

Core and Superdifferential of a Fuzzy TU -Cooperative Game

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Abstract—In the paper, we consider conditions providing the coincidence of the cores and superdifferentials of fuzzy cooperative games with side payments. It turned out that weak homogeneity is one of the simplest sufficient conditions. Moreover, by applying the so-called S^* -representation of a fuzzy game introduced by the author, we show that for any v with nonempty core $C(v)$ there exists some game u such that $C(v)$ coincides with the superdifferential of u . By applying the subdifferential calculus, we describe a structure of the core for classical fuzzy extensions of the ordinary cooperative game (e.g., the Aubin and Owen extensions) as well as for some new continuations, like the generalized Airport game.

Keywords: fuzzy cooperative game, S^* -representation, superdifferential, the core of a fuzzy game, weak homogeneity

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1. INTRODUCTION

The paper deals with the analysis of the relationship between the cores and superdifferentials of fuzzy TU -cooperative games of n persons. The requirements ensuring the coincidence of these cores and superdifferentials are studied. The simplest sufficient conditions include some weak analog of the homogeneity of fuzzy games. Particular attention is paid to the consideration of the inhomogeneous case. Using the so-called S^* -representation of a fuzzy game proposed by the author [3], we manage to show that for an arbitrary game v with a nonempty core there exists a game u such that the core of v coincides with the superdifferential of u . The resulting general theorem about the representation of the core in the form of the superdifferential of a suitable modification of the original game permits one to use the technique of subdifferential calculus [5, 6, 9] to describe the structure of cores of both classical fuzzy extensions of ordinary games (for example, for Aubin's extensions [9]) and some new extensions like a generalization of the well-known Airport game [4, 11].

The main content of the paper is divided into three sections. The first of them (Sec. 2) contains the notation and definitions necessary in what follows. It also states a criterion for the nonemptiness of the core of a fuzzy game with side payments as well as the definition of the S^* -representation of such a game and some properties of this representation. In particular, the coincidence of cores of fuzzy games with the cores of their S^* -representations is noted. The second one (Sec. 3) establishes the main result, a theorem on the coincidence of the core of a fuzzy V -balanced game with the superdifferential of its homogeneous modification. Finally, in the third one (Sec. 4), possible applications of subdifferential calculus are illustrated by an example in which we analyze the core of the Aubin extension v_{Aub} of an “almost positive” cooperative game v and the anticore of one generalization of the well-known Airport game, which simulates the rational distribution of costs for the construction of a runway (see [4, 11] for details). The description of the core of the Aubin

extension v_{Aub} established here deserves a special mention: this core is a singleton and consists of the Shapley value of the game v .

2. MAIN DEFINITIONS

To keep the presentation self-contained, we start from the definition of a fuzzy TU -cooperative game of n persons. Let us first recall the definition of a fuzzy coalition [8, 9]. Let $N = \{1, \dots, n\}$ be the set of players in the game under consideration. Let I^N denote the unit hypercube defined by the formula $I^N := \{\tau = (\tau_1, \dots, \tau_n) \in \mathbf{R}^N \mid \tau_i \in [0, 1], i \in N\}$. Fuzzy coalitions are elements of the set $\sigma_F := I^N \setminus \{0\}$. We also recall that each standard coalition $S \subseteq N$ is identified with its indicator function e_S defined by the formula $(e_S)_i = 1$ for $i \in S$, and $(e_S)_i = 0$ for $i \in N \setminus S$. Further, for each fuzzy coalition $\tau = (\tau_1, \dots, \tau_n) \in \sigma_F$ by $N(\tau)$ we denote its support $N(\tau) := \{i \in N \mid \tau_i > 0\}$. As usual, for a vector $x = (x_1, \dots, x_n) \in \mathbf{R}^N$ and a set $S \subseteq N$, by $x_S \in \mathbf{R}^S$ we denote the restriction of x to S , $(x_S)_i = x_i, i \in S$. The restriction of $\tau \in \sigma_F$ to its support $N(\tau)$ is denoted by $\tau^+ := \tau_{N(\tau)}$. For $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m) \in \mathbf{R}^m$, by $a \cdot b$, as usual, we denote the inner product of the vectors a and b , $a \cdot b = \sum_{k=1}^m a_k b_k$. Finally, setting $\mathbf{R}^\tau := \mathbf{R}^{N(\tau)}$, we introduce the following definition.

Definition 2.1. The fuzzy TU -cooperative game of n persons generated by a generalized characteristic function $v : \sigma_F \rightarrow \mathbf{R}$ is the multivalued mapping $\tau \mapsto G_v(\tau)$ that takes each fuzzy coalition τ to the set of payoffs attainable by τ ; this set is defined by $G_v(\tau) = \{x \in \mathbf{R}^\tau \mid \tau^+ \cdot x \leq v(\tau)\}$.

Thus, according to Definition 2.1 and the terminology adopted in [9, 11], a fuzzy TU -cooperative game v is a fuzzy NTU -game of a special type in which the sets of payoffs attainable by coalitions τ are some half-spaces with normals τ^+ in the corresponding spaces \mathbf{R}^τ .

Let us recall [3, 9] the main concepts of the present paper, namely, the definition of blocking in a fuzzy game v and the notion of core of this game. Throughout the following, as usual, the payoffs attainable by the “grand coalition” e_N will be called the payoffs of the game v .

Definition 2.2. A fuzzy coalition $\tau = (\tau_1, \dots, \tau_n)$ is said to *block* a payoff $x = (x_1, \dots, x_n)$ of the game v if there exists a vector $y = (y_i)_{i \in N(\tau)} \in \mathbf{R}^\tau$ such that

$$(b.1) \quad \sum_{i \in N(\tau)} \tau_i y_i \leq v(\tau).$$

$$(b.2) \quad y_i > x_i, \quad i \in N(\tau).$$

Definition 2.3. The *core of a fuzzy cooperative game* v is the set of all payoffs of this game that are not blocked by any coalition $\tau \in \sigma_F$. The core of the game v will be denoted by $C(v)$.

Passing to the statement of the criterion for the nonemptiness of the core $C(v)$, recall [3] that a finite family of coalitions $\{\tau_k\}_{k \in K} \subseteq \sigma_F$ is said to be *F-balanced* if there exist numbers $\lambda_k \geq 0, k \in K$, such that $\sum_{k \in K} \lambda_k \tau_k = e_N$. Just as in [1], the nonnegative numbers λ_k occurring in this relation will be called the *weights* corresponding to the family $\{\tau_k\}_{k \in K}$.

Definition 2.4. A fuzzy TU -cooperative game $v : \sigma_F \rightarrow \mathbf{R}$ is said to be *V-balanced* if for any *F-balanced* family of fuzzy coalitions $\{\tau_k\}_{k \in K}$ and the corresponding weights $\lambda_k, k \in K$, one has the inequality $\sum_{k \in K} \lambda_k v(\tau_k) \leq v(e_N)$.

One can readily verify (see, e.g., [3]) that the core of a fuzzy TU -cooperative game v has the form

$$C(v) = \{x \in \mathbf{R}^N \mid x \cdot e_N = v(e_N), \quad x \cdot \tau \geq v(\tau), \quad \tau \in \sigma_F\}. \tag{2.1}$$

Based on this representation, the following criterion for the nonemptiness of $C(v)$ was obtained in [3].

Theorem 2.1. *The core $C(v)$ of a fuzzy TU-cooperative game v is nonempty if and only if the game v is V -balanced.*

To obtain the main result of the present paper concerning the cores and superdifferentials of fuzzy games, it is expedient to use the so-called S^* -representation of such games introduced in [3]. Recall [3] that the S^* -representation v^* of a game v is given by the formula

$$v^*(\tau^*) := \sup \left\{ v(t\tau^*)/t \mid t \in \left(0, \frac{1}{\|\tau\|_\infty} \right] \right\}, \quad \tau^* \in \sigma_F^*, \tag{2.2}$$

where $\|\tau\|_\infty = \max\{|\tau_i| \mid i \in N\}$ for any $\tau = (\tau_1, \dots, \tau_n) \in \sigma_F$ and the simplex σ_F^* is part of the hypercube I^N , $\sigma_F^* := \{\tau \in \sigma_F \mid \sum_{i \in N} \tau_i = 1\}$.

In what follows, we focus on the important class of S^* -regular fuzzy cooperative games.

Definition 2.5. A game v is said to be S^* -regular if its S^* -representation v^* satisfies the following conditions:

- ($S^*.1$) $v^*(\tau^*) < \infty$ for each $\tau^* \in \sigma_F^*$.
- ($S^*.2$) $v^*(e_N/n) = v(e_N)/n$.

As was established in [3], the S^* -regularity of a game v is a necessary condition for its core to be nonempty.

Proposition 2.1. *If a fuzzy TU-cooperative game v has a nonempty core, then it is S^* -regular.*

One more property, useful in what follows, of the S^* -representation v^* of the game v has the following form (Theorem 5.2 in [3]).

Theorem 2.2. *If a fuzzy game v satisfies condition ($S^*.2$), then its core $C(v)$ is nonempty if and only if so is the core $C(v^*)$ of its S^* -representation v^* .¹ In this case, one has $C(v) = C(v^*)$.*

Finally, a useful criterion for the nonemptiness of the core of a fuzzy cooperative game can be stated in terms of its homogeneity and weakened concavity.

Theorem 2.3. *If the fuzzy cooperative game $v : \sigma_F \rightarrow \mathbf{R}$ is homogeneous, then a necessary and sufficient condition that the core $C(v)$ be nonempty is that the restriction of v to the simplex σ_F^* is a concave game with respect to the center of gravity e_N^* .²*

3. ON THE REPRESENTATION OF THE CORE IN THE FORM OF A SUPERDIFFERENTIAL

Passing to the representation of the core of a fuzzy game in the form of the superdifferential of a suitable modification of this game, we recall the corresponding concepts of subdifferential calculus [5, 7, 9]. For convenience, the superdifferential (which differs from the subdifferential only in the sense of the corresponding inequalities) is used as the main object instead of the subdifferential.

Definition 3.1. Let v be a fuzzy TU-cooperative game of n persons. We say that a vector $x \in \mathbf{R}^n$ is a supergradient of the game v at a point $\bar{\tau} \in \sigma_F$ if

$$v(\tau) - v(\bar{\tau}) \leq x \cdot (\tau - \bar{\tau}), \quad \tau \in \sigma_F. \tag{3.1}$$

¹ Recall [3] that $C(v^*) = \{x \in \mathbf{R}^N \mid e_N^* \cdot x = v^*(e_N^*), \tau^* \cdot x \geq v^*(\tau^*), \tau^* \in \sigma_F^*\}$.

² A function v on σ_F^* is said to be concave with respect to the center of gravity e_N^* [3] if for any representation of e_N^* in the form of a convex combination $e_N^* = \sum_{k \in K} \lambda_k \tau_k, \tau_k \in \sigma_F^*, k \in K$, one has the inequality $v(e_N^*) \geq \sum_{k \in K} \lambda_k v(\tau_k)$.

Remark 3.1. It is clear that the gradient of a smooth concave game v at an interior point of the set σ_F is its only supergradient at this point [5, 7].

Definition 3.2. The set of all supergradients of a game v at a point $\bar{\tau}$ will be denoted by $\hat{\partial}v(\bar{\tau})$ and will be called the superdifferential of v at that point.

Below, as before, we use the notation $e_N^* = \frac{1}{n}e_N$. One basic concept of the paper is given by the following definition.

Definition 3.3. The superdifferential of a fuzzy TU -cooperative game v is the superdifferential of v at the point e_N^* .

Formula (3.1) and the definition of the subdifferential [5, 7] imply a simple formula relating the superdifferential $\partial(v)(\bar{\tau})$ of a fuzzy game v at a point $\bar{\tau}$,

$$\hat{\partial}v(\bar{\tau}) = \{x \in \mathbf{R}^n \mid v(\tau) - v(\bar{\tau}) \leq x \cdot (\tau - \bar{\tau}), \quad \tau \in \sigma_F\}$$

with its subdifferential

$$\partial(v)(\bar{\tau}) = \{x \in \mathbf{R}^n \mid v(\tau) - v(\bar{\tau}) \geq x \cdot (\tau - \bar{\tau}), \quad \tau \in \sigma_F\}$$

at the same point. Namely, the following relations are true:

$$\hat{\partial}v(\bar{\tau}) = -\partial(-v)(\bar{\tau}), \quad \bar{\tau} \in \sigma_F.$$

The superdifferential and the core of a fuzzy game v are closely related. In particular, the following simple but important result follows directly from their definition and from the representation of the core $C(v)$ as a solution of a system of linear inequalities (formula (2.1) in the preceding part; see also Proposition 3.1 in [3]): if $v(e_N^*) = v(e_N)/n$, then each element of the core $C(v)$ is a supergradient of the game v at the point e_N^* .

Proposition 3.1. For any fuzzy TU -cooperative game v satisfying the condition $v(e_N^*) = \frac{v(e_N)}{n}$, one has the embedding $C(v) \subseteq \hat{\partial}v(e_N^*)$.

Proof. The case of $C(v) = \emptyset$ does not need justification. Therefore, we will assume that the core $C(v)$ is not empty. To verify the embedding $C(v) \subseteq \hat{\partial}v(e_N^*)$, consider an arbitrary payoff $x \in C(v)$ and some fuzzy coalition τ . Since $x \cdot \tau \geq v(\tau)$, we see that the following inequality holds: $x \cdot \tau - v(e_N^*) \geq v(\tau) - v(e_N^*)$. Hence, by virtue of the condition $v(e_N^*) = v(e_N)/n$ and the equality $v(e_N)/n = x(e_N^*)$ following from the inclusion $x \in C(v)$, we obtain the desired relation $x \cdot \tau - x(e_N^*) \geq v(\tau) - v(e_N^*)$. Since τ is arbitrary, we see that x belongs to the superdifferential $\hat{\partial}v(e_N^*)$, as desired. ■

For more meaningful statements, some additional conditions are required. Recall [3] that a fuzzy game v is said to be *homogeneous* if $v(t\tau) = tv(\tau)$ for all $t > 0$ and $\tau \in \sigma_F$ such that $t\tau$ belongs to σ_F . We introduce two weakened versions of homogeneity.

Definition 3.4. We say that a game v is *weakly homogeneous* if there exist positive numbers μ and ν such that $\mu < 1 < \nu \leq n$ and moreover, $v(\mu e_N^*) = \mu v(e_N^*)$ and $v(\nu e_N^*) = \nu v(e_N^*)$.

Definition 3.5. A fuzzy game v is said to be *diagonally homogeneous* (*D-homogeneous*) if we have $v(te_N) = tv(e_N)$ for each $t \in [0, 1]$.

Remark 3.2. One can readily verify that the condition occurring in the definition of diagonal homogeneity is equivalent to the following: $v(te_N^*) = tv(e_N^*)$ for each $t \in [0, n]$.

The following statement is simple but important.

Theorem 3.1. *For any V -balanced weakly homogeneous fuzzy TU -cooperative game v satisfying the condition $v(e_N^*) = v(e_N)/n$, one has $C(v) = \hat{\partial}v(e_N^*)$.*

Proof. The relation $C(v) \subseteq \hat{\partial}v(e_N^*)$ was established in Proposition 3.1. Let us prove the embedding $\hat{\partial}v(e_N^*) \subseteq C(v)$. Let x be an arbitrary element of $\hat{\partial}v(e_N^*)$. It follows from the weak homogeneity of v that there exist numbers $\mu \in (0, 1)$ and $\nu \in (1, n]$ such that $v(\mu e_N^*) = \mu v(e_N^*)$ and $v(\nu e_N^*) = \nu v(e_N^*)$. Set $\delta = 1 - \mu$ and $\gamma = \nu - 1$. By the definition of the supergradient at the point e_N^* , taking into account the weak homogeneity of the game v , we have

$$\begin{aligned} x \cdot \delta e_N^* &= x \cdot e_N^* - x \cdot \mu e_N^* \leq v(e_N^*) - v(\mu e_N^*) = \delta v(e_N^*), \\ x \cdot \gamma e_N^* &= x \cdot \nu e_N^* - x \cdot e_N^* \geq v(\nu e_N^*) - v(e_N^*) = \gamma v(e_N^*), \end{aligned}$$

and therefore, since the numbers δ and γ are positive, we obtain the relation $x \cdot e_N^* = v(e_N^*)$. Hence, in view of the inequalities

$$x \cdot \tau - x \cdot e_N^* \geq v(\tau) - v(e_N^*), \quad \tau \in \sigma_F^*,$$

following from the inclusion $x \in \hat{\partial}v(e_N^*)$, we have $x \cdot \tau \geq v(\tau)$ for all coalitions $\tau \in \sigma_F$. To complete the proof of the inclusion $x \in C(v)$, it remains to note that the condition $v(e_N^*) = v(e_N)/n$ and the equation $x \cdot e_N^* = v(e_N^*)$ established above imply the relation $x \cdot e_N = v(e_N)$. ■

Corollary 3.1. *If $C(v) \neq \emptyset$ and v is a diagonally homogeneous game, then $C(v) = \hat{\partial}v(e_N^*)$.*

Corollary 3.2. *If $C(v) \neq \emptyset$ and v is a homogeneous game, then $C(v) = \hat{\partial}v(e_N^*)$.*

Remark 3.3. One can readily verify that for a diagonally homogeneous game v (with nonempty core), its superdifferentials at all points of the interval $(0, ne_N^*) := \{te_N^* \mid t \in (0, n)\}$ coincide with each other. This is true because they are equal to the core $C(v)$. (The proof reproduces the argument in Theorem 3.1 almost word for word.) Therefore, we can replace e_N^* by any other point of the interval $(0, ne_N^*)$ in the definition of the superdifferential of a diagonally homogeneous game.

Let us proceed to the general (not necessarily weakly homogeneous) case³ and show that for any V -balanced fuzzy TU -cooperative game v there exists a fuzzy game u whose superdifferential at the point e_N^* coincides with the core of v , $C(v) = \hat{\partial}u(e_N^*)$. Here one version of the game u can be constructed using the S^* -representation v^* of the fuzzy game v . Namely, from the S^* -representation v^* of the game v we construct the so-called *homogeneous extension* \hat{v} of the game v to σ_F and show that this extension can play the role of the above-indicated game u .

Definition 3.6. The homogeneous extension of a game v is the function \hat{v} defined by the formula

$$\hat{v}(\tau) := tv^*(\tau^*) \quad \text{for } \tau = t\tau^* \in \sigma_F, \tau^* \in \sigma_F^*. \tag{3.2}$$

Remark 3.4. The analysis of the basic properties of the homogeneous extension \hat{v} of the game v is of considerable independent interest. In addition to the relationship between the cores of the games v and \hat{v} considered below, we note here the theoretically important property of preserving the concavity of the game v^* : if v^* is a concave game, then so is the game \hat{v} . Indeed, let v^* be a concave game. Consider arbitrary fuzzy coalitions $\tau^*, \tau'^* \in \sigma_F^*$, $\tau = t\tau^*$, $\tau' = t'\tau'^*$ and nonnegative numbers λ and λ' such that $\lambda + \lambda' = 1$. Let us show that $\hat{v}(\lambda\tau + \lambda'\tau') \geq \lambda\hat{v}(\tau) + \lambda'\hat{v}(\tau')$. To this end, using formula (3.2), we obtain

$$\hat{v}(\lambda\tau + \lambda'\tau') = \hat{v} \left((\lambda t + \lambda' t') \left[\frac{\lambda t \tau^* + \lambda' t' \tau'^*}{\lambda t + \lambda' t'} \right] \right) = (\lambda t + \lambda' t') v^*(\bar{\tau}^*), \tag{3.3}$$

³ For example, the well-known multilinear Owen extension [10] is inhomogeneous.

where

$$\bar{\tau}^* = \frac{\lambda t}{\lambda t + \lambda' t'} \tau^* + \frac{\lambda' t'}{\lambda t + \lambda' t'} \tau'^*.$$

It is clear that $\bar{\tau}^*$, being a convex combination of the elements τ^* and τ'^* of σ_F^* , belongs to the simplex σ_F^* . Therefore, relation (3.3) is defining for $\hat{v}(\lambda\tau + \lambda'\tau')$. Namely, according to the construction of \hat{v} , we have $\hat{v}(\lambda\tau + \lambda'\tau') = (\lambda t + \lambda' t')v^*(\bar{\tau}^*)$. Hence, owing to the concavity of the function v^* , we obtain

$$\hat{v}(\lambda\tau + \lambda'\tau') \geq (\lambda t + \lambda' t') \left[\frac{\lambda t}{\lambda t + \lambda' t'} v^*(\tau^*) + \frac{\lambda' t'}{\lambda t + \lambda' t'} v^*(\tau'^*) \right].$$

Therefore, taking into account the relations $\hat{v}(t\tau^*) = tv^*(\tau^*)$, $\hat{v}(t'\tau'^*) = t'v^*(\tau'^*)$ and $\tau = t\tau^*$, $\tau' = t'\tau'^*$, we obtain the desired result $\hat{v}(\lambda\tau + \lambda'\tau') \geq \lambda\hat{v}(\tau) + \lambda'\hat{v}(\tau')$.

Let us proceed to the analysis of the relationship between the cores of the games v and \hat{v} .

Theorem 3.2. *For every function v , the game \hat{v} is homogeneous. Moreover, if the core $C(v)$ is nonempty, then the equality $C(v) = C(\hat{v})$ holds.*

Proof. The homogeneity of \hat{v} readily follows from the construction of this game. Indeed, let $\tau = r\tau^*$, and let t be an arbitrary nonnegative number. Then $t\tau = t'\tau^*$, where $t' = tr$. Therefore, by the definition of \hat{v} , we have $\hat{v}(t\tau) = t'v^*(\tau^*) = trv^*(\tau^*) = t\hat{v}(r\tau^*) = t\hat{v}(\tau)$.

Now let us show that $C(\hat{v}) = C(v)$. By Theorem 2.2 (in particular, proving the equality $C(v) = C(v^*)$ if $C(v)$ is nonempty), it suffices to verify that $C(\hat{v}) = C(v^*)$.

1. $C(v^*) \subseteq C(\hat{v})$.

Consider an arbitrary element x in $C(v^*)$. It satisfies the relations $x \cdot \tau^* \geq v^*(\tau^*)$, $\tau^* \in \sigma_F^*$, and $x \cdot e_N^* = v^*(e_N^*)$.

Let us fix some fuzzy coalition $\tau = t\tau^* \in \sigma_F$. By virtue of the above relations we have $x \cdot t\tau^* \geq tv^*(\tau^*) = \hat{v}(\tau)$, and in addition, $x \cdot e_N = v(e_N) = \hat{v}(e_N)$. Indeed, based on the equality $v^*(e_N^*) = v(e_N)/n$, which follows in view of Proposition 2.1 from the assumption that $C(v) \neq \emptyset$, we have $\hat{v}(e_N) = nv^*(e_N^*) = v(e_N)$. Hence $x \cdot e_N = \hat{v}(e_N)$. Since $\tau \in \sigma_F$ is arbitrary, we obtain the desired relation $C(v^*) \subseteq C(\hat{v})$.

2. $C(\hat{v}) \subseteq C(v^*)$.

Consider an arbitrary element $x \in C(\hat{v})$, some fuzzy coalition $\tau^* \in \sigma_F^*$, and a number $t > 0$ such that $\tau = t\tau^*$ belongs to σ_F . By the definition of \hat{v} and the fact that x belongs to the core $C(\hat{v})$, we have the relations $x \cdot \tau = x \cdot t\tau^* \geq \hat{v}(\tau) = tv^*(\tau^*)$ and $x \cdot e_N = x \cdot ne_N^* = \hat{v}(e_N) = nv^*(e_N^*)$.

From the last relations, we obtain $x \cdot e_N = nv^*(e_N^*) = v(e_N)$ and hence $x \cdot e_N^* = v^*(e_N^*)$ (in view of the already mentioned equality $v^*(e_N^*) = v(e_N)/n$). Dividing the first of the above relations by t , we obtain the inequality $x \cdot \tau^* \geq v^*(\tau^*)$. Since the choice of $x \in C(\hat{v})$ and $\tau^* \in \sigma_F^*$ is arbitrary, this completes the proof of the embedding $C(\hat{v}) \subseteq C(v^*)$. ■

Summarizing Corollary 3.2 and Theorem 3.2, we obtain the main result of the paper.

Theorem 3.3. *The formula $C(v) = \hat{\partial}\hat{v}(e_N^*)$ holds for any V -balanced fuzzy TU-cooperative game v .*

4. APPLICATIONS OF THEOREM 3.3

Thus, according to Theorem 3.3, for any fuzzy cooperative game v with side payments that has a nonempty core $C(v)$ there exists a representation of this core in the form of the superdifferential $\hat{\partial}u(e_N^*)$ of a suitable fuzzy game u (for which we can take the game v itself if it is homogeneous or its homogeneous extension $u = \hat{v}$ otherwise). As was already noted, the existence of such a representation permits widely using the technique of subdifferential calculus, from questions concerning

the existence of nonblocking payoffs for specific classes of fuzzy games to describing the structure of their cores. Here are two examples demonstrating this capability. We point out right away that the nonemptiness of the cores (anticores) of the games considered in these examples follows from their superlinearity in the first example and sublinearity in the second example (Theorem 2.3 in the preceding section; see also Corollary 5.3 in [3]).

Example 4.1. We start from a description of the core of the classical Aubin extension [9] for “almost positive” cooperative games v characterized by the fact that their Harsanyi dividends corresponding to more than one-element coalitions are nonnegative (the set of such games will be denoted by V_{+2}). Let us show that the core of the Aubin extension for any game $v \in V_{+2}$ consists of a single element, the Shapley value $\Phi(v)$ of this game: $C(v) = \{\Phi(v)\}$.

Let us proceed to a detailed consideration. Recall (see, e.g., [2]) that the *Harsanyi dividends* of the usual game v are the numbers v_T uniquely determined from the system of linear equations

$$\sum_{T \subseteq S} v_T = v(S), \quad S \subseteq N$$

(by definition, $v(\emptyset) = 0$). The *Aubin extension* [9] of the game v is the fuzzy game v_{Aub} defined by the formula

$$v_{Aub}(\tau) := \sum_{T \subseteq N} v_T \prod_{i \in T} \tau_i^{1/|T|}, \quad \tau = (\tau_1, \dots, \tau_n) \in \sigma_F \tag{4.1}$$

where the number of elements of a finite set T is denoted by $|T|$. It is clear that if the dividends v_T are nonnegative for $|T| \geq 2$, then the function v_{Aub} is concave on the set $I^N = \{x \in \mathbf{R}^N \mid x_i \in [0, 1], i \in N\}$. We also recall that by Theorem 25.1 in [7], in the case of concavity of the function f , its differentiability at the point x^* implies the equality $\hat{\partial}f(x^*) = \{\nabla f(x^*)\}$, where $\nabla f(x^*)$ is the gradient of the function f at the point x^* (i.e., in the indicated case, the superdifferential of the function f at the point x^* consists of a single element, namely, the gradient $\nabla f(x^*)$). Hence, taking into account the differentiability of the extension v_{Aub} at the point e_N^* , we obtain

$$\hat{\partial}v_{Aub}(e_N^*) = \{\nabla v_{Aub}(e_N^*)\}, \quad v \in V_{+2},$$

where, as was already noted, V_{+2} is the collection of all (ordinary) TU -cooperative games of n persons v such that $v_T \geq 0$ for $|T| \geq 2$. Therefore, on the basis of the V -balance and homogeneity of the game v_{Aub} , Theorem 3.1 implies the relation

$$C(v_{Aub}) = \{\nabla v_{Aub}(e_N^*)\}, \quad v \in V_{+2}. \tag{4.2}$$

Finally, according to one well-known formula for the Shapley value $\Phi(v)$ of the game v (see, e.g., [6]), we have

$$\Phi(v)_i = \sum_{T \in \sigma_i} v_T / |T|, \quad i \in N, \tag{4.3}$$

where $\sigma_i := \{S \subseteq N \mid i \in S\}$, $i \in N$. Using formula (4.3), we find that the core $C(v_{Aub})$ of the Aubin extension for any game $v \in V_{+2}$ consists of a single element, the Shapley value of this game,

$$C(v_{Aub}) = \{\Phi(v)\}, \quad v \in V_{+2}. \tag{4.4}$$

Indeed, calculating the partial derivatives of the function v_{Aub} defined by formula (4.1) at the point e_N^* , we have

$$\begin{aligned} \frac{\partial v_{Aub}}{\partial \tau_i}(e_N^*) &= \sum_{T \in \sigma_i} \frac{1}{|T|} v_T \left[\left(\prod_{j \in T \setminus i} \tau_j^{1/|T|} \right) \tau_i^{(1-|T|)/|T|} \right] (e_N^*) \\ &= \sum_{T \in \sigma_i} \frac{1}{|T|} v_T \left[(n^{(1-|T|)/|T|}) n^{(|T|-1)/|T|} \right] = \sum_{T \in \sigma_i} \frac{v_T}{|T|}, \end{aligned}$$

which, by virtue of formula (4.3), gives the desired equality (4.4).

Summarizing the above, we obtain the following statement.

Proposition 4.1. *Let the Harsanyi dividends of the cooperative game $v : S \rightarrow \mathbf{R}$, $S \subseteq N$, corresponding to coalitions with two or more participants be nonnegative. Then the core of the Aubin extension v_{Aub} of this game is a singleton and consists of its Shapley value, $C(v_{Aub}) = \{\Phi(v)\}$.*

Example 4.2. In conclusion, let us give a description of the anticores of fuzzy games such as the Airport game [4, 11] arising in the framework of the cooperative analysis of the rational distribution of costs in the implementation of joint projects. We recall the appropriate analogs of the classical blocking and the core for such games and indicate the resulting modification of the representation (2.1).

Definition 4.1. A fuzzy coalition $\tau = (\tau_1, \dots, \tau_n)$ *a-blocks* a payoff $x = (x_1, \dots, x_n) \in G_v(e_N)$, if there exists a $y = (y_i)_{i \in N(\tau)} \in \mathbf{R}^\tau$ such that

$$(ab.1) \quad \sum_{i \in N(\tau)} \tau_i y_i \geq v(\tau).$$

$$(ab.2) \quad y_i < x_i, \quad i \in N(\tau).$$

Definition 4.2. The *anticore* (briefly, *a-core*) of a fuzzy cooperative game v is the set of all payoffs of this game that are not *a-blocked* by any coalition $\tau \in \sigma_F$. The anticore of a game v will be denoted by $C^-(v)$.

One can readily verify that the *a-core* of a fuzzy TU-cooperative game v has the form

$$C^-(v) = \{x \in \mathbf{R}^N \mid x \cdot e_N = v(e_N), \quad x \cdot \tau \leq v(\tau), \quad \tau \in \sigma_F\}.$$

In addition, the relationship between the *a-core* and the ordinary core is given by the formula

$$C^-(v) = -C(-v), \quad v \in V.$$

Let us give an example of application of Theorem 3.3 and one well-known result of subdifferential calculus to the description of the anticore of the so-called generalized airport game v_A . The latter is determined by a finite set of vectors $A = \{a^k\}_{k \in K} \subseteq \mathbf{R}^n$ by the formula

$$v_A(\tau) := \max_{k \in K} a^k \cdot \tau, \quad \tau \in \sigma_F$$

(a fuzzy analog of the classical Airport game [4, 11] defined by the parameters $K = \{1, \dots, n\}$, $a^k = c_k e^k$, $k \in K$, where the vector e^k is the k th unit vector in the space \mathbf{R}^n and c_k is a positive number). Taking into account the homogeneity of the game v_A , on the basis of Theorem 3.3 and the well-known result of subdifferential calculus (the subdifferential form of the clearing theorem in [5, p. 50]) we obtain the following description of the anticore of the game v_A .

Proposition 4.2. *For any finite set of vectors $A = \{a^k\}_{k \in K}$, one has the formula*

$$C^-(v_A) = \text{co} \{a^r \mid r \in R(N)\},$$

where $R(N) := \{r \in K \mid a^r \cdot e_N = \max_{k \in K} a^k \cdot e_N\}$.

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