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Stabilizing the Oscillations of a Controlled Mechanical System with n Degrees of Freedom

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Abstract—A mechanical system with n degrees of freedom subjected to the action of positional forces and a small smooth control is considered. It is assumed that in the absence of control, the system may have a family of single-frequency oscillations. A universal control—a nonlinear force that implements and simultaneously stabilizes a cycle in the system—is found. An illustrative example is given. In the previous paper [5], the universal control was designed for a two-dimensional manifold of the system.

Keywords: mechanical system, universal small smooth control, natural stabilization

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1. INTRODUCTION

Research into the relaxation modes of a regenerative receiver shows how oscillations can be controlled. A triode embedded into a linear circuit allows obtaining a control force for the implementation of an isolated stable oscillation in the system. The force is nonlinear, dissipative in nature and is produced by the anode current of the triode. The model itself is described by the well-known van der Pol equation

$$\ddot{x} + x = \mu(1 - x^2)\dot{x},$$

where μ denotes a small parameter.

Due to mechanical analogy, van der Pol's can be applied to a nonlinear system as well. For instance, a Hamiltonian system in the plane that may have a family of periodic motions was studied by Pontryagin in the paper [1]. He obtained conditions to be imposed on perturbations that guarantee the existence of a limit cycle in the perturbed system. In [2], the Duffing oscillator was considered, and the nonlinear dissipation $v|v|^{p-1}$, $p = \overline{1,4}$, proportional to the velocity v was used for it. In micromechanics, nonlinear dissipation is taken into account in Duffing-like models; for example, see [3]. In the previous paper [4], van der Pol's approach was applied to a holonomic mechanical system subjected to the action of positional forces (potential and non-conservative positional forces) with a possible occurrence of a single-frequency oscillation (periodic motion). A two-dimensional manifold of periodic motions on which the dynamics are described by a single-degree-of-freedom system was separated. A universal autonomous control that guarantees the existence of a cycle in the controlled mechanical system and simultaneously stabilizes it was constructed. For a system with an arbitrary number of degrees of freedom, the trajectories were attracted to the manifold using linear dissipative forces.

For a system with $n > 1$ degrees of freedom, the separation of the manifold of periodic motions is an independent difficult problem. Therefore, a topical problem is to construct a control in the $2n$-dimensional phase space, where the original system is defined. This problem will be solved below. For a mechanical system with n degrees of freedom, a time-invariant control (without an explicit dependence on time) is obtained; this control is used with a small gain. In the special case $n = 1$, the resulting control coincides with the one designed in [4].

Note that in [5] the orbital stabilization problem for the periodic solutions of underactuated nonlinear systems was solved. (In such systems, the number of independent actuators is one less than the number of degrees of freedom of an uncontrolled conservative system.) The nonlinear feedback controller constructed therein is time dependent.

2. FAMILY OF SYMMETRIC PERIODIC MOTIONS

Consider a holonomic scleronomic mechanical system with n degrees of freedom and stationary geometric constraints

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s(q), \quad s = 1, \ldots, n,$$

that is subjected to the action of positional forces $Q_s(q)$ (potential and non-conservative positional forces). Here T denotes the kinetic energy of the system.

For this system, the kinetic energy T represents a positive definite form of velocities, and the motion equations are invariant with respect to the change of variables

$$(q, \dot{q}, t) \to (q, -\dot{q}, -t).$$

Therefore, the model belongs to the class of reversible mechanical systems [6], and the phase portrait of the system is symmetric with respect to the stationary set $M = \{q, \dot{q} : \dot{q} = 0\}$. From the symmetry of the phase portrait it follows that:

1) The periodic motion of the system is symmetric and intersects the stationary set M with the zero velocity $\dot{q}$.

2) The periodic motion of the system cannot be asymptotically stable.

The velocity $\dot{q} = \dot{q}(q_1^0, \ldots, q_n^0, t)$ on a symmetric periodic motion (SPM) depends on the initial point $q^0 \in M$ only. Hence, the necessary and sufficient conditions for the existence of an SPM in the form of a single-frequency oscillation with a period τ can be written as

$$\dot{q}_s(q_1^0, \ldots, q_n^0, \tau/2) = 0, \quad s = 1, \ldots, n. \tag{1}$$

For details, see [7].

These equalities (n in total) have $n + 1$ unknown variables. Consequently, SPMs of a reversible mechanical system always form families. Construct the matrix

$$A = \left\| \frac{\partial \dot{q}_s(q_1^0, \ldots, q_n^0, \tau/2)}{\partial q_j^0} \right\|,$$

where partial derivatives are calculated along the solution of the system (1). Then the definition given in [4] can be used.

Definition 1. The case $\operatorname{rank} A = n$ is said to be nondegenerate for a symmetric periodic motion, and the SPM itself is said to be nondegenerate as well.

The nondegenerate SPMs of a system always form two-dimensional manifolds on which the period monotonically depends on one parameter; see [7]. A controlled mechanical system for such a manifold was constructed in [4].

3. EXISTENCE OF CYCLE IN SYSTEM

In accordance with Section 2, the cycle is not implemented if the additional forces acting on the system preserve the symmetry of the phase space. Symmetry can be destroyed using an appropriately designed control, which results in a controlled mechanical system. For making the cycle of this system close to the oscillation of the original (uncontrolled) system, the control is applied with a small gain. The necessary and (separately) sufficient conditions for the existence of a cycle will be presented below. These conditions lead to the requirements to be imposed on the requisite control.

Denote by $\Sigma(h)$ the family of SPMs on which the coordinate has the formula $q = \varphi(h, t + \gamma)$. Here h is the family parameter, and γ is the initial point shift along the trajectory. The time when the solution intersects the stationary set is assumed to be $\gamma = 0$. The period τ on the family Σ is treated as a function of h, i.e., $\tau = \tau(h)$.

Let the mechanical system under consideration be subjected to the control $\mu R(q, \dot{q})$, where μ is a small parameter. Then the solution of the controlled system depends on μ. The derivative of this solution with respect to μ at the point $\mu = 0$ satisfies a linear inhomogeneous system of differential equations. For this system, the necessary and sufficient conditions for the existence of a τ^*-periodic solution with zero initial conditions, $\tau^* = \tau(h^*)$, lead to the amplitude (bifurcation) equation

$$I(h) \equiv \int_0^{\tau^*} R(\varphi(h,t), \dot{\varphi}(h,t))\psi(h,t)dt = 0, \tag{2}$$

where ψ denotes the periodic solution of the adjoint linear system.

As is well known [8, 9], a root $h = h^*$ of the amplitude equation such that $dI(h^*)/dh \neq 0$ guarantees the existence of the function $q(\mu, q^0, t)$ describing a cycle.

Thus, the force R is found using two conditions. They were adopted in [4] to construct a control on a two-dimensional manifold of periodic motions. In this case, the control destroys the family of oscillations; therefore, it must be an odd function of the velocity. Also, the control must be applicable for all points of the family, including the limit point (equilibrium). Other considerations, including the ease of control, suggest the choice of nonlinear dissipation control.

4. CONTROL DESIGN

The original control design conditions from [4] will be used below for the system with n degrees of freedom. In addition, recall that the control must be an odd function of the vector velocity $\dot{q}$. Due to the universal form of control for the equilibrium and SPMs, the function R must have a factor described by a quadratic form of the velocities. Another factor (a) in the function R, which generally depends on the vectors q and $\dot{q}$, characterizes the SPM with the parameter h. Therefore, a contains some characteristic K, i.e., a function of h. In the case of equilibrium $a = 1$, and for the cycle in the neighborhood of the SPM with $h = h^*$, the value of this characteristic is calculated: $K = K(h^*)$. Finally, the amplitude Eq. (2) must hold for R.

As a result, the function $R = (R_1, \ldots, R_n)$ takes the form

$$R_s = [1 - K(h^*)b(q, \dot{q})] \sum_{j=1}^{n} l_{sj}\dot{q}_j, \quad a = 1 - K(h^*)b, \quad b > 0, \quad l_{sj} = \text{const}, \quad s = 1, \ldots, n. \tag{3}$$

The dependence $K(h)$ acts as a characteristic of the family of SPMs. It can be found from the identity

$$\int_0^{\tau(h)} [1 - K(h)b(\varphi(h,t), \dot{\varphi}(h,t))] \sum_{s,j=1}^{n} l_{sj}\dot{\varphi}_j(h,t)\psi_s(h,t)dt \equiv 0. \tag{4}$$

Hence,

$$K(h) = \frac{\int\limits_0^{\tau(h)} \sigma(h,t)dt}{\int\limits_0^{\tau(h)} b(\varphi(h,t),\dot\varphi(h,t))\sigma(h.t)dt}, \quad \sigma = \sum_{s,j=1}^{n} l_{sj}\dot\varphi_j(h,t)\psi_s(h,t).$$

Taking into account the odd functions $\dot\varphi_j$ in (4), differentiate this identity with respect to h. Then the derivative of the integral with respect to the upper limit vanishes, and for $\tau = \tau^*$ the other terms yield a constructively verifiable root simplicity criterion in the form

$$\frac{dI(h^*)}{dh} = \chi\nu, \quad \chi = \frac{dK(h^*)}{dh}, \quad \nu = \int\limits_0^{\tau^*} b(\varphi(h,t),\dot\varphi(h,t))\sigma(h,t)dt. \tag{5}$$

In accordance with (5), the sufficient existence conditions of a cycle are not satisfied at the points from the family of SPMs for which $dK = 0$. Therefore, by analogy with the previous paper [4], Definition 2 will be used below.

Definition 2. A point h from the family of SPMs of a mechanical system at which the derivative of the function $K(h)$ vanishes is called singular.

Thus, the following result is true.

Theorem 1. *Assume that the mechanical system may have the h-family of SPMs with the characteristic $K(h)$. Then the controlled system subjected to the action of positional forces with the control function* (3) *always has a cycle at each non-singular point $h = h^*$ with the characteristic $K(h)$ under the condition $\nu \neq 0$.*

Remark 1. For the single-degree-of-freedom system, Theorem 1 was established in [4].

Remark 2. The characteristic $K(h)$ can be constructively calculated using formula (4). For example, $K = 4/h^2$ for the van der Pol equation, and $K(h)$ is monotonically decreasing for the simple pendulum [4].

For the sake of definiteness, let the cycle period be equal to $\tau^* = 2\pi$.

5. STABILIZING CYCLE OF CONTROLLED MECHANICAL SYSTEM

Being subjected to the action of the control function (3), the mechanical system remains autonomous; the cycle of the system is an isolated periodic solution. Find the control conditions ensuring the orbital asymptotic stability of the cycle. As a result, the issue of cycle stabilization will be solved simultaneously with obtaining the stability conditions. In this case, stabilization involves no additional controls, thereby being natural. The characteristics of SPMs are divided into the pairs $\pm\,\lambda$ and can be calculated from quadratic equations; for example, see [7]. In the controlled mechanical system, the coefficients of the characteristic polynomial for calculating these characteristics continuously depend on the parameter μ. Therefore, the characteristics of SPMs belonging to the positive (negative) half-plane for $\mu = 0$ will remain in the positive (negative) half-plane for a sufficiently small $\mu > 0$ as well. Consequently, the cycle of the perturbed mechanical system can be orbitally stable if all characteristics of the reference SPM, for which $h = h^*$, have zero real parts. In other words, for the controlled mechanical system a stable cycle is possible only in the neighborhood of a stable SPM in the linear approximation.

The characteristic of a cycle of the controlled mechanical system is calculated using the following procedure.

First, construct variational equations for the cycle. This yields a linear periodic system parameterized by μ. For $\mu = 0$ the system is reversible and coincides with the variational equations for the SPMs. The last equations are reduced by an appropriate Lyapunov transformation to a system with constant coefficients. Note that the requisite transformation must preserve reversibility. The transformation of the system of variational equations for the cycle is described in the Appendix. The symbols δq and $\delta\dot{q}$ denote the variations of the coordinates and velocities, respectively.

This transformation yields the linear 2π-periodic system

$$\dot{x} = Py + \mu X(x,y,t), \quad \dot{y} = Gx + \mu Y(x,y,t), \tag{6}$$

in which

$$X = \Sigma_x(t)\delta R(\delta q, \delta\dot{q}), \quad Y = \Sigma_y(t)\delta R(\delta q, \delta\dot{q}),$$
$$\delta R_s(\delta q, \delta\dot{q}) = \sum_{j=1}^{n}\left[\left(\frac{\partial R_s}{\partial q_j}\right)_*\delta q_j + \left(\frac{\partial R_s}{\partial \dot{q}_j}\right)_*\delta\dot{q}_j\right], \quad s = 1,\ldots,n.$$

Here $\Sigma_x(t)$ and $\Sigma_y(t)$ are 2π-periodic functions; star indicates that derivatives are calculated along the solution $q = q(h^*, t)$.

By definition, each characteristic $\lambda + \mu\alpha$ of the system (6) corresponds to the solution

$$x = \zeta(t)\exp(\lambda + \mu\alpha)t, \quad y = \chi(t)\exp(\lambda + \mu\alpha)t$$

with 2π-periodic functions $\zeta(t), \chi(t)$. In view of this fact, apply the change of variables

$$x = w\exp(\lambda + \mu\alpha)t, \quad y = z\exp(\lambda + \mu\alpha)t$$

to the system (6). This gives the system

$$\dot{w} = Pz - (\lambda + \mu\alpha)w + \mu X(w,z,t), \quad \dot{z} = Gw - (\lambda + \mu\alpha)z + \mu Y(w,z,t). \tag{7}$$

The system (7) has at least one zero characteristic and a corresponding 2π-periodic solution. To find it, represent the solution of the system (6) in the form

$$w = w_0(t) + \mu w_1(t) + o(\mu), \quad z + z_0(t) + \mu z_1(t) + o(\mu).$$

As a result, the variables w_0 and z_0 satisfy the system of differential equations

$$\dot{w}_0 = Pz_0 - \lambda w_0, \quad \dot{z}_0 = Gw_0 - \lambda z_0, \tag{8}$$

which is split into k subsystems (by the number of paired characteristics of SPMs) with the variables w_{0s} and z_{0s}.

Write the equations for the variables w_1 and z_1:

$$\begin{aligned}\dot{w}_1 &= Pz_1 - \lambda w_1 - \alpha w_0 + X(w_0,z_0,t),\\ \dot{z}_1 &= Gw_1 - \lambda z_1 - \alpha z_0 + Y(w_0,z_0,t).\end{aligned} \tag{9}$$

Clearly, with the substitution $w_{0j}(t) = 0$, $z_{0j}(t) = 0$, $j \neq s$, the system (9) contains the manifold Υ_s corresponding to the sth subsystem in (8). Hence, the number α for the sth subsystem can be calculated from the existence condition of a periodic solution on the manifold Υ_s. Now, considering all possible manifolds Υ_s, find all characteristics of the system (7).

Three types of manifolds may exist in the system (9). The first type corresponds to a pair of zero characteristics in a Jordan cell. In the system (8), $\lambda = 0$ and the corresponding subsystem may have a unique periodic solution (up to a constant factor).

Note that a cycle of a perturbed mechanical system is always associated with one zero characteristic. Therefore, a bifurcation of a Jordan cell occurs with the birth of a real-valued characteristic. The corresponding value for α is obtained from (9).

On the manifolds corresponding to a pair of pure imaginary numbers λ or to a pair of simple zero numbers λ, the subsystems in (8) may have two periodic solutions. Therefore, the two numbers α are found from the existence conditions of periodic solutions in (9). Explicit formulas for calculating α are obtained from the period-average values of the functions $X_0(t) = X(w_0, z_0, t)$ and $Y_0(t) = Y(w_0, z_0, t)$.

In each of these cases of paired characteristics, the functions

$$X_0(t) = \Sigma_x(t)\delta R(\delta q_0, \delta\dot{q}_0), \quad Y_0(t) = \Sigma_y(t)\delta R(\delta q_0, \delta\dot{q}_0) \tag{10}$$

are obtained using the Lyapunov transformation. Denote by $(\delta q_0, \delta\dot{q}_0)$ the periodic solution of the system (6) for $\mu = 0$. As concerns the function δR, it contains a constant square matrix $L = ||l_{sj}||$ of dimensions $n \times n$. In accordance with formulas (10), the average values $\bar{X}_0(t)$ and $\bar{Y}_0(t)$ linearly depend on the elements l_{sj} $(s, j = \overline{1, n})$.

For an asymptotically orbitally stable cycle, all numbers α, except for the zero one, must be negative. From the system (9) it follows that the inequalities $\alpha < 0$ lead to inequalities for the average values $\bar{X}_0(t)$ and $\bar{Y}_0(t)$. Therefore, the stability conditions hold if a system of linear inequalities in l_{sj} is consistent.

In the mechanical system with one degree of freedom, there is a single pair of zero characteristics in a Jordan cell, the matrix consists of one element L, and the condition $\alpha < 0$ is achieved by an appropriately chosen sign of L. In the mechanical system with two degrees of freedom, the matrix L consists of four elements, and the numbers α (which depend on four constants) must satisfy three negativity conditions; in other words, the conditions are consistent. In the general case with n degrees of freedom, the asymptotically orbital stability of a cycle is ensured by the $2n - 1$ negativity conditions for the numbers α, which depend on n^2 constants. Obviously, $n^2 \geq 2n - 1$, and the equality holds only if $n = 1$.

Thus, the system of inequalities for finding the negative numbers α is always consistent. As a result, the functions R_s given by (3) ensure the asymptotical orbital stability of the cycle.

Theorem 2. *Assume that the mechanical system may have the h-family of SPMs. Then a cycle of the controlled system subjected to the action of positional forces with the control function* (3) *and an appropriately chosen constant matrix L is asymptotically orbitally stable.*

Remark 3. Under the hypotheses of Theorem 2, a cycle of the controlled mechanical system is stabilized in a "natural" way, i.e., without additional controls.

Remark 4. The control μR is used to design a cycle and also a stable cycle regardless of the positional forces applied to the system (potential, non-conservative positional, jointly acting potential and non-conservative positional ones). The conclusions on the existence of a cycle (Theorem 1) and its stabilization (Theorem 2) are equally valid for any family of SPMs—the family of isochronous oscillations (e.g., a linear oscillator) or the family of nondegenerate oscillations (e.g., a simple pendulum). The control μR gives a solution of the cycle and asymptotically stable cycle design problems irrespective of a specific mechanical system. Finally, the force μR has a rather simple form and also a natural analog. Due to these reasons, the control function (3) is called universal.

6. EXAMPLE

Following [10], consider the system of equations

$$\begin{gathered}
\ddot{\theta}_1 + \sin\theta_1 + \kappa(1 - 1/f)\left(\frac{c^2}{4}\sin(\theta_1 - \theta_2) - \frac{c}{2}\cos\theta_1\right) = 0, \\
\ddot{\theta}_2 + \sin\theta_2 + \kappa(1 - 1/f)\left(-\frac{c^2}{4}\sin(\theta_1 - \theta_2) + \frac{c}{2}\cos\theta_2\right) = 0, \\
f^2 = 1 + \frac{c^2}{2} - c\sin\theta_1 + c\sin\theta_2 - \frac{c^2}{2}\cos(\theta_2 - \theta_1),
\end{gathered} \tag{11}$$

which describes the dynamics of two uncontrolled identical pendulums connected by a spring, the suspension points of which lie on a horizontal line. Here $\theta_{1,2}$ denote the deviation angles of the pendulums from the vertical; κ and c are positive constants. The system (11) admits of the integral manifold $\tilde{\Xi}$

$$\ddot{\eta}_1 + \sin\eta_1 = 0, \quad \eta_1 = (\theta_1 + \theta_2)/2, \quad \eta_2 = (\theta_1 - \theta_2)/2 \equiv 0,$$

which describes the dynamics of the two independent pendulums (the spring is unstrained).

Now consider the controlled mechanical system. On the manifold Ξ of this system that corresponds to $\tilde{\Xi}$, the dynamics are described by the equation

$$\ddot{\eta}_1 + \sin\eta_1 = \mu[1 - K\eta_1^2]\dot{\eta}_1. \tag{12}$$

(For details, see [4].)

For $\mu = 0$ the oscillations in (12) form a family $\eta_1 = \eta_1(h, t)$ in the constant h of the energy integral. In other words, $K(h)$ is the characteristic of this family. As was demonstrated in [4], the control with $K = K(h^*)$ and $\eta_1 = \eta_1(h, t)$ guarantees the existence and stabilization of a cycle of the controlled system (12) in the neighborhood of an oscillation with the parameter $h = h^*$. In addition, the trajectories of the controlled system are attracted to the manifold Ξ due to linear dissipation.

Write variational equations for a cycle in the variable η_2 :

$$\delta\ddot{\eta}_2 + (1 - 2c\cos\eta_1(h^*, t))\delta\eta_2 = 0. \tag{13}$$

Clearly, Eq. (13) with $c = 0$ has a pair of pure imaginary characteristics equal to $\pm i$. Equation (13) is invariant with respect to the transformation: $(\eta_2, \dot{\eta}_2, t) \to (\eta_2, -\dot{\eta}_2, -t)$. Therefore, the characteristics always form a pair $\pm\lambda$, see [8], and the pure imaginary numbers $\pm\lambda$ exist at least for small c. For these λ, dissipation is introduced using the term $\mu\delta\dot{\eta}_2$ added into the left-hand side of Eq. (13).

In view of the formulas of the variables $\psi_{1,2}$, the controls $\mu R_{1,2}$ with the functions

$$R_1 = \left[1 - K(\theta_1 + \theta_2)^2\right](\dot{\theta}_1 + \dot{\theta}_2) + \left[1 - K(\theta_1 - \theta_2)^2\right](\dot{\theta}_1 - \dot{\theta}_2),$$
$$R_2 = \left[1 - K(\theta_1 + \theta_2)^2\right](\dot{\theta}_1 + \dot{\theta}_2) - \left[1 - K(\theta_1 - \theta_2)^2\right](\dot{\theta}_1 - \dot{\theta}_2)$$

applied to the mechanical system (11) guarantee the existence of an orbitally asymptotically stable cycle in the neighborhood of its oscillation on the manifold $\tilde{\Xi}$ (Theorem 2). In the linear approximation, the dissipative term has the form $\mu(\dot{\eta}_1 - \dot{\eta}_2)$.

This example illustrates well Theorem 2 and the connection between the results established above and the earlier results of the paper [4].

7. CONCLUSIONS

Van der Pol gave an example of oscillation control by adding nonlinear dissipation into the oscillator and performing further transition to a controlled system. His approach can be applied to an arbitrary mechanical system subjected to positional forces.

This paper has found a universal small smooth control, i.e., a nonlinear force of the dissipation type in the van der Pol equation, which guarantees the existence and stabilization of a cycle of a controlled mechanical system with n degrees of freedom. In this case, the controlled mechanical system behaves like a regenerative receiver in radio engineering, whose natural (relaxation) oscillations are described by the van der Pol equation.

APPENDIX

LYAPUNOV TRANSFORMATION FOR SYSTEM OF VARIATIONAL EQUATIONS

A cycle of the controlled mechanical system satisfies variational equations of the form

$$\delta\ddot{q}_s = \sum_{j=1}^{n} p_{sj}(t)\delta q_j + \sum_{j=1}^{n} r_{sj}(t)\delta\dot{q}_j + \mu\delta R_s(\delta q, \delta\dot{q}), \tag{A.1}$$

$$p_{sj}(t) = p_{sj}(-t), \quad r_{sj}(t) = -r_{sj}(-t), \quad p_{sj}(t+2\pi) = p_{sj}(t), \quad r_{sj}(t+2\pi) = r_{sj}(t).$$

For $\mu = 0$ the system (A.1) is a reversible linear periodic system described by

$$\dot{u} = A_-(t)u + A_+(t)v, \quad \dot{v} = B_+(t)u + B_-(t)v, \quad u, v \in \mathrm{R}^n, \tag{A.2}$$

with two stationary sets of the form

$$M_u = \{u, v, t : v = 0, \sin t = 0\}, \quad M_v = \{u, v, t : u = 0, \sin t = 0\}.$$

Here the plus (minus) sign indicates the matrices and vectors containing the even (odd, respectively) 2π-periodic functions.

The adjoint system for (A.2),

$$\dot{\xi} = -A_-^{\mathrm{T}}(t)\xi - B_+^{\mathrm{T}}(t)\eta, \quad \dot{\eta} = -A_+^{\mathrm{T}}(t)\xi - B_-^{\mathrm{T}}(t)\eta, \tag{A.3}$$

where T means transposition, is also reversible and admits of two stationary sets:

$$M_\xi = \{\xi, \eta, t : \eta = 0, \sin t = 0\}, \quad M_\eta = \{\xi, \eta, t : \xi = 0, \sin t = 0\}.$$

The solution $(\xi(t), \eta(t))$ of the system (A.3) is used to write the first integral

$$f = \xi_1(t)u_1 + \ldots + \xi_n(t)u_n + \eta_1(t)v_1 + \ldots + \eta_n(t)v_n \tag{A.4}$$

of the system (A.2). By the way, this result shows that the function $\psi(t)$ in the amplitude Eq. (2) is odd.

The system (A.2) is reduced to a system with constant coefficients using an appropriate Lyapunov transformation. Recall that the system (A.2) contains n solutions symmetrical with respect to M_ξ, as well as the same number of solutions symmetrical with respect to M_η. Choose these $2n$ solutions to obtain $2n$ integrals f for reducing the reversible system (A.2) to a system with constant coefficients. Following [8], select the transformation

$$x_s = p_s^+(t)u + q_s^-(t)v, \quad y_s = p_s^-(t)u + q_s^+(t)v, \quad s = 1, \ldots, n, \tag{A.5}$$

with the 2π-periodic vector functions $p_s^\pm(t)$ and $q_s^\pm(t)$; the transformed system contains the stationary sets

$$M_x = \{x, y, t : y = 0, t = 0\}, \quad M_y = \{x, y, t : x = 0, t = 0\}.$$

The system (A.1) is transformed simultaneously.

The reference SPM has characteristics with zero real parts. Therefore, consider the following cases: (a) a pair of zero characteristics in a Jordan cell; (b) a pair $\pm\, i\omega$, $\omega > 0$, of pure imaginary characteristics; (c) a pair of simple zero characteristics. Write the integrals (A.4) and the corresponding reduced equations in the three cases mentioned.

(a) A pair of zero characteristics in a Jordan cell. This pair corresponds to the first integrals

$$f_1 = g_*^- = \text{const}, \quad f_2 = tg_*^- - g_*^+ = \text{const},$$
$$g_*^\pm = \xi_{*1}^\pm(t)\delta q_1 + \ldots + \xi_{*n}^\pm(t)\delta q_n + \eta_{*1}^\mp(t)\delta\dot{q}_1 + \ldots + \eta_{*n}^\mp(t)\delta\dot{q}_n.$$

Find the derivatives $\dot{f}_1$ and $\dot{f}_2$ along the trajectories of the system (A.1):

$$\dot{f}_1 = \mu\sum_{s=1}^{n}\eta_{*s}^+(t)\delta R_s(\delta q, \delta\dot{q}), \quad \dot{f}_2 = -\mu\sum_{s=1}^{n}\eta_{*s}^-(t)\delta R_s(\delta q, \delta\dot{q}).$$

Denoting $x_* = g_*^+$ and $y_* = g_*^-$, for $\mu = 0$ obtain $\dot{y}_* = \dot{f}_1 = 0$, $\dot{x}_* = y_* - \dot{f}_2$, and $\dot{f}_2 = 0$. For $\mu \neq 0$, from (A.1) it follows that

$$\dot{x}_* = y_* + \mu\sum_{s=1}^{n}\eta_{*s}^-(t)\delta u_s, \quad \dot{y}_* = \mu\sum_{s=1}^{n}\eta_{*s}^+(t)\delta u_s.$$

(b) A pair $\pm\, i\omega$ of pure imaginary characteristics. In this case, the first integrals have the complex representation

$$f_\pm = \exp(\pm\, i\omega t)g_\omega^\pm(\delta q, \delta\dot{q}),$$
$$g_\omega^\pm = \xi_{\omega 1}^\pm(t)\delta q_1 + \ldots + \xi_{\omega n}^\pm(t)\delta q_n + \eta_{\omega 1}^\mp(t)\delta\dot{q} + \ldots + \eta_{\omega n}^\mp(t)\delta\dot{q}_n.$$

Calculate the total derivatives of the functions $f_\pm$ along the trajectories of the system (A.1):

$$\dot{f}_\pm = \mu\exp(\pm\, i\omega t)\sum_{s=1}^{n}\eta_{\omega s}^\mp(t)\delta R_s(\delta q, \delta\dot{q}).$$

Next, find

$$\dot{g}_\omega^\pm = \dot{f}_\pm\exp(\mp\, i\omega t) \pm i\omega f_\pm\exp(\mp\, i\omega t).$$

Then the variables $x_\omega = g_\omega^+$ and $y_\omega = ig_\omega^-$ satisfy the equations

$$\dot{x}_\omega = \omega y_\omega + \mu\sum_{s=1}^{n}\eta_{\omega s}^-(t)\delta u_s, \quad \dot{y}_\omega = -\omega x_\omega + \mu\sum_{s=1}^{n}\eta_{\omega s}^+(t)\delta u_s.$$

Finally, transition to the real variables yields two equations for the system (7).

(c) A pair of simple zero characteristics. Here the first integrals are given by

$$f_{1,2} = g^\pm = \xi_1^\pm(t)\delta q_1 + \ldots + \xi_n^\pm(t)\delta q_n + \eta_1^\mp(t)\delta\dot{q}_1 + \ldots + \eta_n(t)^\mp\delta\dot{q}_n.$$

The corresponding equations of the system (A.1) take the form

$$\dot{x}_+ = \mu\sum_{s=1}^{n}\eta_s^-(t)\delta R_s, \quad \dot{y}_- = \mu\sum_{s=1}^{n}\eta_s^+(t)\delta R_s.$$

Thus, the groups of variables (x_*, y_*), (x_ω, y_ω) and (x_+, y_-) can be adopted for reducing the system (A.1) to a convenient form for further calculation of characteristics. For this purpose, apply the transformation (A.5) in which $u = \delta q$, $v = \delta\dot{q}$, the vector $x(y)$ consists of the vectors x_*, x_ω, x_+ $(y_*, y_\omega, y_+$, respectively), and the functions $\delta R(\delta q, \delta\dot{q})$ and the vectors δq and $\delta\dot{q}$ are replaced by the vectors x and y using the inverse of (A.5).

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REFERENCES

1. Pontryagin, L.S., On Dynamic Systems That Are Close to Hamiltonian Systems, *Zh. Eksper. Teor. Fiz.*, 1934, vol. 4, no. 9, pp. 883–885.
2. Patidar, V., Sharma, A., and Purohit, G., Dynamical Behaviour of Parametrically Driven Duffing and Externally Driven Helmholtz–Duffing Oscillators under Nonlinear Dissipation, *Nonlin. Dynam.*, 2016, vol. 83, nos. 1–2, pp. 375–388.
3. Zaitsev, S., Shtempluck, O., and Gottlieb, E.B., Nonlinear Damping in a Micromechanical Oscillator, *Nonlin. Dynam.*, 2016, vol. 67, no. 1, pp. 859–883.
4. Tkhai, V.N., Stabilizing the Oscillations of a Controlled Mechanical System, *Autom. Remote Control*, 2019, vol. 80, no. 11, pp. 1996–2004.
5. Shiriaev, A., Perram, J.W., and Canudas-de-Wit, C., Constructive Tool for Orbital Stabilization of Underactuated Nonlinear Systems: Virtual Constraints Approach, *IEEE Trans. Autom. Control*, 2005, vol. 50, no. 8, pp. 1164–1176.
6. Tkhai, V.N., The Reversibility of Mechanical Systems, *J. Appl. Math. Mech.*, 1991, vol. 55, no. 4, pp. 461–468.
7. Tkhai, V.N., The Behaviour of the Period of Symmetrical Periodic Motions, *J. Appl. Math. Mech.*, 2012, vol. 76, no. 4, pp. 446–450.
8. Tkhai, V.N., Periodic Motions of a Perturbed Reversible Mechanical System, *J. Appl. Math. Mech.*, 2015, vol. 79, no. 2, pp. 122–131.
9. Malkin, I.G., *Nekotorye zadachi teorii nelineinykh kolebanii* (Some Problems in the Theory of Nonlinear Oscillations), Moscow: Gostekhizdat, 1956.
10. Evdokimenko, A.P., On Equilibrium Configurations and Their Stability for a System of Two Coupled Pendulums, *Mech. Solids.*, 2017, vol. 52, no. 3, pp. 266–277.

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