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Asymptotic Approximations to the Solution of the Singularly Perturbed Linear-Quadratic Optimal Control Problem with Terminal Path Constraints

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Abstract—The minimum-energy control problem for a linear singularly perturbed system with linear constraints imposed on the right endpoint of the trajectories is considered. Asymptotic approximations in the form of open loop and feedback controls to the optimal control (solution of this problem) are constructed. The main advantage of the algorithms proposed below consists in the decomposition of the original problem into two unperturbed optimal control problems of smaller dimension.

Keywords: optimal control, linear system, quadratic performance criterion, singular perturbations, asymptotic approximations, suboptimal feedback design

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1. INTRODUCTION

Systems of ordinary differential equations with small parameters at some derivatives are usually called singularly perturbed systems. Within the mathematical theory of optimal processes, considerable attention is paid to the optimization problems of such systems; for example, see [1–4]. This class of problems provoked much interest due to the efficiency of asymptotic methods for their solution, which allow decomposing the original optimal control problems into the ones of smaller dimension. In addition, the asymptotic approach makes it possible to avoid the integration of singularly perturbed systems, which are rigid systems [5].

This paper is devoted to the construction of asymptotic approximations in the form of open loop and feedback controls to the solution of the singularly perturbed linear-quadratic optimal control problem with linear terminal constraints on the trajectories. It can be interpreted as a control problem with minimum energy cost. The core of the algorithms suggested below consists in the asymptotic expansion of the Lagrange multipliers in terms of the integer powers of the small parameter; in accordance with the maximum principle [6], these multipliers correspond to optimal control. The computational procedures developed in this paper are another implementation of the methodological approach to the optimization problems of perturbed dynamic systems, which is based on the idea of special finite-dimensional parametrization of optimal controls; the details can be found in [2].

Note that a significant number of publications were devoted to singularly perturbed linear-quadratic optimal control problems; for example, see [7–11]. However, with the exception of [11], no constraints on system trajectories were imposed. This paper generalizes the results established in [11], where the problem with a fixed right endpoint of the trajectories was considered.

2. PROBLEM STATEMENT

In the class of r-dimensional controls $u(t)$, $t \in T = [t_*, t^*]$, with piecewise continuous components, consider an optimal control problem of the form

$$\begin{aligned} \dot{y} &= A_1(t)y + A_2(t)z + B_1(t)u, \quad y(t_*) = y_*, \\ \mu\dot{z} &= A_3(t)y + A_4(t)z + B_2(t)u, \quad z(t_*) = z_*, \end{aligned} \tag{2.1}$$

$$H_1 y(t^*) = g_1, \quad H_2 z(t^*) = g_2, \tag{2.2}$$

$$J(u) = \frac{1}{2}\int\limits_{t_*}^{t^*} u^{\mathrm{T}} P(t) u dt \to \min, \tag{2.3}$$

with the following notations: μ as a small positive parameter; t_* and t^* as given time instants ($t_* < t^*$); y as the n-dimensional vector of slow variables; z as the m-dimensional vector of fast variables; g_1 and g_2 as vectors of dimensions n_1 and m_1, respectively ($n_1 \le n, m_1 \le m$); H_1 and H_2 as full rank matrices of compatible dimensions; finally, $P(t)$ as a positive definite and symmetric matrix of compatible dimensions, for all $t \in T$.

Assumption 1. All eigenvalues of the matrix $A_4(t)$, $t \in T$, have negative real parts.

Assumption 2. The elements of all matrices figuring in the problem statement are indefinitely differentiable.

A control with piecewise continuous components is said to be admissible if the trajectory of system (2.1) induced by this control satisfies the terminal constraints (2.2). An admissible control on which the performance criterion (2.3) achieves minimum is said to be optimal. In addition to the conventional notions, we define the asymptotic approximations to the solution of the stated problem in the following way.

Definition 1. A control $u^{(N)}(t, \mu)$, $t \in T$, with piecewise continuous components will be called an asymptotically suboptimal open loop control of order N ($N = 0, 1, 2, \ldots$) in problem (2.1)–(2.3) if it deviates from the optimal control in terms of the performance criterion (2.3) by a value of order $O\left(\mu^{N+1}\right)$ and also the trajectory $y\left(t, \mu\right), z\left(t, \mu\right), t \in T$, of system (2.1) induced by it satisfies the terminal constraints (2.2) with the same infinitesimal order.

Definition 2. A vector function $u^{(N)}(y, z, t, \mu)$ will be called an asymptotically suboptimal feedback control of order N if, for any initial state (y_*, z_*, t_*), $t_* < t^*$,

$$u^{(N)}(y_*, z_*, t_*, \mu) = u^{(N)}(t_*, \mu),$$

where $u^{(N)}(t, \mu)$, $t \in T$, is an asymptotically suboptimal open loop control of order N in problem (2.1)–(2.3).

In this paper, we will propose and justify an algorithm to construct an asymptotically suboptimal open loop control of a given order N in the optimal control problem under study. Also, an asymptotically suboptimal feedback control of zeroth order will be constructed.

For compact presentation, the following notations will be used below:

$$\begin{aligned} A\left(t, \mu\right) &= \begin{pmatrix} A_1\left(t\right) & A_2\left(t\right) \\ A_3\left(t\right)/\mu & A_4\left(t\right)/\mu \end{pmatrix}, \\ B\left(t, \mu\right) &= \begin{pmatrix} B_1\left(t\right) \\ B_2\left(t\right)/\mu \end{pmatrix}, \quad x_* = \begin{pmatrix} y_* \\ z_* \end{pmatrix}. \end{aligned} \tag{2.4}$$

3. FIRST BASIC PROBLEM

The calculations for constructing asymptotically suboptimal controls begin with solving the degenerate problem

$$\dot{y} = A_0(t)y + B_0(t)u, \quad y(t_*) = y_*, \quad H_1 y(t^*) = g_1,$$
$$J_1(u) = \frac{1}{2}\int_{t_*}^{t^*} u^{\mathrm{T}} P(t) u dt \to \min, \tag{3.1}$$

where

$$A_0(t) = A_1(t) - A_2(t)A_4^{-1}(t)A_3(t), \quad B_0(t) = B_1(t) - A_2(t)A_4^{-1}(t)B_2(t). \tag{3.2}$$

In the sequel, this problem will be called the first basic problem.

Assumption 3. The dynamic system in problem (3.1) is controllable on a time interval $[\tau,\ t^*]$ with respect to the subspace $H_1 y = 0$ for any $\tau \in [t_*, t^*)$ [12].

This assumption holds if and only if, for any $\tau \in [t_*, t^*)$ and any nonzero vector l of dimension n_1,

$$l^{\mathrm{T}} H_1 F_0(t) B(t) \not\equiv 0, \quad \tau \le t \le t^*, \tag{3.3}$$

where $F_0(t)$, $t \in T$, is a matrix function of dimensions $(n \times n)$ that satisfies the original differential equation

$$\dot{F}_0 = -F_0 A_0\left(t\right), \quad F_0(t^*) = E_n \tag{3.4}$$

with an identity matrix E_n of compatible dimensions; for example, see [13]. Note that condition (3.3) is often called the implicit controllability criterion with respect to a subspace and, for a time-invariant dynamic system, is equivalent to the requirement [12]

$$\operatorname{rank}\Big(H_1 B_0,\ H_1 A_0 B_0,\ \ldots,\ H_1 A_0^{n-1} B_0\Big) = n_1. \tag{3.5}$$

Under Assumption 3 there exist admissible controls in the first basic problem; moreover, this problem has a unique solution [14], representing a normal extremal. This means that the maximum principle [6, 15] can be formulated in the following way: let $u^0(t)$, $y^0(t)$, $t \in T$, be an optimal control and a corresponding optimal trajectory in problem (3.1); then there exists the n_1-dimensional vector of Lagrange multipliers λ_0 such that

$$\psi^{0\mathrm{T}}(t)B_0(t)u^0(t) - \frac{1}{2}u^{0\mathrm{T}}(t)P(t)u^0(t)$$
$$= \max_{u \in \mathrm{R}^r}\left(\psi^{0\mathrm{T}}(t)B_0(t)u - \frac{1}{2}u^{\mathrm{T}}P(t)u\right), \quad t \in T,$$

where $\psi^0(t)$, $t \in T$, is the solution of the conjugate system $\dot{\psi} = -A_0^{\mathrm{T}}(t)\psi$, $\psi(t^*) = H_1^{\mathrm{T}}\lambda_0$. This immediately gives

$$u^0(t) = P^{-1}(t)B_0^{\mathrm{T}}(t)\psi^0(t), \quad t \in T. \tag{3.6}$$

Note that

$$\psi^{0\mathrm{T}}\left(t\right) = \lambda_0^{\mathrm{T}} H_1 F_0\left(t\right), \quad \psi^{0\mathrm{T}}(t)B_0(t) = \lambda_0^{\mathrm{T}} H_1 \Phi_0\left(t\right), \quad t \in T, \tag{3.7}$$

where

$$\Phi_0(t) = F_0(t)\, B_0(t), \quad t \in T. \tag{3.8}$$

Due to (3.6), (3.7) and the Cauchy formula, we have the equality

$$H_1 y^0(t^*) = H_1 F_0(t_*)\, y_* + H_1 C_1 H_1^{\mathrm{T}} \lambda_0 = g_1, \tag{3.9}$$

where

$$C_1 = \int\limits_{t_*}^{t^*} \Phi_0(t)\, P^{-1}(t)\, \Phi_0^{\mathrm{T}}(t)\, dt. \tag{3.10}$$

Under Assumption 3 the matrix $H_1 C_1 H_1^{\mathrm{T}}$ is nonsingular, which can be easily checked using the implicit controllability criterion with respect to a subspace. Since λ_0 is uniquely defined, by the maximum principle a unique vector of the conjugate variables is associated with the optimal control in the first basic problem.

4. SECOND BASIC PROBLEM

The second stage of the calculations is to solve the infinite-horizon optimal control problem

$$\begin{gathered} \frac{dz}{ds} = A_4(t^*) z + B_2(t^*) u, \\ H_2 z(0) = H_2 A_4^{-1}(t^*) \Big(A_3(t^*) y^0(t^*) + B_2(t^*) u^0(t^*) \Big) + g_2, \\ z(-\infty) = 0, \quad J_2(u) = \frac{1}{2} \int\limits_{-\infty}^{0} u^{\mathrm{T}} P(t^*) u ds \to \min, \end{gathered} \tag{4.1}$$

which will be called *the second basic problem.*

Assumption 4. The controllability criterion with respect to a subspace is satisfied:

$$\operatorname{rank}\Big(H_2 B_2(t^*),\ H_2 A_4(t^*) B_2(t^*),\ \ldots,\ H_2 A_4^{m-1}(t^*) B_2(t^*) \Big) = m_1.$$

This assumption guarantees the existence of admissible controls in the second basic problem. In turn, from this fact it follows that problem (4.1) has a unique solution [14] and is normal. In this case, the maximum principle [6] can be stated in the following way: let $u^*(s)$, $z^*(s)$, $s \le 0$, be an optimal control and an optimal trajectory in problem (4.1); then there exists the m_1-dimensional vector of Lagrange multipliers ν_0 such that

$$\begin{gathered} \Pi\psi^{\mathrm{T}}(s)\, B_2(t^*)\, u^*(s) - \frac{1}{2} u^{*\mathrm{T}}(s)\, P(t^*)\, u^*(s) \\ = \max_{u \in \mathrm{R}^r} \left(\Pi\psi^{\mathrm{T}}(s)\, B_2(t^*)\, u - \frac{1}{2} u^{\mathrm{T}} P(t^*)\, u \right), \quad s \le 0, \end{gathered}$$

where $\Pi\psi(s)$, $s \le 0$, is the solution of the conjugate system

$$\frac{d}{ds}\Pi\psi = -A_4^{\mathrm{T}}(t^*)\Pi\psi, \quad \Pi\psi(0) = H_2^{\mathrm{T}} \nu_0.$$

This immediately gives

$$u^*(s) = P^{-1}(t^*) B_2^{\mathrm{T}}(t^*) \Pi\psi m(s), \quad s \le 0. \tag{4.2}$$

Note that

$$\Pi\psi^{\mathrm{T}}(s) = \nu_0^{\mathrm{T}} H_2 G(s), \quad \Pi\psi^{\mathrm{T}}(s) B_2(t^*) = \nu_0^{\mathrm{T}} H_2 \Pi\Phi(s), \quad s \le 0, \tag{4.3}$$

where

$$\Pi\Phi(s) = G(s) B_2(t^*), \quad s \le 0, \tag{4.4}$$

and $G(s)$, $s \le 0$, is a matrix function of dimensions $(m \times m)$ that satisfies the differential equation

$$\frac{dG}{ds} = -GA_4(t^*), \quad G(0) = E_m. \tag{4.5}$$

Introduce the notation

$$C_3 = \int\limits_{-\infty}^{0} \left(\Pi\Phi(s) P^{-1}(t^*) \Pi\Phi^{\mathrm{T}}(s)\right) ds. \tag{4.6}$$

Due to Assumption 4, the matrix $H_2 C_3 H_2^{\mathrm{T}}$ is nonsingular. Moreover, from (4.2), (4.3), and the Cauchy formula it follows that

$$H_2 z^*(0) = H_2 C_3 H_2^{\mathrm{T}} \nu_0, \tag{4.7}$$

and hence a unique vector of the conjugate variables is associated with the optimal control in the second basic problem.

We emphasize that the vector of Lagrange multipliers ν_0 is the only information about the solution of the second basic problem that will be used below for constructing asymptotically suboptimal controls. There is no need to construct the optimal control $u^*(s)$, $s \le 0$. By the way, this is even impossible if the problem is solved numerically.

As soon as the basic problems are solved, we have to form the matrix

$$I_0 = \begin{pmatrix} H_1 C_1 H_1^{\mathrm{T}} & 0_{n_1 \times m_1} \\ H_2 C_2 H_1^{\mathrm{T}} & H_2 C_3 H_2^{\mathrm{T}} \end{pmatrix} \tag{4.8}$$

of dimensions $(n_1 + m_1) \times (n_1 + m_1)$. The matrices C_1 and C_3 are given by formulas (3.10), (4.6), and

$$C_2 = -A_4^{-1}(t^*) \left(A_3(t^*) C_1 + B_2(t^*) P^{-1}(t^*) B_0^{\mathrm{T}}(t^*)\right).$$

As it has been mentioned, $H_1 C_1 H_1^{\mathrm{T}}$ and $H_2 C_3 H_2^{\mathrm{T}}$ are nonsingular matrices; in this case, $\det I_0 \ne 0$.

5. ASYMPTOTIC ANALYSIS OF SOLUTION OF ORIGINAL PROBLEM

It makes sense to speak about asymptotic suboptimal controls only if the original problem has a solution. We will demonstrate that under the assumptions above, there exists an optimal control in problem (2.1)–(2.3) with a sufficiently small parameter μ. The proof of this fact will be constructive, predetermining further calculations for asymptotically suboptimal control design.

Let $\psi_1(t, \lambda, \nu, \mu)$, $\psi_2(t, \lambda, \nu, \mu)$, $t \in T$, be the solution of the original problem

$$\begin{aligned} \dot{\psi}_1 &= -A_1^{\mathrm{T}}(t)\psi_1 - A_3^{\mathrm{T}}(t)\psi_2, \quad \psi_1(t^*) = H_1^{\mathrm{T}}\lambda, \\ \mu\dot{\psi}_2 &= -A_2^{\mathrm{T}}(t)\psi_1 - A_4^{\mathrm{T}}(t)\psi_2, \quad \psi_2(t^*) = H_2^{\mathrm{T}}\nu, \end{aligned} \tag{5.1}$$

where λ and ν are vectors of dimensions n_1 and m_1, respectively. Denote by $y(t, \lambda, \nu, \mu)$, $z(t, \lambda, \nu, \mu)$, $t \in T$, the trajectory of system (2.1) that is induced by the control

$$u(t, \lambda, \nu, \mu) = P^{-1}(t)\left(B_1^{\mathrm{T}}(t)\psi_1(t, \lambda, \nu, \mu) + B_2^{\mathrm{T}}(t)\psi_2(t, \lambda, \nu, \mu)\right), \quad t \in T. \tag{5.2}$$

Theorem. *Under Assumptions 1–4 there exists a unique optimal control in problem* (2.1)–(2.3) *with a sufficiently small parameter* μ, *which is a normal extremal and can be represented as*

$$u^0(t,\mu) = u(t,\lambda(\mu),\nu(\mu),\mu), \quad t \in T. \tag{5.3}$$

In accordance with the maximum principle, the vector of conjugate variables

$$(\psi_1(t,\lambda(\mu),\mu\nu(\mu),\mu), \mu\psi_2(t,\lambda(\mu),\mu\nu(\mu),\mu)), \quad t \in T,$$

is associated with the optimal control; the vectors $\lambda(\mu)$ *and* $\nu(\mu)$, *representing the solution of the system of equations*

$$H_1 y(t^*,\lambda,\nu,\mu) - g_1 = 0, \quad H_2 z(t^*,\lambda,\nu,\mu) - g_2 = 0, \tag{5.4}$$

have the asymptotic expansions

$$\lambda(\mu) \sim \lambda_0 + \sum_{k=1}^{\infty} \mu^k \lambda_k, \quad \nu(\mu) \sim \nu_0 + \sum_{k=1}^{\infty} \mu^k \nu_k, \tag{5.5}$$

in which the leading coefficients are the vectors of Lagrange multipliers in the basic problems.

Proof. First of all, we show that the left-hand sides of Eqs. (5.4) have asymptotic expansions in terms of the integer powers of the small parameter, and derive the coefficients of these expansions. For the sake of brevity, introduce the notations $\eta = (\lambda,\nu)$, $\eta_0 = (\lambda_0,\nu_0)$, and let $x(t,\eta,\mu) = (y(t,\eta,\mu), z(t,\eta,\mu))$, $t \in T$. By the Cauchy formula, using notations (2.4) we obtain

$$x(t^*,\eta,\mu) = F(t_*,\mu)\, x_* + \int_{t_*}^{t^*} F(t,\mu)\, B(t,\mu)\, u(t,\eta,\mu)\, dt, \tag{5.6}$$

where $F(t,\mu)$, $t \in T$, is a matrix function of dimensions $(n+m)\times(n+m)$ that satisfies the original equation

$$\dot F = -FA(t,\mu), \quad F(t^*) = E_{n+m}. \tag{5.7}$$

For convenience, the solution of this singularly perturbed equation can be written in the block form

$$F(t,\mu) = \begin{pmatrix} F_1(t,\mu) & F_2(t,\mu) \\ F_3(t,\mu) & F_4(t,\mu) \end{pmatrix},$$

where F_1, F_2, F_3, and F_4 are matrices of dimensions $n \times n$, $n \times m$, $m \times n$, and $m \times m$, respectively. Using the method of boundary functions [16], we expand them into the asymptotic series

$$F_i(t,\mu) \sim \sum_{k=0}^{\infty} \mu^k \left(F_{ik}(t) + \Pi_k F_i(s)\right), \quad s = (t-t^*)/\mu, \quad t \in T, \quad i = 1,2,3,4. \tag{5.8}$$

We emphasize that expressions (5.8) are asymptotic expansions uniform in $t \in T$. Another important fact is that the matrix functions $\Pi_k F_i(s)$, $s \le 0$, called boundary terms, satisfy the upper bounds

$$\|\Pi_k F_i(s)\| \le \alpha_k \exp(\beta_k s), \quad i = 1,2,3,4, \quad k = 0,1,\ldots, \tag{5.9}$$

where α_k and β_k are positive constants. Some leading coefficients of expansions (5.8), which will be employed for proving this theorem, have the form

$$\begin{gathered}F_{10}=F_0\left(t\right),\quad F_{20}=0_{n\times m},\quad F_{30}=-A_4^{-1}\left(t^*\right)A_3\left(t^*\right)F_0\left(t\right),\\ F_{40}=0_{m\times m},\quad F_{21}=-F_0\left(t\right)A_2\left(t\right)A_4^{-1}\left(t\right),\\ F_{41}=A_4^{-1}\left(t^*\right)A_3\left(t^*\right)F_0\left(t\right)A_2\left(t\right)A_4^{-1}\left(t\right),\\ \Pi_0F_1=0_{n\times n},\quad \Pi_0F_2=0_{n\times m},\quad \Pi_0F_3=G\left(s\right)A_4^{-1}\left(t^*\right)A_3\left(t^*\right),\\ \Pi_0F_4=G\left(s\right),\quad \Pi_1F_2=A_2\left(t^*\right)A_4^{-1}\left(t^*\right)G\left(s\right),\end{gathered}\tag{5.10}$$

where $F_0\left(t\right),\ t\in T$, and $G\left(s\right),\ s\leq 0$, are the solutions of the original problems (3.4) and (4.5), respectively.

We write (5.6) in the block form

$$\begin{aligned}y(t^*,\eta,\mu)&=F_1(t_*,\mu)y_*+F_2(t_*,\mu)z_*+\int\limits_{t_*}^{t^*}(F_1(t,\mu)B_1(t)+F_2(t,\mu)B_2(t)/\mu)u(t,\eta,\mu)dt,\\ z(t^*,\eta,\mu)&=F_3(t_*,\mu)y_*+F_4(t_*,\mu)z_*+\int\limits_{t_*}^{t^*}(F_3(t,\mu)B_1(t)+F_4(t,\mu)B_2(t)/\mu)u(t,\eta,\mu)dt.\end{aligned}\tag{5.11}$$

Note that

$$\begin{gathered}\psi_1^{\mathrm{T}}(t,\eta,\mu)=\lambda^{\mathrm{T}}H_1F_1\left(t,\mu\right)+\mu\nu^{\mathrm{T}}H_2F_3\left(t,\mu\right),\\ \psi_2^{\mathrm{T}}(t,\eta,\mu)=\lambda^{\mathrm{T}}H_1F_2\left(t,\mu\right)/\mu+\nu^{\mathrm{T}}H_2F_4\left(t,\mu\right).\end{gathered}$$

In view of formulas (5.8) and (5.10), this leads to the asymptotic expansions

$$\psi_i(t,\eta,\mu)\sim\sum_{k=0}^{\infty}\mu^k\left(\psi_{ik}(t,\eta)+\Pi_k\psi_i\left(s,\eta\right)\right),\quad s=\left(t-t^*\right)/\mu,\quad t\in T,\quad i=1,2,\tag{5.12}$$

in which

$$\begin{gathered}\psi_{10}^{\mathrm{T}}(t,\eta)=\lambda^{\mathrm{T}}H_1F_0\left(t\right),\quad \psi_{20}^{\mathrm{T}}(t,\eta)=-\lambda^{\mathrm{T}}H_1F_0\left(t\right)A_2\left(t\right)A_4^{-1}\left(t\right),\\ \Pi_0^{\mathrm{T}}\psi_1(s,\eta)=0,\quad \Pi_0^{\mathrm{T}}\psi_2\left(s,\eta\right)=\left(\lambda^{\mathrm{T}}H_1A_2\left(t^*\right)A_4^{-1}\left(t^*\right)+\nu^{\mathrm{T}}H_2\right)G\left(s\right),\\ \psi_{1k}^{\mathrm{T}}(t,\eta)=\lambda^{\mathrm{T}}H_1F_{1k}\left(t\right)+\nu^{\mathrm{T}}H_2F_{3,k-1}\left(t\right),\\ \psi_{2k}^{\mathrm{T}}(t,\eta)=\lambda^{\mathrm{T}}H_1F_{2,k+1}\left(t\right)+\nu^{\mathrm{T}}H_2F_{4k}\left(t\right),\\ \Pi_k^{\mathrm{T}}\psi_1(s,\eta)=\lambda^{\mathrm{T}}H_1\Pi_kF_1\left(s\right)+\nu^{\mathrm{T}}H_2\Pi_{k-1}F_3\left(s\right),\\ \Pi_k^{\mathrm{T}}\psi_2(s,\eta)=\lambda^{\mathrm{T}}H_1\Pi_{k+1}F_2\left(s\right)+\nu^{\mathrm{T}}H_2\Pi_kF_4\left(s\right),\\ k\geq 0,\quad t\in T,\quad s\leq 0.\end{gathered}\tag{5.13}$$

The asymptotic expansions are uniform in the domain $\|\eta-\eta_0\|<\varepsilon_0,\ t\in T$, where ε_0 is some positive number. Note that due to formulas (3.7) and (4.3), we have the equalities

$$\begin{gathered}\psi_{10}^{\mathrm{T}}(t,\eta_0)=\psi^0\left(t\right),\quad \psi_{20}^{\mathrm{T}}(t,\eta_0)=-\left(A_2\left(t\right)A_4^{-1}\left(t\right)\right)^{\mathrm{T}}\psi^0\left(t\right),\quad t\in T,\\ \Pi_0\psi_2\left(s,\eta_0\right)=\Pi\psi\left(s\right),\quad s\leq 0.\end{gathered}\tag{5.14}$$

In addition, control (5.2) has the uniform asymptotic expansion

$$u(t,\eta,\mu) \sim \sum_{k=0}^{\infty} \mu^k \left(u_k(t,\eta) + \Pi_k u\left(s,\eta\right)\right), \quad s = \left(t - t^*\right)/\mu, \quad t \in T, \tag{5.15}$$

where

$$u_k\left(t,\eta\right) = P^{-1}\left(t\right)\left(B_1^{\mathrm{T}}\left(t\right)\psi_{1k}\left(t,\eta\right) + B_2^{\mathrm{T}}\left(t\right)\psi_{2k}\left(t,\eta\right)\right),$$

$$\Pi_k u\left(s,\eta\right) = \sum_{j=0}^{k} \frac{s^j}{j!}\left(\frac{d^j}{dt^j}\left(P^{-1}B_1^{\mathrm{T}}\right)(t^*)\Pi_{k-j}\psi_1(s,\eta) + \frac{d^j}{dt^j}\left(P^{-1}B_2^{\mathrm{T}}\right)(t^*)\Pi_{k-j}\psi_2(s,\eta)\right), \tag{5.16}$$

$$t \in T, \quad s \le 0, \quad k = 0, 1, \ldots.$$

Besides formulas (3.2), (3.6), (4.2), and (5.14), the equality $\Pi_0\psi_1(s,\eta) = 0$ holds; consequently,

$$u_0(t,\eta_0) = u^0\left(t\right), \quad t \in T, \quad \Pi_0 u\left(s,\eta_0\right) = u^*\left(s\right), \quad s \le 0. \tag{5.17}$$

From (5.8)–(5.10), (5.12), (5.13), (5.15), and (5.16) it follows that the vector functions (5.11) can be expanded into the asymptotic series

$$y(t^*,\eta,\mu) \sim \sum_{k=0}^{\infty} \mu^k y_k\left(\eta\right), \quad z(t^*,\eta,\mu) \sim \sum_{k=0}^{\infty} \mu^k z_k\left(\eta\right), \tag{5.18}$$

in which

$$y_0(\eta) = F_0\left(t_*\right)y_* + \int_{t_*}^{t^*} F_0(t)B_0(t)u_0\left(t,\eta\right)dt,$$

$$z_0(\eta) = -A_4^{-1}(t^*)A_3(t^*)y_0(\eta) + \int_{-\infty}^{0} G(s)B_2(t^*)(u_0(t^*,\eta) + \Pi_0 u(s,\eta))ds,$$

$$\begin{aligned} y_k(\eta) = {} & F_{1k}\left(t_*\right)y_* + F_{2k}\left(t_*\right)z_* + \int_{t_*}^{t^*}\sum_{j=0}^{k} F_{1j}\left(t\right)B_1\left(t\right)u_{k-j}\left(t,\eta\right)dt \\ & + \int_{t_*}^{t^*}\sum_{j=1}^{k+1} F_{2j}\left(t\right)B_2\left(t\right)u_{k-j+1}\left(t,\eta\right)dt \\ & + \int_{-\infty}^{0}\sum_{p=0}^{k-1}\sum_{j=0}^{k-p-1}\frac{s^j}{j!}\frac{d^j}{dt^j}\left(F_{1,k-p-j-1}B_1\right)\left(t^*\right)\Pi_p u\left(s,\eta\right)ds \\ & + \int_{-\infty}^{0}\sum_{p=0}^{k}\sum_{j=0}^{k-p}\frac{s^j}{j!}\frac{d^j}{dt^j}\left(F_{2,k-p-j-1}B_2\right)\left(t^*\right)\Pi_p u\left(s,\eta\right)ds \\ & + \int_{-\infty}^{0}\sum_{p=0}^{k-1}\sum_{j=0}^{k-p-1}\frac{s^j}{j!}\Pi_{k-p-j-1}F_1(s)\left(\frac{d^jB_1}{dt^j}\left(t^*\right)\Pi_p u\left(s,\eta\right) + \frac{\partial^j}{\partial t^j}(B_1u_p)\left(t^*,\eta\right)\right)ds \\ & + \int_{-\infty}^{0}\sum_{p=0}^{k}\sum_{j=0}^{k-p}\frac{s^j}{j!}\Pi_{k-p-j}F_2\left(s\right)\left(\frac{d^jB_2}{dt^j}\left(t^*\right)\Pi_p u\left(s,\eta\right) + \frac{\partial^j}{\partial t^j}\left(B_2u_p\right)\left(t^*,\eta\right)\right)ds, \quad k \ge 1. \end{aligned} \tag{5.19}$$

For $z_k(\eta)$, $k \geq 1$, we have a similar formula, with the only difference that F_{1j}, F_{2j}, $\Pi_j F_1$, and $\Pi_j F_2$ are replaced by F_{3j}, F_{4j}, $\Pi_j F_3$, and $\Pi_j F_4$, respectively, where $j = 0, 1, \ldots$.

Using the implicit function theorem, we check that the system of Eqs. (5.4) is uniquely solvable in η for sufficiently small numbers μ. We write (5.4) as

$$R\,(\eta, \mu) = 0. \tag{5.20}$$

Due to (5.18), the asymptotic expansion

$$R\,(\eta, \mu) \sim \sum_{k=0}^{\infty} \mu^k R_k\,(\eta)\,, \tag{5.21}$$

where $R_0(\eta) = (H_1 y_0(\eta) - g_1, H_2 z_0(\eta) - g_2)$ and $R_k(\eta) = (H_1 y_k(\eta), H_2 z_k(\eta))$, $k = 1, 2, \ldots$, holds uniformly in the domain $\|\eta - \eta_0\| < \varepsilon_0$. Let $R(\eta, 0) = R_0(\eta)$; then the vector function $R(\eta, \mu)$ is continuous together with its partial derivatives with respect to the components of the vector η in the domain $\|\eta - \eta_0\| < \varepsilon_0$, $0 \leq \mu < \mu_0$, where μ_0 is a sufficiently small positive number.

In accordance with (5.17), (5.19), and the Cauchy formula, we have $H_1 y_0\,(\eta_0) = H_1 y^0\,(t^*) = g_1$. The matrix function $G\,(s)\,,\ s \leq 0$, is the solution of the original problem (4.5); since $G\,(s) \to 0$ as $s \to -\infty$,

$$\begin{aligned}
&\int\limits_{-\infty}^{0} G\,(s)\,B_2\,(t^*)\,u_0(t^*, \eta_0)ds \\
&= -\int\limits_{-\infty}^{0} \frac{dG}{ds}\,(s)\,A_4^{-1}\,(t^*)\,B_2\,(t^*)\,u^0(t^*)ds = -A_4^{-1}\,(t^*)\,B_2\,(t^*)\,u^0(t^*).
\end{aligned} \tag{5.22}$$

At the same time, from (5.17) and the Cauchy formula it follows that

$$\int\limits_{-\infty}^{0} G\,(s)\,B_2\,(t^*)\,\Pi_0 u(s, \eta_0)ds = z^*\,(0)\,.$$

In combination with expressions (4.1), (5.19), and (5.22), this equality gives $H_2 z_0\,(\eta_0) = g_2$. Thus, $R\,(\eta_0, 0) = R_0\,(\eta_0) = 0$.

Taking into account (3.7), (4.4), (5.10), (5.13), and (5.16), we perform direct differentiation of the vector function (5.19) to check that $\partial R_0\,(\eta_0, 0)\,/\partial\eta = \partial R_0\,(\eta_0)\,/\partial\eta = I_0$ (see (4.8)). Because this Jacobian matrix is nonsingular, system (5.20) (equivalently, system (5.4)) satisfies the conditions of the implicit function theorem. In accordance with this theorem, in some right-sided neighborhood of the origin $0 \leq \mu < \mu_1$ there exists a uniquely defined vector function $\eta\,(\mu) = (\lambda\,(\mu)\,, \nu\,(\mu))$ satisfying Eqs. (5.4). Moreover, it is continuous and $\eta\,(0) = \eta_0 = (\lambda_0, \nu_0)\,$.

As it has been mentioned, expansion (5.21) is uniform in the domain $\|\eta - \eta_0\| < \varepsilon_0$, and its coefficients are linear functions. Then the solution $(\lambda\,(\mu)\,, \nu\,(\mu))$ of system (5.4) has the asymptotic expansions (5.5).

Consider control (5.3). It is admissible in problem (2.1)–(2.3), since the corresponding trajectory $y^0\,(t, \mu) = y\,(t, \lambda\,(\mu)\,, \nu\,(\mu)\,, \mu)$, $z^0\,(t, \mu) = z\,(t, \lambda\,(\mu)\,, \nu\,(\mu)\,, \mu)$, $t \in T$, induced by this control satisfies the conditions $H_1 y^0(t^*, \mu) = g_1$ and $H_2 z^0(t^*, \mu) = g_2$. By construction, control (5.3) is the normal Pontryagin extremal with the vector of Lagrange multipliers $(\lambda\,(\mu)\,, \mu\nu\,(\mu))$. In accordance with the maximum principle, the vector of conjugate variables $\psi_1^0\,(t, \mu) = \psi_1(t, \lambda\,(\mu)\,, \mu\nu\,(\mu)\,, \mu)$, $\psi_2^0\,(t, \mu) = \mu\psi_2(t, \lambda\,(\mu)\,, \mu\nu\,(\mu)\,, \mu)$, $t \in T$, is associated with this control. Due to the results proved above, there exists an admissible control in the original problem with a sufficiently small number μ. Then, this problem has a unique solution [14], represented by control (5.3). Really, the normal extremal is an optimal control in the minimization problems of convex integral functionals on the trajectories of linear systems; details can be found, e.g., in [17]. The proof of this theorem is complete.

6. DESIGN OF ASYMPTOTICALLY SUBOPTIMAL CONTROLS

In this section, we further describe the algorithm for constructing asymptotic approximations to the solution of problem (2.1)–(2.3), based on the theorem from the previous section and associated formulas.

The vector function

$$u^{(0)}(t,\mu) = P^{-1}(t)\left(B_1^{\mathrm{T}}(t)\,\psi_1(t,\eta_0,\mu) + B_2^{\mathrm{T}}(t)\,\psi_2(t,\eta_0,\mu)\right), \quad t \in T,$$

is an asymptotically suboptimal open loop control of zeroth order in the original problem. Note that it can be obtained directly after solving the basic problems. An asymptotically suboptimal open loop control of order N ($N \geq 1$) has the form

$$u^{(N)}(t,\mu) = P^{-1}(t)\left(B_1^{\mathrm{T}}(t)\,\psi_1(t,\eta^{(N)}(\mu),\mu) + B_2^{\mathrm{T}}(t)\,\psi_2(t,\eta^{(N)}(\mu),\mu)\right), \tag{6.1}$$

where

$$\eta^{(N)}(\mu) = \sum_{k=0}^{N} \mu^k \eta_k, \quad \eta_k = (\lambda_k, \nu_k), \quad k = 1,\ldots,N. \tag{6.2}$$

For calculating control (6.1), we have to find the coefficients λ_k, ν_k, $k = 1,\ldots,N$, of the asymptotic series (5.5), which can be done by the method of undetermined coefficients using expansion (5.21). For this purpose, we expand the vector function

$$\sum_{k=0}^{N} \mu^k R_k\left(\eta^{(N)}(\mu)\right)$$

into the Taylor series in terms of the powers of μ up to order N inclusive, equating to zero all the coefficients starting from the one at μ. As a result, the vectors η_k, $k = 1,\ldots,N$, can be found sequentially from the following nondegenerate systems of linear algebraic equations:

$$I_0\eta_1 = -R_1(\eta_0), \quad I_0\eta_k = -R_2(\eta_0) - \sum_{i=1}^{k-1} \frac{\partial R_i}{\partial \eta}(\eta_0)\,\eta_{k-i}, \quad k \geq 2. \tag{6.3}$$

(Recall that the coefficients of expansions (5.19) are linear vector functions, and this fact has been used in the expressions above.) Due to structure (4.8) of the Jacobian matrix I_0, systems (6.3) are split. Solving these systems sequentially, we find the vectors η_k, $k = 1,\ldots,N$, and compile polynomials (6.2). For obtaining control (6.1), we have to solve the original problem (5.1) for the conjugate system with

$$\lambda = \sum_{k=0}^{N} \mu^k \lambda_k, \quad \nu = \sum_{k=0}^{N} \mu^k \nu_k.$$

The conjugate system is singularly perturbed and hence rigid. The integration of rigid systems can be avoided by replacing the vector functions $\psi_i(t,\eta,\mu)$, $i=1,2$, in (6.1) by their asymptotic approximations

$$\psi_i^{(N)}(t,\eta,\mu)=\sum_{k=0}^{N}\mu^k\left(\psi_{ik}(t,\eta)+\Pi_k\psi_i(s,\eta)\right),\quad s=(t-t^*)/\mu,\quad t\in T,\quad i=1,2.$$

The vector function

$$\bar{u}^{(N)}(t,\mu)=P^{-1}(t)\left(B_1^{\mathrm{T}}(t)\,\psi_1^{(N)}\left(t,\eta^{(N)}(\mu),\mu\right)+B_2^{\mathrm{T}}(t)\,\psi_2^{(N)}\left(t,\eta^{(N)}(\mu),\mu\right)\right),\quad t\in T,$$

together with (6.1), is an asymptotically suboptimal open loop control of order N in problem (2.1)–(2.3). In particular, as it follows from (3.6), (3.7), (4.2), (5.13), and (5.14),

$$\begin{aligned}\bar{u}^{(0)}(t,\mu)&=P^{-1}(t)\left(B_0^{\mathrm{T}}(t)\psi^0(t)+B_2^{\mathrm{T}}(t)\Pi\psi\left((t-t^*)/\mu\right)\right)\\&=u^0(t)+u^*\left((t-t^*)/\mu\right),\quad t\in T.\end{aligned}\tag{6.4}$$

The vector function (6.4) is an asymptotically suboptimal open loop control of zeroth order, which is also clear from formulas (5.15) and (5.17). Note that control (6.4) does not depend on the initial state z_* of the vector of fast variables; for small numbers μ, this control considerably differs from the solution $u^0(t)$, $t\in T$, of the first basic problem only in the boundary layer, i.e., some left-sided neighborhood of the point t^*.

Remark 1. For constructing an asymptotically suboptimal control of order N in the original problem, it suffices to find an asymptotic approximation to $R(\eta,\mu)$ with an accuracy of order μ^{N+1}. This applies the following smoothness requirement to the elements of all matrices figuring in the problem statement [16]: they must have continuous derivatives up to order $N+1$ inclusive.

Remark 2. As follows from the proof of the theorem above, the admissible control (5.3) also exists if the elements of all matrices forming the dynamic system in problem (2.1)–(2.3) are continuously differentiable. In this problem, (y_*,z_*) is an arbitrary initial state; hence, such a hypothesis together with Assumptions 1 and 3 guarantees the controllability of the dynamic system on the segment $[t_*,t^*]$ with respect to the subspace $H_1y=0,\ H_2z=0$. In the case of complete controllability $(H_1=E_n,H_2=E_m)$, this hypothesis leads to the result established in [18].

The asymptotic approximations to the Lagrange multipliers, which satisfy the system of Eqs. (5.20), can be adopted for obtaining an optimal control in problem (2.1)–(2.3) with a given value of μ. This is achieved by the so-called refinement procedure [19]: the roots of system (5.20) are calculated using the Newton method, with $\eta^{(N)}(\mu)$ taken as an initial approximation. For avoiding the integration of rigid systems, the matrix $\partial R(\eta,\mu)/\partial\eta$ can be replaced by its asymptotic approximation I_0.

7. ASYMPTOTICALLY SUBOPTIMAL FEEDBACK CONTROL DESIGN

Of course, asymptotically suboptimal open loop controls depend on the initial state (y_*,z_*,t_*) of the dynamic system. In the previous considerations, such a dependence has been neglected due to a given value of the initial state. In this section, which is devoted to the construction of an asymptotically suboptimal feedback control of zeroth order, we will study precisely this dependence. Hereinafter, let the matrix C_1 be a function of t_*, $t_*<t^*$. As before, the time instant t^* is given.

From equality (3.9) it follows that

$$\lambda_0=\left(H_1C_1(t_*)\,H_1^{\mathrm{T}}\right)^{-1}\left(g_1-H_1F_0(t_*)y_*\right);\tag{7.1}$$

since $\psi^0(t_*) = F_0^{\mathrm{T}}(t_*)\,\psi^0(t^*) = F_0^{\mathrm{T}}(t_*)\,H_1^{\mathrm{T}}\lambda_0$,

$$\psi^0(t_*) = M_1(t_*)\,(g_1 - H_1F_0(t_*)y_*)\,, \tag{7.2}$$

where $M_1(t) = F_0^{\mathrm{T}}(t)_1^{\mathrm{T}}\left(H_1C_1(t)\,H_1^{\mathrm{T}}\right)^{-1}$.

Due to (4.1) and (4.7), we have the equality

$$\nu_0 = \left(H_2C_3H_2^{\mathrm{T}}\right)^{-1}H_2z^*(0) = \left(H_2C_3H_2^{\mathrm{T}}\right)^{-1}\left(H_2A_4^{-1}(t^*)\left(A_3(t^*)\,y^0(t^*) + B_2(t^*)\,u^0(t^*)\right) + g_2\right),$$

which together with formulas (3.6), (3.10), (7.1), and (7.2) yields

$$\begin{aligned}\Pi\psi(s) &= G^{\mathrm{T}}(s)\Pi\psi(0) = G^{\mathrm{T}}(s)H_2^{\mathrm{T}}\nu_0\\ &= M_2(s)\Big(H_2A_4^{-1}(t^*)(A_3(t^*)y^0(t^*) + B_2(t^*)u^0(t^*)) + g_2\Big)\\ &= M_2(s)\Big(H_2A_4^{-1}(t^*)(A_3(t^*)F_0(t_*)y_* + M_3(t_*)(g_1 - H_1F_0(t_*)y_*)) + g_2\Big), \quad s \le 0,\end{aligned} \tag{7.3}$$

where

$$M_2(s) = G^{\mathrm{T}}(s)\,H_2^{\mathrm{T}}\left(H_2C_3H_2^{\mathrm{T}}\right)^{-1},$$

$$M_3(t) = \left(A_3(t^*)\,C_1(t) + B_2(t^*)\,P^{-1}(t^*)\,B_0^{\mathrm{T}}(t^*)\right)H_1^{\mathrm{T}}\left(H_1C_1(t)\,H_1^{\mathrm{T}}\right)^{-1}.$$

In accordance with (6.4), (7.2), and (7.3), at the initial time instant the asymptotically suboptimal open loop control of zeroth order can be represented as

$$\begin{aligned}\bar u^{(0)}(t_*,\mu) = P^{-1}(t_*)\Big(&B_0^{\mathrm{T}}(t_*)M_1(t_*)\,H_1 - B_2^{\mathrm{T}}(t_*)M_2\left((t_* - t^*)/\mu\right)\\ &\times H_2A_4^{-1}(t^*)\,(A_3(t^*) - M_3(t_*)\,H_1)\Big)F_0(t_*)\,y_*\\ &- P^{-1}(t_*)\Big(M_1(t_*)\,g_1 + M_2\left((t_* - t^*)/\mu\right)(g_2 + M_3(t_*)\,g_1)\Big).\end{aligned}$$

Because (y_*, z_*, t_*) is an arbitrary initial state of the dynamic system, by Definition 2 the vector function

$$\begin{aligned}&u^{(0)}(t,y,z,\mu)\\ &= P^{-1}(t)\Big(B_0^{\mathrm{T}}(t)M_1(t)H_1 - B_2^{\mathrm{T}}(t)M_2((t-t^*)/\mu)H_2A_4^{-1}(t^*)(A_3(t^*) - M_3(t)H_1)\Big)F_0(t)y\\ &\quad - P^{-1}(t)\Big(M_1(t)g_1 + M_2((t-t^*)/\mu)(g_2 + M_3(t)g_1)\Big)\end{aligned} \tag{7.4}$$

is an asymptotically suboptimal feedback control of zeroth order in the original problem. Interestingly, this feedback control does not depend on the current position of the vector of fast variables z.

8. EXAMPLE

Consider the reorientation problem for a dynamically symmetric rigid body rotating about its axis of symmetry:

$$\begin{gathered}\dot y_1 = z_1, \quad \dot y_2 = z_2, \quad \mu\dot z_1 = -cz_1 - kz_2 + bu_1, \quad \mu\dot z_2 = kz_1 - cz_2 + bu_2,\\ y_1(t_*) = y_{*1}, \quad y_2(t_*) = y_{*2},\\ z_1(t_*) = 0, \quad z_2(t_*) = 0,\\ y_1(t^*) = 0, \quad z_1(t^*) = 0,\\ J(u) = \frac{1}{2}\int\limits_{t_*}^{t^*}\left(u_1^2 + u_2^2\right)dt \to \min.\end{gathered} \tag{8.1}$$

The constants μ, b, c, and k are positive, and also $\mu \ll 1$. We will construct asymptotically suboptimal open loop and feedback controls of zeroth order in this problem. Note that Assumptions 1 and 2 obviously hold.

In the first basic problem

$$\dot{y}_1 = \frac{cbu_1 - kbu_2}{c^2 + k^2}, \quad \dot{y}_2 = \frac{kbu_1 + cbu_2}{c^2 + k^2},$$
$$y_1(t_*) = y_{*1}, \quad y_2(t_*) = y_{*2}, \quad y_1(t^*) = 0,$$
$$J_1(u) = \frac{1}{2}\int\limits_{t_*}^{t^*} \left(u_1^2 + u_2^2\right) dt \to \min,$$

the dynamic system satisfies requirement (3.5); hence, Assumption 3 holds. The optimal control in this problem can be represented as

$$u_1^0(t) = -\frac{cy_{*1}}{b(t^* - t_*)}, \quad u_2^0(t) = \frac{ky_{*1}}{b(t^* - t_*)}, \quad t \in T = [t_*, t^*],$$

being independent of time.

The second basic problem has the form

$$dz_1/ds = -cz_1 - kz_2 + bu_1, \quad dz_2/ds = kz_1 - cz_2 + bu_2 z_1(0) = y_{*1}/(t^* - t_*),$$
$$z_i(-\infty) = 0, \quad i = 1, 2,$$
$$J_2(u) = \frac{1}{2}\int\limits_{-\infty}^{0} \left(u_1^2 + u_2^2\right) ds \to \min.$$

In this case, Assumption 4 holds. The solution of the second basic problem is the control

$$u_1^*(s) = \frac{2c\exp(cs)\cos(ks)}{b(t^* - t_*)} y_{*1}, \quad u_2^*(s) = \frac{2c\exp(cs)\sin(ks)}{b(t^* - t_*)} y_{*1}, \quad s \le 0.$$

In accordance with formula (6.4), the asymptotically suboptimal open loop control of zeroth order in problem (8.1) can be represented as

$$\bar{u}_i^{(0)}(t, \mu) = u_i^0(t) + u_i^*((t - t^*)/\mu), \quad t \in T, \quad i = 1, 2. \tag{8.2}$$

The asymptotically suboptimal feedback control of zeroth order (see formula (7.4)) is given by

$$\begin{aligned} u_1^{(0)}(y, z, t, \mu) &= \frac{2c\exp(c(t - t^*)/\mu)\cos(k(t - t^*)/\mu) - c}{b(t^* - t)} y_1, \\ u_2^{(0)}(y, z, t, \mu) &= \frac{2c\exp(c(t - t^*)/\mu)\sin(k(t - t^*)/\mu) + k}{b(t^* - t)} y_1. \end{aligned} \tag{8.3}$$

Note that in this example, expression (8.3) for the asymptotically suboptimal feedback control follows not only from the general formula (7.4), but also from formula (8.2) and the above considerations for solving the basic problems.

Finally, we estimated the quality of the asymptotic approximations to the solution of problem (7.2). For this purpose, we calculated the states $(y_1(t^*, \mu), z_1(t^*, \mu))$ obtained by applying the open loop control (8.2) (the feedback control (8.3)) to the dynamic system with specific values of the small parameter in the case $b = 4$, $c = 3$, $t_* = 0$, $t^* = 4$, $k = 1$, $y_{*1} = -2$, and $y_{*2} = 1$. In particular, it turned out that

$$y_1(4,\, 0.1) = -0.03, \quad z_1(4,\, 0.1) = 0, \quad y_1(4,\, 0.001) = 0.0003, \quad z_1(4,\, 0.001) = 0.$$

The results of the calculations are presented with an accuracy of 10^{-6}.

9. CONCLUSIONS

In this paper, computational procedures for constructing asymptotic approximations (open loop and feedback controls) to the solution of the singularly perturbed linear-quadratic optimal control problem with linear terminal constraints on the trajectories have been proposed and substantiated. When implementing the corresponding algorithms, the original optimal control problem splits into two unperturbed optimal control problems of smaller dimension. With such a decomposition, it is possible to efficiently solve optimization problems for dynamic systems with a large number of state-space variables. In addition, the computational procedures of the algorithms do not contain the integration of rigid systems.

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