatorname{int}(E_t^a(x))$ for all $x \in B_{t-1}$. Further, $0 \in D_t(\cdot) \bigcap \operatorname{int}(E_t^a(\cdot)) \ne \emptyset$, and the sets $D_t(\cdot)$ and $E_t^a(\cdot)$ are convex; consequently, we can apply Proposition 2.54 in [9], in accordance with which the multivalued mapping $\hat{D}_t^a(\cdot) = D_t(\cdot) \bigcap E_t^a(\cdot)$ is lower semicontinuous.

According to Lemma 3.3, the multivalued mapping $x \mapsto \hat{D}_t^a(x)$ is closed. By item 1 in Remark 3.1, taking into account the fact that closed multivalued mappings take closed values (Remark 2.12 in [9]), the sets $D_t(\cdot)$ are closed, with $\hat{D}_t^a(\cdot)$ being compact according to item 1 in Remark 3.1. Since the sets $\hat{D}_t^a(\cdot)$ are convex, we can apply Theorem 2.102 in [9], according to which a closed lower semicontinuous multivalued mapping with argument in a metric space and with values being path-connected compact subsets of a finite-dimensional Euclidean space is continuous. ■

Theorem 3.2. *Assume that for $t = 1, \ldots, N$ the numerical functions $\bar{x}_t \mapsto g_t(\bar{x}_t)$ are continuous, the multivalued mappings $\bar{x}_{t-1} \mapsto K_t(\bar{x}_{t-1})$ are continuous, condition RNDSAUP is satisfied, and the multivalued mappings $\bar{x}_{t-1} \mapsto D_t(\bar{x}_{t-1})$ are lower semicontinuous and closed. Then assertions* 1 *and* 2 *in Theorem 2.7 hold true.*

Proof. Fix an $a > C$ and choose $\hat{D}_t^a(\cdot)$ instead of $D_t(\cdot)$ in formulas (BA) using Lemma 3.1. By Proposition 3.3, the compact-valued mapping $x \mapsto \hat{D}_t^a(x)$ is continuous, so we can readily apply Theorem 2.7. ■

[46] In the terminology of the book [2], the weakly continuous map $x \mapsto D_t(\cdot)$ is simultaneously upper and lower weakly semicontinuous; moreover, weak upper semicontinuity is equivalent to closedness (see [2, Theorem 14.7]) and weak lower semicontinuity coincides with (ordinary) lower semicontinuity (see [2, Remark 14.1]).

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