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STOCHASTIC SYSTEMS

On the Partial Stability Problem for Nonlinear Discrete-Time Stochastic Systems

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Abstract—We consider a system of nonlinear discrete-time equations subject to the influence of a discrete random process of the "white" noise type. It is assumed that the system admits a "partial" (with respect to some part of state variables) zero equilibrium. The problem of partial stability in probability is posed—the stability of a given equilibrium is not with respect to all but only to part of the variables determining it. To solve the problem, a discrete-stochastic version of the Lyapunov function method is used with the appropriate refinement of the requirements for the Lyapunov function. To expand the capabilities of the method used, it is proposed to correct the domain in which the auxiliary Lyapunov function is constructed; this is achieved by introducing an additional (vector, generally speaking) auxiliary function. Conditions of partial and asymptotic stability in probability in the indicated form are obtained. An example is given showing the specific features of the proposed approach.

Keywords: system of nonlinear discrete-time (finite-difference) stochastic equations, partial stability, method of Lyapunov functions

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1. INTRODUCTION

Stability problems are among the main problems of qualitative analysis and synthesis of nonlinear dynamical systems subjected to random disturbances and structural changes. As in the case of deterministic systems, the Lyapunov function method is used to solve these problems.

The development of the stochastic version of the method of Lyapunov functions was greatly influenced by the idea due to Kats and Krasovskii [1] about using the *averaged derivative* of the Lyapunov function, which can be calculated knowing only the right-hand sides of the system and the probabilistic characteristics of the random process affecting it. This approach, proposed for systems of differential equations with right-hand side containing a homogeneous Markov chain with finitely many states, largely predetermined many subsequent studies, including systems of stochastic differential equations in the Itô form [2, 3] as well as more general classes of stochastic systems with random parameters and/or structure [4, 5].

A separate line of research is associated with the analysis of the stability of *discrete-time* (finite-difference) systems subjected to random factors. The increased interest in discrete-time systems is associated with the use of digital control systems, problems of financial mathematics, biocenose dynamics, as well as problems of the numerical solution of systems of stochastic differential equations. In this way, the corresponding discrete-stochastic version of the method of Lyapunov functions was developed [2, 6–13] as applied to the highly general problem on the stability of the zero equilibrium *with respect to all variables*. The problem of stability of compact sets in the system state space was also considered [14]. The averaged derivative (or differential generating operator [2, 3, 5]) is replaced in these cases by the *averaged finite difference* of the Lyapunov function [6].

Starting from Rumyantsev's publications [15, 16], the theory of stability of deterministic systems and then of stochastic systems with continuous dynamics considers *partial stability* problems (see the survey [17]): stability of the zero equilibrium with respect to some of the variables, as well as stability of a "partial" (zero) equilibrium with respect to all or some of the variables. From the formal mathematical point of view, the problem of stability of a "partial" equilibrium with respect to all variables belongs to the problem of stability of noncompact (closed but unbounded) sets, while problems of stability with respect to some of the variables have independent significance and, generally speaking, are not reducible to any problems of stability of sets. The point is that stability with respect to some of the variables does not imply the proximity of the trajectories corresponding to perturbed motions and the unperturbed motion (equilibrium) of the system.

Substantively, the indicated partial stability problems naturally arise in applications both based on the requirement of normal operation and in assessing the capabilities of the system to be designed. They can also be viewed as auxiliary problems in the analysis of stability of selected equilibria with respect to all variables. In addition, there arise related partial stabilization problems for nonlinear control systems, which have been actively examined in recent years. However, partial stability and stabilization problems have practically not been studied for systems of stochastic discrete-time equations.

The present paper discusses a system of nonlinear discrete-time (finite-difference) equations of general form subject to the influence of a discrete random process such as white noise. It is assumed that the system admits a partial (with respect to some part of the state variables) zero equilibrium. The statement of the stability in probability problem for this equilibrium is given; stability is considered with respect to part of the state variables determining it. The possibility of solving the problem on the basis of the Lyapunov function method is analyzed.

2. DEFINITIONS. STATEMENT OF THE PROBLEM

Consider a linear finite-dimensional space of vectors $\mathbf{x}$ with Euclidean norm $\|\mathbf{x}\|$. Introduce a partitioning of a vector $\mathbf{x}$ into two parts, $\mathbf{x} = (\mathbf{y}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}$ (where "T" stands for transposition). By $\mathbb{Z}_+ = \{k = 0, 1, 2, \dots\}$ we denote the set of nonnegative integers.

Assume that we are given a finite-dimensional nonlinear system of stochastic discrete (finite-difference) equations [2, 6–14]

$$\mathbf{x}(k+1) = \mathbf{X}\big(k, \mathbf{x}(k), \boldsymbol{\xi}(k)\big),$$

where $k \in \mathbb{Z}_+$ is discrete time; $\mathbf{x}(k)$ is the sequence of state vector values determining the system state; and $\boldsymbol{\xi}(k)$ is a sequence of independent random vectors defined on a probability space $(\Omega,\ F,\ \mathbf{P})$ with identical distribution laws for each $k \in \mathbb{Z}_+$. Here Ω is the space of elementary events $\{\omega\}$ with a σ-algebra of F-measurable sets, filtration F_k, and a probability measure $\mathbf{P} : F \to [0, 1]$ defined on this space.

Taking into account the partitioning $\mathbf{x} = (\mathbf{y}^{\mathrm{T}}, \mathbf{z}^{\mathrm{T}})^{\mathrm{T}}$, we represent the system in question in the form of two groups of equations

$$\mathbf{y}(k+1) = \mathbf{Y}\big(k, \mathbf{y}(k), \mathbf{z}(k), \boldsymbol{\xi}(k)\big), \quad \mathbf{z}(k+1) = \mathbf{Z}\big(k, \mathbf{y}(k), \mathbf{z}(k), \boldsymbol{\xi}(k)\big). \tag{1}$$

Under the condition

$$\mathbf{Y}\big(k, \mathbf{0}, \mathbf{z}(k), \boldsymbol{\xi}(k)\big) \equiv \mathbf{0},$$

the set $M = \{\mathbf{x}(k) : \mathbf{y}(k) = \mathbf{0}\}$ is a "partial" equilibrium of system (1).

Assume also that the vector function $\mathbf{X} = (\mathbf{Y}^{\mathrm{T}}, \mathbf{Z}^{\mathrm{T}})^{\mathrm{T}}$ determining the right-hand side of system (1) is *continuous* in $\mathbf{x}$ and $\boldsymbol{\xi}$ in the domain $\|\mathbf{x}\| < \infty$ for each $k \in \mathbb{Z}_+$. The initial value $\mathbf{x}_0$

of the state vector will be assumed to be *deterministic*. Then (see, e.g., [9, 11]) for all $k_0 \geq 0$ and $\mathbf{x}_0$ there exists a unique random multidimensional Markov process that is consistent with the flow of σ-algebras F_k and is a random vector function $\{\mathbf{x}(k) = \mathbf{x}(k; k_0, \mathbf{x}_0), \boldsymbol{\xi}(k)\}$ in the space $\{\mathbf{x}, \boldsymbol{\xi}\}$ whose realizations $\{\mathbf{x}(k, \omega) = \mathbf{x}(k, \omega; k_0, \mathbf{x}_0), \boldsymbol{\xi}(k, \omega)\}$ satisfy system (1). For all $k \geq k_0$, this random process and the corresponding collection of realizations of the random vector function determine a solution of system (1) with the initial conditions $\mathbf{x}_0 = \mathbf{x}(k_0; k_0, \mathbf{x}_0)$ as well as the collection of sample paths of system (1) corresponding to this solution. The Markov property of solutions of system (1) is used in the sequel to justify the conditions for partial stability in probability.

Under the assumptions made, the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) is an invariant set of this system. The assumption $\mathbf{X}(k, \mathbf{0}, \boldsymbol{\xi}(k)) \equiv \mathbf{0}$ about the existence of a "complete" equilibrium $\mathbf{x}(k) = \mathbf{0}$ is not necessary and can even contradict the meaning of the problems being solved.

Following the approach of partial stability theory, we will analyze the stability of the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ not with respect to all the variables determining it but with respect to some part of those, *defined* in advance. To this end, we assume that $\mathbf{y} = (\mathbf{y}_1^{\mathrm{T}}, \mathbf{y}_2)^{\mathrm{T}}$, with the vector $\mathbf{y}_1$ including the components of $\mathbf{y}$ with respect to which stability is considered.

In this case, the variables occurring in the vector $\mathbf{z}$ are "uncontrolled," although they significantly affect the dynamics of the $\mathbf{y}_1$-variables. To expand the operation capabilities of the notions of $\mathbf{y}_1$-stability of the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ considered in what follows, we arbitrarily introduce a partitioning $\mathbf{z} = (\mathbf{z}_1^{\mathrm{T}}, \mathbf{z}_2^{\mathrm{T}})^{\mathrm{T}}$ of the vector $\mathbf{z}$ into two groups of variables.

By D_δ we denote the domain of $\mathbf{x}_0$ such that $\|\mathbf{y}_0\| < \delta$, $\|\mathbf{z}_{10}\| \leq L$, and $\|\mathbf{z}_{20}\| < \infty$; the domain D_Δ is obtained by replacing δ with Δ.

Definition. For large values of $\mathbf{z}_{10}$ and on the whole with respect to $\mathbf{z}_{20}$, a "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) is

1. $\mathbf{y}_1$-stable in probability if for each $k_0 \in \mathbb{Z}_+$ and any arbitrarily small numbers $\varepsilon > 0$ and $\gamma > 0$, as well as for each number $L > 0$ given in advance, there exists a number $\delta(\varepsilon, \gamma, L, k_0) > 0$ such that for all $k \geq k_0$ and $\mathbf{x}_0 \in D_\delta$ one has the relation

$$\mathbf{P}\left\{\sup_{k \geq k_0} \left\|\mathbf{y}_1(k; k_0, \mathbf{x}_0)\right\| > \varepsilon\right\} < \gamma. \tag{2}$$

2. Uniformly $\mathbf{y}_1$-stable if $\delta = \delta(\varepsilon, \gamma, L)$.
3. Asymptotically $\mathbf{y}_1$-stable if it is uniformly $\mathbf{y}_1$-stable in probability and, in addition, for each $k_0 \in \mathbb{Z}_+$ and for each number $L > 0$ given in advance there exists a number $\Delta(L) > 0$ such that for all $k \geq k_0$ and $\mathbf{x}_0 \in D_\Delta$ one has the limit relation

$$\lim_{\|\mathbf{y}_0\| \to 0} \mathbf{P}\left\{\lim_{k \to +\infty} \left\|\mathbf{y}_1(k; k_0, \mathbf{x}_0)\right\| = 0\right\} = 1.$$

Remark 1. It can be shown (see, e.g., [5]) that if $\mathbf{x}_0$ is a random variable (independent of $\boldsymbol{\xi}(k)$) and the inclusions $\mathbf{x}_0 \in D_\delta$ and $\mathbf{x}_0 \in D_\Delta$ hold almost certainly (with probability 1), then we obtain definitions equivalent to the above-introduced definitions of partial stability.

Remark 2. The closest to the ones introduced are the notions of partial stability of a "partial" equilibrium of stochastic systems of differential equations in Itô form with respect to all [18, 19] and part [20] of the variables. The assumptions about "on the whole with respect to $\mathbf{z}_0$" or "for large values of $\mathbf{z}_0$" are typical of the definitions of stability (both in all and part of the variables) of a "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) but lead to different requirements for the Lyapunov functions. Partitioning the vector $\mathbf{z}_0$ into two parts gives rise to "intermediate" concepts of $\mathbf{y}_1$-stability in the sense of the above-introduced Definitions 1–3. In this case, the suitable choice of the partition $\mathbf{z} = (\mathbf{z}_1^{\mathrm{T}}, \mathbf{z}_2^{\mathrm{T}})^{\mathrm{T}}$ depends on the structure of system (1) and is a result of finding a trade-off between the informative meaning of the concept of the $\mathbf{y}_1$-stability of the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ and the corresponding requirements for the Lyapunov functions. Moreover, the above-introduced notions of stability arise when passing (by means of the notation $w = k$, $r = k - k_0$) from system (1) to the *time-invariant* discrete system

$$\mathbf{x}(r+1) = \mathbf{X}\big(\mathbf{x}(r), w(r), \boldsymbol{\xi}(r)\big), \quad w(r+1) = w(r) + 1,$$

when the requirements of uniformity (nonuniformity) with respect to k_0 in the problems of $\mathbf{y}_1$-stability for large values of $\mathbf{z}_0$ or on the whole in $\mathbf{z}_0$ of a "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ are replaced by the requirements "on the whole in w_0" ("for large values of w_0").

3. PARTIAL STABILITY CONDITIONS

In the context of the Lyapunov function method, we consider single-valued scalar functions $V = V(k, \mathbf{x})$, $V(k, \mathbf{0}) \equiv 0$, continuous in $\mathbf{x}$ for each $k \in \mathbb{Z}_+$ and defined in the domain

$$\|\mathbf{y}_1\| < h, \quad \|\mathbf{y}_2\| + \|\mathbf{z}\| < \infty. \tag{3}$$

An analog of the derivatives of these functions according to system (1) are their *averaged difference* (increments) calculated by the formula [6, 9]

$$\mathbf{L}V(k, \mathbf{x}) = \mathbf{E}\Big[V\Big(k+1, \mathbf{X}\big(k, \mathbf{x}(k), \boldsymbol{\xi}(k)\big)\Big)|\mathbf{x}(k) = \mathbf{x}\Big] - V(k, \mathbf{x}),$$

where the operator $\mathbf{E}[V(k+1, \mathbf{X}(k, \mathbf{x}(k), \boldsymbol{\xi}(k)))|\mathbf{x}(k) = \mathbf{x}]$ determines the conditional expectation, for $\mathbf{x}(k) = \mathbf{x}$, of the random variable $V(k+1, \mathbf{X}(k, \mathbf{x}(k)\boldsymbol{\xi}(k)))$ generated by the set of realizations $\{\mathbf{x}(k, \omega), \boldsymbol{\xi}(k, \omega)\}$ of a process $\{\mathbf{x}(k), \boldsymbol{\xi}(k)\}$ that is a solution of system (1).

To state the partial stability conditions, we will also additionally use the following auxiliary functions.

1. Scalar functions $V^*(k, \mathbf{y}, \mathbf{z}_1)$, $V^*(\mathbf{y}, \mathbf{z}_1)$ that are required to render concrete (in accordance with the statement of the problem) requirements for the Lyapunov V-function and an auxiliary vector function $\boldsymbol{\mu}(k, x)$, $\boldsymbol{\mu}(k, \mathbf{0}) \equiv \mathbf{0}$, by means of which we correct the domain where the main Lyapunov V-function is constructed. These functions are continuous in $\mathbf{x}$ for each $k \in \mathbb{Z}_+$ in the domain (3).
2. Continuous scalar functions $a_i(r)$, $a_i(0) = 0$ $(i = 1, 2, 3)$ monotone increasing in $r > 0$ (Hahn-type functions [16]) defining the standard requirements for the main Lyapunov V-function.

Introducing, along with the main Lyapunov V-function, the auxiliary function $\boldsymbol{\mu}(k, \mathbf{x})$ is motivated as follows. When studying the $\mathbf{y}_1$-stability in probability of a "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1), in the general case, one has the dependence of Lyapunov V-functions not only on k, $\mathbf{y}_1$ but also on $\mathbf{y}_2$, $\mathbf{z}$. In such a situation, the analysis of the posed problem about $\mathbf{y}_1$-stability in the usually considered domain

$$\|\mathbf{y}_1\| < h_1 < h, \quad \|\mathbf{y}_2\| + \|\mathbf{z}\| < \infty \tag{4}$$

does not always allow one to reveal the desired properties of the Lyapunov V-function or endow it with these properties. The reason lies in the requirement $\|\mathbf{y}_2\| + \|\mathbf{z}\| < \infty$, which considerably complicates producing the necessary estimates for the Lyapunov V-function and its averaged finite difference.

The indicated requirement essentially means counting on the "worst-case" scenario of changes in the variables $\mathbf{y}_2$, $\mathbf{z}$, and it can be replaced by the "softer" requirement

$$\|\mathbf{y}_1\| + \big\|\boldsymbol{\mu}(k,\mathbf{x})\big\| < h_1 < h, \quad \|\mathbf{y}_2\| + \|\mathbf{z}\| < \infty \tag{5}$$

if we mean the "extended" $(\mathbf{y}_1, \boldsymbol{\mu})$-stability of the "partial" equilibrium $\mathbf{y} = \mathbf{0}$ of system (1). In this case, the $\boldsymbol{\mu}$-function is not specified initially but is selected when solving the original $\mathbf{y}_1$-stability problem, with the possibility of the extension of the concept of $\mathbf{y}_1$-stability occurring because of the dependence of the $\boldsymbol{\mu}$-function not only on k and $\mathbf{y}$ but also on $\mathbf{z}$.

Therefore, it is not only possible but also expedient to select a suitable Lyapunov V-function to be consistent with the choice of the domain in the $(k,\mathbf{x})$-space in which this function is considered. This consistency can be achieved by introducing, along with the *main* Lyapunov V-function, an *additional* (generally speaking, vector) auxiliary function $\boldsymbol{\mu}(k,\mathbf{x})$ for correcting the domain in which the main Lyapunov V-function is constructed.

Theorem 1. *Suppose that for system* (1), *along with the main scalar Lyapunov V-function, one can specify an auxiliary vector function $\boldsymbol{\mu}(k,\mathbf{x})$, $\boldsymbol{\mu}(k,\mathbf{0}) \equiv 0$, such that the following conditions are satisfied*:

$$V(k,\mathbf{x}) \geq a_1\Big(\|\mathbf{y}_1\| + \big\|\boldsymbol{\mu}(k,\mathbf{x})\big\|\Big), \tag{6}$$

$$V(k,\mathbf{x}) \leq V^*(k,\mathbf{y},\mathbf{z}_1), \quad V^*(k,\mathbf{0},\mathbf{z}_1) \equiv 0, \tag{7}$$

$$\mathbf{L}V(k,\mathbf{x}) = \mathbf{E}\Big[V(k+1,\mathbf{X}(k,\mathbf{x}(k),\boldsymbol{\xi}(k))|\mathbf{x}(k)=\mathbf{x}\Big] - V(k,\mathbf{x}) \leq 0 \tag{8}$$

for each $k \in \mathbb{Z}_+$ and a sufficiently small $h_1 > 0$ in the domain (5).

Then the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) *is $\mathbf{y}_1$-stable in probability for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$.*

If conditions (7) *are replaced with the conditions*

$$V(k,\mathbf{x}) \leq V^*(\mathbf{y},\mathbf{z}_1), \quad V^*(\mathbf{0},\mathbf{z}_1) \equiv 0, \tag{9}$$

then the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) *is uniformly $\mathbf{y}_1$-stable in probability for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$.*

The proofs of Theorem 1 and the subsequent Theorem 2 are moved to the Appendix.

Within the framework of the considered approach, we can also state the conditions of *asymptotic* $\mathbf{y}_1$-stability in probability for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$ of the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1). Here is one of the versions of such conditions.

Theorem 2. *Suppose that for system* (1), *along with the main scalar Lyapunov V-function, one can indicate an additional vector function $\boldsymbol{\mu}(\mathbf{x})$, $\boldsymbol{\mu}(\mathbf{0}) \equiv \mathbf{0}$, for which the conditions*

$$a_1\Big(\|\mathbf{y}_1\| + \big\|\boldsymbol{\mu}(\mathbf{x})\big\|\Big) \leq V(k,\mathbf{x}) \leq a_2\Big(\|\mathbf{y}_1\| + \big\|\boldsymbol{\mu}(\mathbf{x})\big\|\Big), \tag{10}$$

$$\mathbf{L}V(k,\mathbf{x}) \leq -a_3\Big(\|\mathbf{y}_1\| + \big\|\boldsymbol{\mu}(\mathbf{x})\big\|\Big), \tag{11}$$

as well as conditions (9), *are satisfied for each $k \in \mathbb{Z}_+$ and a sufficiently small $h_1 > 0$ in the domain* (5).

Then the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) *is asymptotically $\mathbf{y}_1$-stable in probability for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$.*

Remark 3. The auxiliary Lyapunov V-function and its averaged difference (increment) $\mathbf{L}V(k,\mathbf{x})$ according to system (1) in Theorems 1 and 2 are, generally speaking, *alternating* functions in the domain (4) for each $k \in \mathbb{Z}_+$. Along with the main Lyapunov V-function, an additional auxiliary $\boldsymbol{\mu}$-function is introduced for the most expedient replacement of the domain (4) by the domain (5).

Conditions (7) are "intermediate" between the less restrictive condition $V(k, \mathbf{0}, \mathbf{z}) \equiv 0$ and the more restrictive conditions $V(k, \mathbf{x}) \leq V^*(k, \mathbf{y})$, $V^*(k, \mathbf{0}) \equiv 0$, under which the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) is, respectively, $\mathbf{y}_1$-stable in probability for large values of $\mathbf{z}_0$ or $\mathbf{y}_1$-stable in probability on the whole with respect to $\mathbf{z}_0$.

Remark 4. Within the framework of the proposed approach, the nonlinear Lyapunov V-functions can be constructed as alternating quadratic forms (or forms of higher order) $V(k, \mathbf{x}) \equiv V^*(k, \mathbf{y}_1, \boldsymbol{\mu}(k, \mathbf{x}))$ of the variables $\mathbf{y}_1$, $\boldsymbol{\mu}$. In this case, the choice of $\boldsymbol{\mu}$-functions must be coordinated with conditions (7), (9); for example, auxiliary $\boldsymbol{\mu}$-functions of the form $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{y}_2, \mathbf{z}_1)$, $\boldsymbol{\mu}(\mathbf{0}, \mathbf{z}_1) \equiv \mathbf{0}$ are admissible.

If a subsystem of the form

$$\mathbf{y}_1(k+1) = \mathbf{Y}_1\Big(k, \mathbf{y}_1(k), \boldsymbol{\mu}(k), \boldsymbol{\xi}(k)\Big),$$
$$\boldsymbol{\mu}(k+1) = \mathbf{Y}_1^*\Big(k, \mathbf{y}_1(k), \boldsymbol{\mu}(k), \boldsymbol{\xi}(k)\Big)$$

can be separated from system (1), then the Lyapunov V-function can be constructed using the numerical method in [11] as applied to the problem of stability with respect to all variables (with respect to $\mathbf{y}_1$, $\boldsymbol{\mu}$) of the zero equilibrium of this subsystem.

Remark 5. If system (1) admits a "complete" equilibrium $\mathbf{x}(k) = \mathbf{0}$, then in the case of $\boldsymbol{\mu}(k, \mathbf{x}) \equiv \mathbf{0}$, $\boldsymbol{\xi}(k) \equiv \mathbf{0}$, $\|\mathbf{x}_0\| < \delta$ and under conditions (6) and (8) we have a discrete version of the classical Rumyantsev theorem [15] on the stability with respect to part of the variables. In the case of $\boldsymbol{\xi}(k) \equiv \mathbf{0}$, Theorem 1 transforms into discrete versions [21, 22] of the corresponding theorems in [23, 24].

Remark 6. The stability with respect to some of the variables "on the average" of the zero equilibrium of systems of discrete-time stochastic equations was studied in [25, 26] by isolating "truncated" subsystems [25], as well as by constructing auxiliary systems [26]. The possibilities of using the Lyapunov function method to solve partial stability (stabilization) problems for systems of stochastic differential equations were analyzed in [27–31], including systems with a random structure [32–34]; a model example is used in [35] to compare optimal stabilization problems with respect to all and part of the variables for quasilinear systems of stochastic differential equations.

4. EXAMPLE

Let the discrete-time system (1) consist of the equations

$$\begin{aligned} y_1(k+1) &= \big[a + \alpha\xi_1(k)\big]y_1(k) + l y_2(k) z_1(k), \\ y_2(k+1) &= \big[b + d y_1(k)\big]y_2(k), \\ z_1(k+1) &= \big[c + e y_1(k)\big]z_1(k), \\ z_2(k+1) &= Z_2\big(k, \mathbf{x}(k)\big), \end{aligned} \tag{12}$$

where $\xi_1(k)$ is a sequence of independent random variables with standard normal distribution for each $k \in \mathbb{Z}_+$; the function Z_2 is arbitrary and satisfies only the general requirements for system (1); and a, b, c, d, e, l, and α are constant parameters.

System (12) admits the "partial" equilibrium

$$y_1(k) = y_2(k) = 0. \tag{13}$$

Along with the main Lyapunov function

$$V(\mathbf{x}) = y_1^2 + 2y_2^2 z_1^2, \tag{14}$$

we also consider the auxiliary function $\mu_1 = y_2 z_1$.

Conditions (9) and (10) are satisfied for the Lyapunov V-function in the domain (5), and its averaged difference (increment) $\mathbf{L}V(\mathbf{x})$ according to system (12) is determined for all $k \in \mathbb{Z}_+$ as follows:

$$\begin{aligned}
\mathbf{L}V(\mathbf{x}) = \mathbf{E}\Big[&\big(ay_1(k) + ly_2(k)z_1(k) + \alpha y_1(k)\xi_1(k)\big)^2 \\
&+ 2y_2^2(k)z_1^2(k)(b + dy_1(k))^2(c + ey_1(k))^2 | \mathbf{x}(k) = \mathbf{x}\Big] - y_1^2 - 2y_2^2 z_1^2 \\
= {}& a^2y_1^2 + 2aly_1y_2z_1 + l^2y_2^2z_1^2 + \alpha^2 y_1^2 + 2b^2c^2y_2^2z_1^2 \\
&+ r_1y_1y_2^2z_1^2 + r_2y_1^2y_2^2z_1^2 + r_3y_1^3y_2^2z_1^2 + 2d^2e^2y_1^4y_2^2z_1^2 - y_1^2 - 2y_2^2z_1^2 \\
= {}& (a^2 + \alpha^2 - 1)y_1^2 + 2aly_1\mu_1 + (l^2 + 2b^2c^2 - 2)\mu_1^2 + r_1y_1\mu_1^2 \\
&+ r_2y_1^2\mu_1^2 + r_3y_1^3\mu_1^2 + 2d^2e^2y_1^4\mu_1^2, \\
r_1 = bcr_0, \quad & r_2 = 2(b^2e^2 + 4bcde + c^2d^2), \quad r_3 = der_0, \quad r_0 = 4(be + cd);
\end{aligned}$$

the conditional expectation has been calculated in view of the relations $\mathbf{E}[\xi_1(k)] = 0$ and $\mathbf{E}[\xi_1^2(k)] = 1$ determining the standard normal distribution of the random variables $\xi_1(k)$.

Under the inequalities

$$a^2 + \alpha^2 < 1, \quad (a^2 + \alpha^2 - 1)(l^2 + 2b^2c^2 - 2) > a^2l^2 \tag{15}$$

satisfied for each $k \in \mathbb{Z}_+$ with a sufficiently small $h_1 > 0$ in the domain (5) (but not in the domain (4)), for any values of the parameters d and e one has the estimate $\mathbf{L}V(\mathbf{x}) \le -\beta(y_1^2 + \mu_1^2)$, $\beta = \text{const} > 0$. It follows that, apart from conditions (9) and (10), the Lyapunov V-function (14) also satisfies condition (11) in the domain (5).

Based on Theorem 2, we conclude that under conditions (15), the "partial" equilibrium (13) of system (12) is asymptotically y_1-stable in probability for large z_{10} on the whole with respect to z_{20}.

Let us geometrically explain this property of partial stability in connection with Definitions 2 and 3 introduced in Sec. 2. For each $k_0 \in \mathbb{Z}_+$ and for any arbitrarily small numbers $\varepsilon > 0$ ($\varepsilon < h_1$) and $\gamma > 0$, as well as for each number $L > 0$ given in advance, the boundary of the admissible domain $\mathbf{x}_0 \in D_\delta$ of initial disturbances in the three-dimensional space $Oy_1y_2z_1$ is the cylinder $\|\mathbf{y}_0\| = \delta$ of height $2L$ lying between two planes $y_1 = \pm\varepsilon$ (Fig. 1) with $\delta = \delta(\varepsilon, \gamma, L)$. If the solutions of system (12) start inside this δ-cylinder for $k = k_0$ (with an arbitrary value of z_{20}), then the sample paths corresponding to the indicated solutions in the space $Oy_1y_2z_1$ will remain between the indicated two ε-planes with probability at least $1 - \gamma$ for all $k \ge k_0$.

We can also indicate a number $\Delta = \Delta(\gamma, L, h_1) > 0$ such that

$$\begin{aligned}
\mathbf{P}\left\{\sup_{k \ge k_0} \big|y_1(k; k_0, \mathbf{x}_0)\big| \ge h_1\right\} < \gamma, \\
\mathbf{P}\left\{\lim_{k \to +\infty} y_1(k; k_0, \mathbf{x}_0) = 0\right\} \ge 1 - \gamma
\end{aligned} \tag{16}$$

for all $k \ge k_0$ and $\mathbf{x}_0 \in D_\Delta$. If the solutions of system (12) start inside a Δ-cylinder $\|\mathbf{y}_0\| = \Delta$ of height $2L$ for $k = k_0$ (with an arbitrary value of z_{20}), then the corresponding sample paths in the space $Oy_1y_2z_1$ will not only stay with a probability at least $1-\gamma$ for all $k \ge k_0$ between the ε-planes but will tend to the plane $y_1 = 0$ as $k \to \infty$.

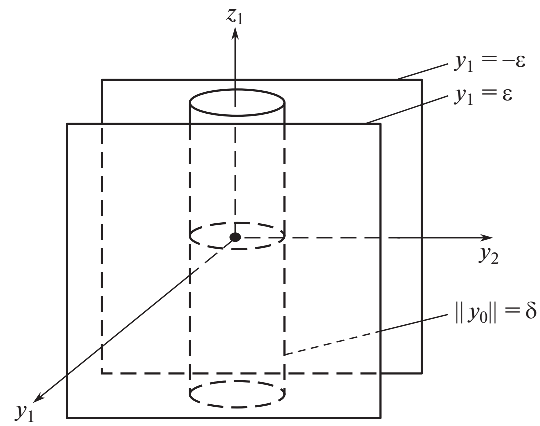

Fig. 1. Domains of admissible initial and current deviations from the invariant set $y_1(k) = y_2(k) = 0$.

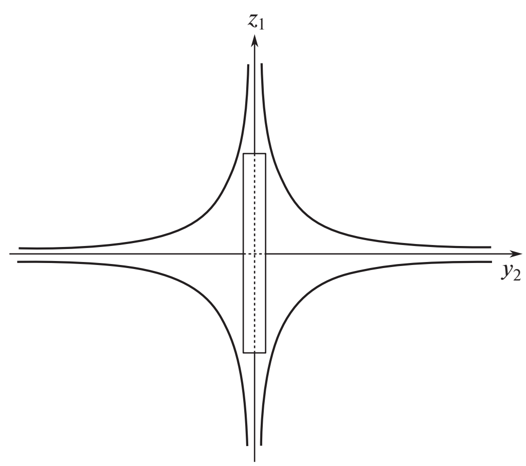

Fig. 2. Domain of admissible initial and current deviations (in projection on the plane Oy_2z_1).

However, within the framework of the approach considered, based on the transition from the domain (4) to the domain (5), one has the "extended" uniform (y_1, μ_1)-stability of the "partial" equilibrium (13) of system (12), ensuring the legitimacy of such a transition. Therefore, the δ-cylinder lies in the domain D^* of the space $Oy_1y_2z_1$ bounded by the surface $y_1^2 + y_2^2z_1^2 = \varepsilon^2$. (In Fig. 2, the location of the δ-cylinder is shown in projection on the plane Oy_2z_1; the corresponding rectangle with sides of length 2δ and $2L$ lies in a domain whose boundaries are the branches of the hyperbolae $y_2z_1 = \pm\,\varepsilon$.) If the solutions of system (12) start inside the δ-cylinder for $k = k_0$ (with an arbitrary value of z_{20}), then the corresponding sample paths in the space $Oy_1y_2z_1$ will remain with probability at least $1-\gamma$ in the domain D^* for all $k \ge k_0$.

An analysis of the structure of system (12) permits one to supplement the conclusions made: under conditions (15), the sample paths in the space $Oy_1y_2z_1$ corresponding to the solutions of system (12) starting inside the Δ-cylinder $\|\mathbf{y}_0\| = \Delta$ of height $2L$ (with an arbitrary value of z_{20}) "focus" with probability at least $1-\gamma$ as $k \to \infty$ or along the Oy_2-axis, or along the Oz_1-axis. Indeed, by virtue of the existing "extended" asymptotic (y_1, μ_1)-stability of the "partial" equilibrium (13) of system (12), the second and third equations in this nonlinear system can be represented in the form of the *linear* recurrence equations

$$y_2\,(k+1) = \big[b + dy_1(k;\, k_0, \mathbf{x}_0)\big]\,y_2(k),$$
$$z_1\,(k+1) = \big[c + ey_1(k; k_0, \mathbf{x}_0)\big]\,z_1(k),$$

Table 1

k	$\xi_1(k)$	$y_1(k)$	$y_2(k)$	$z_1(k)$	$\xi_1(k)$	$y_1(k)$	$y_2(k)$	$z_1(k)$
0	0	0.1	0.1	1	0	0.1	0.1	1
1	0	0.15	0.16	0.4333	−1	0.1167	0.16	0.4333
2	0	0.1443	0.2640	0.2094	1	0.1665	0.2587	0.1950
3	0	0.1275	0.4340	0.1000	1	0.1891	0.4311	0.0975
4	0	0.1072	0.7063	0.0461	0	0.1366	0.7282	0.0509
5	0	0.0861	1.1352	0.0203	−1	0.0599	1.1918	0.0239
6	0	0.0661	1.8005	0.0085	−1	0.0385	1.8590	0.0094
7	0	0.0484	2.8198	0.0034	1	0.0496	2.8601	0.0035
8	0	0.0338	4.3662	0.0013	0	0.0348	4.4320	0.0013
9	0	0.0226	6.6969	0.0005	0	0.0233	6.8022	0.0005
10	0	0.0145	10.197	0.00017	1	0.0227	10.362	0.00017
...	...	...	...	...	...	...	...	...
15	0	0.0009	79.001	7.6×10^{-7}	−1	0.0007	82.073	9.1×10^{-7}
...	...	...	...	...	...	...	...	...
20	0	0.000048	600.69	3.2×10^{-9}	−1	0.000017	623.28	3.7×10^{-9}

and with the component $y_1(k; k_0, \mathbf{x}_0)$ of any solution of system (12) starting inside the Δ-cylinder of height $2L$, one has relations (16). Therefore, under conditions (15), for these solutions of system (12) as $k \to \infty$ with probability at least $1 - \gamma$ one has the relations $|y_2(k)| \to \infty$ and $|z_1(k)| \to 0$ (for $|b| > 1$ and $|c| < 1$) or the relations $|y_2(k)| \to 0$ and $|z_1(k)| \to \infty$ (for $|b| < 1$ and $|c| > 1$).

To render the obtained numerical data concrete, the left-hand side of Table 1 lists the results of calculations based on the recurrence relations (12) in the range $k \in [0, 20]$ for the "unperturbed" case of $\xi_1(k) \equiv 0$ with $y_1(0) = y_2(0) = 0{,}1$ and $z_1(0) = 1$, as well as with the parameter values $a = 1/2$, $b = 3/2$, $c = 1/3$, and $d = e = l = 1$. Under the random action $\xi_1(k)$, the sample paths group around the "unperturbed" trajectory, focused along the Oy_2-axis as $k \to \infty$. To estimate the effect of the random action upon the dynamics of system (12), the right-hand side of Table 1 lists the results of computations for $\alpha = 1/3$ and with the same values of parameters in the case where the admissible realization $\xi_1(k)$ is determined on the interval $k \in [0, 20]$ by the sequence $\{0, -1, 1, 1, 0, -1, -1, 1, 0, 0, 1, 1, -1, 0, 0, -1, 1, 0, -1, 1, -1\}$.

5. CONCLUSIONS

For a nonlinear system of stochastic discrete-time (finite-difference) equations subject to the action of a discrete random process of the white noise type, a statement of the problem of stability (asymptotic stability) with respect to the variables of the "partial" zero equilibrium is given. The initial values of the "uncontrolled" variables that do not determine the considered "partial" equilibrium are assumed to be large (bounded in the norm by any predetermined number) in one part and arbitrary in the other.

Sufficient conditions are given for the solvability of this problem in the context of the discrete-stochastic version of the Lyapunov function method in the corresponding modification. Along with the main Lyapunov V-function, we consider an additional (vector, generally speaking) auxiliary $\boldsymbol{\mu}$-function for correcting the domain in which the main Lyapunov V-function is constructed. The expediency of this approach lies in the fact that, as a result, the main Lyapunov V-function, as well as its averaged difference (increment) according to the system in question, can be of alternating signs.

APPENDIX

Proof of Theorem 1.

I. Assume that conditions (6)–(8) are satisfied in the domain (5) for each $k \in \mathbb{Z}_+$ and for a sufficiently small $h_1 > 0$.

Take an arbitrary number ε $(0 < \varepsilon < h_1)$ and consider an arbitrary time k_0 and an initial point $\mathbf{x}_0$ in the domain

$$D_\varepsilon = \Big\{\|\mathbf{y}_0\| < \varepsilon,\ \|\mathbf{z}_{10}\| \le L,\ \|\mathbf{z}_{20}\| < \infty\Big\}.$$

Consider a random process $\mathbf{x}(k; t_0, \mathbf{x}_0)$ $(k \ge k_0)$ that is a solution of system (1) and denote the "integer" time of the first exit of this process from the domain $\|\mathbf{y}_1\| \le \varepsilon$ by τ_ε. If some trajectories leave the domain $\|\mathbf{y}_1\| \le \varepsilon$ in no finite time, then τ_ε for these trajectories is taken to be ∞. Set

$$\tau(k) = \min(\tau_\varepsilon, k); \quad \tau(k_0) = k_0.$$

One has the relations

$$\begin{aligned}
V\Big(\tau(k), \mathbf{x}\big(\tau(k); k_0, \mathbf{x}_0\big)\Big) - V(k_0, \mathbf{x}_0) &= V\Big(\tau(k), \mathbf{x}\big(\tau(k); k_0, \mathbf{x}_0\big)\Big) - V\Big(\tau(k-1), \mathbf{x}\big(\tau(k-1); k_0, \mathbf{x}_0\big)\Big)\\
&+ V\Big(\tau(k-1), \mathbf{x}\big(\tau(k-1); k_0, \mathbf{x}_0\big)\Big) - V\Big(\tau(k-2), \mathbf{x}\big(\tau(k-2); k_0, \mathbf{x}_0\big)\Big) + \cdots\\
&+ V\Big(\tau(k_0+1), \mathbf{x}\big(\tau(k_0+1); k_0, \mathbf{x}_0\big)\Big) - V(k_0, \mathbf{x}_0) = \sum_{s=k_0}^{k-1} \Delta V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big);
\end{aligned}$$

$$\Delta V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big) = V\Big(\tau(s+1), \mathbf{x}\big(\tau(s+1); k_0, \mathbf{x}_0\big)\Big) - V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big).$$

It follows from these equalities that for the sequence $v(k)$ of random variables $v(k) = V(\tau(k), \mathbf{x}(\tau(k); k_0, \mathbf{x}_0))$ generated by the realizations $\{\mathbf{x}(k,\omega), \boldsymbol{\xi}(k,\omega)\}$ of the random process $\{\mathbf{x}(k), \boldsymbol{\xi}(k)\}$ determined by system (1) one has the "averaged" relations

$$\begin{aligned}
\mathbf{E}\Big[V\big(\tau(k), \mathbf{x}(\tau(k); k_0, \mathbf{x}_0)\big) - V(k_0, \mathbf{x}_0)\Big] &= \mathbf{E}V\big(\tau(k), \mathbf{x}(\tau(k); k_0, \mathbf{x}_0)\big) - V(k_0, \mathbf{x}_0)\\
&= \sum_{s=k_0}^{k-1} \mathbf{E}\Delta V\big(\tau(s), \mathbf{x}(\tau(s); k_0, \mathbf{x}_0)\big).
\end{aligned}$$

Taking into account the equalities (obtained with allowance for the rule of calculating the repeated expectation)

$$\begin{aligned}
&\mathbf{E}\Big[\Delta V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big]\\
&\quad = \mathbf{E}\Big[V\Big(\tau(s+1), \mathbf{X}\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big), \boldsymbol{\xi}\big(\tau(s)\big)\Big) - V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big)\Big]\\
&\quad = \mathbf{E}\Big\{\mathbf{E}\Big[V\Big(\tau(s+1), \mathbf{X}\big(\tau(s), \mathbf{x}\big(\tau(s), \boldsymbol{\xi}(\tau(s))|\mathbf{x}(\tau(s))\big)\big)\big) = \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big]\Big\}\\
&\qquad - \mathbf{E}\Big[V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big]
\end{aligned}$$

$$= \mathbf{E}\Big\{ \mathbf{E}\Big[V\Big(\tau(s+1), \mathbf{X}\big(\tau(s), \mathbf{x}(\tau(s), \boldsymbol{\xi}(\tau(s)))\big) | \mathbf{x}(\tau(s)) = \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big]$$

$$- V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big\} = \mathbf{E}\Big[\mathbf{L}V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big],$$

we arrive at the relation (a discrete-time version of the Dynkin formula) [9]

$$\mathbf{E}V\Big(\tau(k), \mathbf{x}\big(\tau(k); k_0, \mathbf{x}_0\big)\Big) - V(k_0, \mathbf{x}_0) = \sum_{s=k_0}^{k-1} \mathbf{E}\Big[\mathbf{L}V\Big(\tau(s), \mathbf{x}\big(\tau(s); k_0, \mathbf{x}_0\big)\Big)\Big].$$

As a result, based on condition (8), we obtain the inequality

$$\mathbf{E}V\Big(\tau(k), \mathbf{x}\big(\tau(k); k_0, \mathbf{x}_0\big)\Big) \le V(k_0, \mathbf{x}_0) < \infty. \tag{A.1}$$

If the inequality $k > \tau_\varepsilon$ holds (in this case, $\tau(k) = \tau_\varepsilon$), then we have the relations

$$\big\|\mathbf{y}_1(\tau(k); k_0, \mathbf{x}_0)\big\| = \big\|\mathbf{y}_1(\tau_\varepsilon; k_0, \mathbf{x}_0)\big\| \ge \varepsilon.$$

If, however, we have the inequality $k < \tau_\varepsilon$ (in this case,e $\tau(k) = k$), then, based on the Chebyshev–Markov inequality and the estimate (A.1), we find

$$\begin{aligned}
\mathbf{P}\Big[\big\|\mathbf{y}_1(k; k_0, \mathbf{x}_0)\big\| > \varepsilon\Big] &\le a_1^{-1}(\varepsilon)\mathbf{E}\Big[a_1\Big(\big\|\mathbf{y}_1(k; k_0, \mathbf{x}_0)\big\|\Big)\Big] \\
&\le a_1^{-1}(\varepsilon)\mathbf{E}\Big[a_1\Big(\big\|\mathbf{y}_1(k; k_0, \mathbf{x}_0)\big\| + \Big\|\boldsymbol{\mu}\big(k, \mathbf{x}(k; k_0, \mathbf{x}_0)\big)\Big\|\Big)\Big] \\
&\le a_1^{-1}(\varepsilon)\mathbf{E}\Big[V\big(k, \mathbf{x}(k; k_0, \mathbf{x}_0)\big)\Big] \\
&= a_1^{-1}(\varepsilon)\mathbf{E}\Big[V\Big(\tau(k), \mathbf{x}\big(\tau(k); k_0, \mathbf{x}_0\big)\Big)\Big] \le a_1^{-1}(\varepsilon)V(k_0, \mathbf{x}_0).
\end{aligned} \tag{A.2}$$

Since the Lyapunov function $V(k, \mathbf{x})$ is continuous for each $k \in \mathbb{Z}_+$, $V(t, \mathbf{0}) \equiv 0$, and conditions (7) are satisfied for all $k_0 \ge 0$ and for each given number $L > 0$, we conclude that the limit relation

$$\lim_{\|\mathbf{y}_0\| \to 0} V(k_0, \mathbf{x}_0) = 0 \tag{A.3}$$

holds for $\|\mathbf{z}_{10}\| \le L$ uniformly with respect to $\|\mathbf{z}_{20}\| < \infty$.

Therefore, for all $k_0 \ge 0$ and for each given number $L > 0$, based on inequalities (A.2) and (A.3), we have the limit relation

$$\lim_{\|\mathbf{y}_0\| \to 0} \mathbf{P}\left[\sup_{k > k_0} \big\|\mathbf{y}_1(k; t_0, \mathbf{x}_0)\big\| > \varepsilon\right] = 0,$$

which holds for $\|\mathbf{z}_{10}\| \le L$ uniformly with respect to $\|z_{20}\| < \infty$.

As a result, for each $k_0 \le 0$ and for any arbitrarily small numbers $\varepsilon > 0$ and $\gamma > 0$, as well as for each number $L > 0$ given in advance, there exists a number $\delta(\varepsilon, \gamma, L, k_0) > 0$ such that inequality (2) holds for all $k \ge k_0$ and $\mathbf{x}_0 \in D_\delta$. Consequently, for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$ the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) is $\mathbf{y}_1$-stable in probability.

II. If conditions (9) are satisfied instead of conditions (7), then for each given number $L > 0$ the limit relation (A.3) is satisfied for $\|\mathbf{z}_{10}\| \le L$ uniformly not only with respect to $\|\mathbf{z}_{20}\| < \infty$ but also with respect to $k_0 \geq 0$. As a result, for each $k_0 \geq 0$ and for any arbitrarily small numbers $\varepsilon > 0$ and $\gamma > 0$, as well as for each number $L > 0$ given in advance there exists a number $\delta(\varepsilon, \gamma, L) > 0$ independent of k_0 such that inequality (2) holds for all $k \geq k_0$ and $\mathbf{x}_0 \in D_\delta$. Consequently, for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$ the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) is uniformly $\mathbf{y}_1$-stable in probability. The proof of Theorem 1 is complete. ∎

Proof of Theorem 2. Under the assumptions of the theorem, the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) is uniformly $\mathbf{y}_1$-stable in probability for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$.

Based on inequality (A.1), the sequence $v(k)$ of random variables $v(k) = V(\tau(k), \mathbf{x}(\tau(k); k_0, \mathbf{x}_0))$ generated by realizations $\{\mathbf{x}(k,\omega), \boldsymbol{\xi}(k,\omega)\}$ of the random process $\{\mathbf{x}(k), \boldsymbol{\xi}(k)\}$ determined by system (1) forms a nonnegative supermartingale, which is an analog of a monotone decreasing sequence in the deterministic case. Therefore, based on inequality (A.1), for each initial point $\mathbf{x}_0$ in the domain D_ε, with probability 1 one has the limit relation [36]

$$v(k) = V\big(\tau(k), \mathbf{x}(\tau(k); k_0, \mathbf{x}_0)\big) \to v_*, \quad k \to \infty.$$

Using the well-known scheme of analysis [2, 3, 9], it can be shown that the equality $v_* = 0$ is satisfied with probability 1. Indeed, applying the operation of expectation and passing to the limit on both sides of inequality (11), with probability 1 we obtain

$$\mathbf{E}\left[a_3\left(\left\|\mathbf{y}_1\big(\tau(k); k_0, \mathbf{x}_0\big)\right\|\right)\right] \to 0, \quad k \to \infty.$$

Therefore, by virtue of conditions (10), (11) and the Fatou lemma the equality $\mathbf{E}[a_3(a_2^{-1}(v_*))] = 0$ holds, which implies that $v_* = 0$. However, in the case of $v_* = 0$, as $\|\mathbf{y}_0\| \to 0$ one has the limit relation

$$\mathbf{P}\left\{\lim_{k\to+\infty} \left\|\mathbf{y}_1(k; k_0, \mathbf{x}_0)\right\| = 0\right\} = 1,$$

and consequently, the "partial" equilibrium $\mathbf{y}(k) = \mathbf{0}$ of system (1) is asymptotically $\mathbf{y}_1$-stable in probability for large values of $\mathbf{z}_{10}$ on the whole with respect to $\mathbf{z}_{20}$. The proof of Theorem 2 is complete. ∎

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