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Optimality Region for Job Permutation in Single-Machine Scheduling with Uncertain Processing Times

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Abstract—The problem of scheduling optimally a given set of jobs on a single machine is studied. The lower and upper bounds on the admissible duration of each job are known. The optimality criterion of the schedule is the minimum total completion time of a given set of jobs. Some properties of the optimality region for a job permutation are investigated. Polynomial algorithms for constructing the optimality region for a job permutation and also for calculating the volume of this region are developed. The existence conditions of an empty optimality region for a job permutation are determined. A criterion for the existence of a job permutation with the maximum possible volume of the optimality region is established.

Keywords: scheduling theory, uncertain processing times, minimum total completion time, optimality region for job permutation

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1. INTRODUCTION

Production planning includes the stage of constructing schedules for the execution of incoming customers' orders (scheduling for a given set of jobs) on existing equipment (a given set of machines). The optimal schedule is an important factor for the efficiency of any manufacturing process: the production expenses of the enterprise are decreased, the execution time of customers' orders for the final products is reduced, and the raw materials and components required for the manufacture of the final products of the enterprise are supplied in due time. The optimal schedule of the manufacturing process allows reducing the cost of storing raw materials and components, thereby increasing the efficiency of available resources (machines) and capital.

For scheduling problems arising in practice, the exact values of job durations (processing times), as a rule, cannot be determined in advance. However, it is possible to determine some lower and upper bounds for job durations. To solve such problems, algorithms for constructing almost optimal schedules under uncertain numerical parameters are required; see [1–9].

In Section 2 of this paper, we consider the construction problem of an almost optimal schedule for a set $\mathcal{J} = \{J_1, \ldots, J_n\}$ of jobs processed on a single machine, with the minimum total sum $\sum C_i$ of the completion times C_i of all jobs $J_i \in \mathcal{J}$ under given lower $p_i^L > 0$ and upper $p_i^U \geqslant p_i^L$ bounds on the admissible duration $p_i \in [p_i^L, p_i^U]$ of each job J_i. In Section 3, we define the optimality region for a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n})$ of jobs from the set $\mathcal{J}$ that contains the optimality box for the same permutation. The optimality box for a permutation π_k was investigated in [10–17]. In Section 4, we develop polynomial algorithms for determining the optimality region for a permutation π_k and also for calculating the volume of this region. We prove necessary and sufficient conditions under which the optimality region for a job permutation is an empty set. In Section 5, we study some properties of a permutation π_k of jobs from the set $\mathcal{J}$ that have the maximal volume of the optimality region.

2. PROBLEM STATEMENT AND A SURVEY OF KNOWN RESULTS

The set of jobs $\mathcal{J} = \{J_1, \ldots, J_n\}$ has to be processed on a single machine. The exact value of the duration p_i of each job $J_i \in \mathcal{J}$ is unknown before scheduling of the set $\mathcal{J}$. No preemptions of job processing are allowed. When implementing the schedule of the set $\mathcal{J}$, the duration p_i of each job $J_i \in \mathcal{J}$ can take any real value from a segment $\left[p_i^L, p_i^U\right]$, where $p_i^U \geqslant p_i^L > 0$. The exact value of the duration $p_i \in \left[p_i^L, p_i^U\right]$ becomes known only at the time C_i of completing the job J_i. We denote by R^n the space of n-dimensional real vectors, and by R^n_+ the subspace of R^n consisting of all nonnegative n-dimensional real vectors, $R^n_+ \subset R^n$. In the space R^n, the set of all vectors $(p_1, \ldots, p_n)$ of the admissible durations of all jobs from the set $\mathcal{J}$ is the n-dimensional box, i.e., the set of all vectors $p = (p_1, \ldots, p_n) \in R^n_+$ satisfying the system of inequalities

$$p_1^L \leqslant p_1 \leqslant p_1^U; \ \ldots; \ p_n^L \leqslant p_n \leqslant p_n^U.$$

The set of the admissible durations $(p_1, \ldots, p_n) = p \in R^n_+$ of all jobs from the set $\mathcal{J}$ can be represented as the Cartesian product of the segments $[p_i^L, p_i^U]$: $\times_{i=1}^n \left[p_i^L, p_i^U\right] := \left[p_1^L, p_1^U\right] \times \ldots \times \left[p_n^L, p_n^U\right] = T = \left\{p \in R^n_+ \colon p_i^L \leqslant p_i \leqslant p_i^U, \ i \in \{1, \ldots, n\}\right\}$. A vector $p \in T$ will be called a scenario. Let the set $\Pi = \{\pi_1, \ldots, \pi_{n!}\}$ contain all permutations $\pi_k = (J_{k_1}, \ldots, J_{k_n})$ of the jobs from the set $\mathcal{J}$. For a fixed scenario $p \in T$ and a fixed permutation $\pi_k \in \Pi$, we denote by $C_i = C_i(\pi_k, p)$ the completion time of a job $J_i \in \mathcal{J}$ in the semi-active schedule [5, 18], which is uniquely determined by the permutation π_k. Hereinafter, the criterion $\sum C_i$ is the minimum total completion time of all jobs from the set $\mathcal{J}$:

$$\sum_{J_i \in \mathcal{J}} C_i(\pi_t, p) = \min_{\pi_k \in \Pi} \left\{ \sum_{J_i \in \mathcal{J}} C_i(\pi_k, p) \right\}. \tag{1}$$

The permutation $\pi_t = (J_{t_1}, \ldots, J_{t_n}) \in \Pi$ figuring in (1) is an optimal permutation of jobs from the set $\mathcal{J}$ for a fixed scenario $p \in T$.

Since the number of jobs $n = |\mathcal{J}|$ is known before constructing a schedule π_t for the set of jobs $\mathcal{J}$, the criterion $\sum C_i$ also corresponds to the minimum average duration $\frac{\sum_{J_i \in \mathcal{J}} C_i(\pi_k, p)}{n}$ of all jobs from the set $\mathcal{J}$.

The uncertain problem stated above can be written as $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ in the standard three-position notation $\alpha|\beta|\gamma$ of scheduling problems; see [19]. Hereinafter, α specifies the type of the processing system (the number of machines), β the processing conditions of jobs, and γ the objective function.

If a scenario $p \in T$ is known before scheduling (i.e., for each job $J_i \in \mathcal{J}$, $[p_i^L, p_i^U] = [p_i, p_i]$), then the uncertain problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ turns into the deterministic problem $1||\sum C_i$. The notation $1|p|\sum C_i$ will be used to indicate an instance of the deterministic problem $1||\sum C_i$ with a fixed scenario $p \in T$ by the time of optimal schedule construction. As was demonstrated in [20], an instance $1|p|\sum C_i$ can be solved in the time $O(n \log n)$ using the following optimality criterion of a permutation $\pi_k \in \Pi$.

Theorem 1. *A permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ of jobs from the set $\mathcal{J}$ is optimal for the instance $1|p|\sum C_i$ if and only if*

$$p_{k_1} \leqslant \ldots \leqslant p_{k_n}. \tag{2}$$

Under the condition $p_{k_u} < p_{k_v}$, the job J_{k_u} precedes the job J_{k_v} in any permutation $\pi_k \in \Pi$ that is optimal for the instance $1|p|\sum C_i$.

In the uncertain problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$, the scenario $p \in T$ is not fixed before constructing a permutation $\pi_k \in \Pi$ of jobs from the set $\mathcal{J}$. Hence, the exact completion time C_i of each job $J_i \in \mathcal{J}$ cannot be determined until the job J_i is completed. Consequently, the value of the objective function $\sum C_i$ for a permutation π_k of jobs from the set $\mathcal{J}$ remains unknown until all jobs from $\mathcal{J}$ processed, provided that the jobs $J_i \in \mathcal{J}$ satisfy the strict inequalities $p_i^L < p_i^U$.

As a rule, in the uncertain problem $\alpha|p_i^L \leqslant p_i \leqslant p_i^U|\gamma$ there exists no schedule that would be optimal for all scenarios from a set T of cardinality greater than 1. Therefore, additional objective functions and/or assumptions are often introduced for such ill-posed scheduling problems in the literature. In particular, the problem $\alpha|p_i^L \leqslant p_i \leqslant p_i^U|\gamma$ can be solved by the stochastic approach under the assumption that all job durations are random variables with known probability distributions [7, 18].

If sufficient information on the probability distributions of random durations is absent, other methods can be used [5, 21]. For example, in accordance with the robust approach [2, 22–24], the decision maker prefers to eliminate the worst-case scenario for the requisite schedule to be implemented [4, 9, 23]. For any permutation $\pi_k \in \Pi$ and any scenario $p \in T$, the difference $\gamma_p^k - \gamma_p^t = r(\pi_k, p)$ is called the regret for π_k with the value γ_p^k of the objective function γ for p. The value $Z(\pi_k) = \max\{r(\pi_k, p) : p \in T\}$ is called the worst-case absolute regret, and the value $Z^*(\pi_k) = \max\left\{\frac{r(\pi_k,p)}{\gamma_p^t} : p \in T\right\}$ the worst-case relative regret.

Despite the fact that the problem $1|p|\sum w_iC_i$ is polynomially solvable [20] for any weights $w_i > 0$ defined for the set of jobs $J_i \in \mathcal{J}$, the construction of a permutation $\pi_t \in S$ with the smallest value $Z(\pi_t)$ (or a permutation with the smallest value $Z^*(\pi_t)$) for the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ is NP-hard even in the case of two admissible scenarios [21, 24, 25]. The branch-and-bound method for constructing a permutation π_t with the minimum absolute regret value $Z(\pi_t)$ for the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum w_iC_i$ was developed in [3]. In accordance with the experimental evidence of computer simulations, the suggested method yields such a permutation π_t for the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum w_iC_i$ if the number n of jobs does not exceed 40.

The fuzzy approach can be used to construct optimal schedules for the set of jobs $\mathcal{J}$ with fuzzy durations processed on a given set of machines $\mathcal{M}$; see [8, 9, 26]. This approach is applicable to the uncertain scheduling problems with a small number n of jobs only.

Several heuristics for solving the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum w_iC_i$ were tested in [1]. Different probability distributions for the factual durations of jobs were chosen for the numerical experiments with $n \in \{100, 300, 400, 600, 800, 1000\}$. As a result, the best heuristic $U2$ was identified, which yielded an average error of 1.1% over all test examples considered in comparison with the values of the objective function $\sum w_iC_i$ calculated for the factual durations of jobs.

The stability approach to scheduling [6, 10–17] includes the stability analysis of optimal schedules with respect to possible variations of the durations of all jobs from the set $\mathcal{J}$ and further construction of the minimal dominant set of schedules. For any scenario $p \in T$, the minimal dominant set contains at least one optimal schedule. In contrast to the stochastic, robust and fuzzy approaches, the stability approach to scheduling is to find an optimal schedule for a maximum possible number of admissible scenarios from a given set T. In particular, if there exists a dominant singleton $\{\pi_t\}$ for the problem $\alpha|p_i^L \leqslant p_i \leqslant p_i^U|\gamma$, then the schedule π_t is optimal for the instance $\alpha|p|\gamma$ for all scenarios $p \in T$, despite the uncertain durations of all jobs from the set $\mathcal{J}$.

In [10–12, 14, 15], the stability approach was used for solving the scheduling problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum w_iC_i$. In [15], a criterion for the existence of a dominant singleton for the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum w_iC_i$ was established. In [12], difficult instances of the problem

$1|p_i^L \leqslant p_i \leqslant p_i^U|\sum w_i C_i$ with $n \in \{50, 100, 500, 1000, 5000, 10\,000\}$ were randomly generated and then approximately solved using the MAX-OPTBOX algorithm (developed therein as well) with an average error of 1.5%. In [10], some instances of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ with $n \in \{10, 20, \ldots, 100, 200, \ldots, 1000, 2000, \ldots, 10\,000\}$ were randomly generated and then approximately solved using Algorithm 3 (developed therein as well) with an average error of 0.74%.

The MAX-OPTBOX algorithm (Algorithm 3) yields a permutation $\pi_k \in \Pi$ for which the optimality box has the greatest half-perimeter (the greatest relative half-perimeter, respectively). Algorithm 3 takes into account the following distinction of the objective function $\sum C_i$: an increase δ in a term of the objective function caused by violating inequality (2) due to the factual duration p_{k_i} of the job J_{k_i} finally leads to the increase $\delta(n-i+1)$ of the value of the objective function. Therefore, the error in choosing an appropriate order of processing for the job J_{k_i} in the schedule is more important than the error in choosing an appropriate order of processing for the job J_{k_j} if $i < j$.

The definition of the optimality box was given in [10, 12]. Let $M = (k_{i_1}, \ldots, k_{i_{|M|}})$ be an ordered subset of the set $\{1, \ldots, n\}$ that satisfies the relations $\{k_1, \ldots, k_{|M|}\} \subseteq \{1, \ldots, n\}$, $|M| \leqslant n$, and $k_{i_1} < \ldots < k_{i_{|M|}}$.

Definition 1. An inclusion-maximal box

$$\mathcal{OB}(\pi_k, T) = \left[l^{opt}_{k_{i_1}}, u^{opt}_{k_{i_1}}\right] \times \ldots \times \left[l^{opt}_{k_{i_{|M|}}}, u^{opt}_{k_{i_{|M|}}}\right] =: \times_{k_{i_r} \in M} \left[l^{opt}_{k_{i_r}}, u^{opt}_{k_{i_r}}\right] \subseteq T$$

is called the optimality box for a permutation $\pi_k = (J_{k_{i_1}}, \ldots, J_{k_{i_n}}) \in \Pi$ if π_k, being optimal for the instance $1|p|\sum C_i$ with a scenario $p = (p_1, \ldots, p_n) \in T$, remains optimal for the instance $1|p'|\sum C_i$ with any scenario $p' = (p'_1, \ldots, p'_n) \in [p_1, p_1] \times \ldots \times [p_{k_{i_r}-1}, p_{k_{i_r}-1}] \times [l^{opt}_{k_{i_r}}, u^{opt}_{k_{i_r}}] \times [p_{k_{i_r}+1}, p_{k_{i_r}+1}] \times \ldots \times [p_n, p_n] \in T$. If there exists no scenario $p \in T$ for which the permutation π_k is optimal for the instance $1|p|\sum C_i$, then let $\mathcal{OB}(\pi_k, T) = \emptyset$.

Any variation $p'_{k_{i_r}}$ of the duration $p_{k_{i_r}}$ of a job $J_{k_{i_r}} \in \mathcal{J}$ that belongs to the inclusion-maximal segment $\left[l^{opt}_{k_{i_r}}, u^{opt}_{k_{i_r}}\right]$ (see Definition 1) guarantees the optimality of the permutation $\pi_k \in \Pi$ for any scenario $p' = (p'_1, \ldots, p'_n)$ if it satisfies the inclusion $p'_{k_{i_r}} \in \left[l^{opt}_{k_{i_r}}, u^{opt}_{k_{i_r}}\right]$. The maximal segment $\left[l^{opt}_{k_{i_r}}, u^{opt}_{k_{i_r}}\right]$ of the length $u^{opt}_{k_{i_r}} - l^{opt}_{k_{i_r}} \geqslant 0$, $l^{opt}_{k_{i_r}} \leqslant u^{opt}_{k_{i_r}}$ (see Definition 1) will be called the optimality segment for the job $J_{k_{i_r}} \in \mathcal{J}$ in the permutation π_k. If there exists no optimality segment $\left[l^{opt}_{k_{i_r}}, u^{opt}_{k_{i_r}}\right]$, $l^{opt}_{k_{i_r}} \leqslant u^{opt}_{k_{i_r}}$, for a job $J_{k_{i_r}} \in \mathcal{J}$, then we will say that the job $J_{k_{i_r}}$ has no optimality segment in the permutation π_k.

3. OPTIMALITY REGION FOR JOB PERMUTATION $\pi_k \in \Pi$

We define the optimality region $\mathcal{OR}(\pi_k, T)$ for a permutation $\pi_k \in \Pi$ containing the optimality box for π_k: $\mathcal{OB}(\pi_k, T) \subseteq \mathcal{OR}(\pi_k, T)$.

Definition 2. An inclusion-maximal closed subset $\mathcal{OR}(\pi_k, T) \subseteq T$ of the set R^n_+ is called the optimality region for a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ with respect to T if the permutation π_k is optimal for the instance $1|p|\sum C_i$ for any scenario $p = (p_1, \ldots, p_n) \in \mathcal{OR}(\pi_k, T)$. If there exists no scenario $p \in T$ for which the permutation π_k is optimal for the instance $1|p|\sum C_i$, then let $\mathcal{OR}(\pi_k, T) = \emptyset$.

Due to Theorem 1, we will discriminate among three types of segments for each job $J_{k_r} \in \mathcal{J}$ in a fixed permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ as follows:

the optimality segment $\left[l^{opt}_{k_r}, u^{opt}_{k_r}\right] \subseteq \left[p^L_{k_r}, p^U_{k_r}\right]$;

the conditional optimality segment $\left[l_{k_r}^{copt}, u_{k_r}^{copt}\right] \subseteq \left[p_{k_r}^L, p_{k_r}^U\right]$;

the nonoptimality segment $\left[l_{k_r}^{non}, u_{k_r}^{non}\right] \subseteq \left[p_{k_r}^L, p_{k_r}^U\right]$.

The optimality segment $\left[l_{k_r}^{opt}, u_{k_r}^{opt}\right]$ for a job J_{k_r} in a permutation π_k has been specified in Definition 1 of the optimality box.

The nonoptimality segment for a job J_{k_r} in a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ is an inclusion-maximal segment $\left[l_{k_r}^{non}, u_{k_r}^{non}\right] \subseteq \left[p_{k_r}^L, p_{k_r}^U\right]$ for which the permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n})$ cannot be optimal for the instance $1|p'|\sum C_i$ for any scenario $p' = (p'_1, \ldots, p'_n) \in T$ such that

$$(p'_1, \ldots, p'_n) \in \left\{\times_{i=1}^{r-1}\left[p_{k_i}^L, p_{k_i}^U\right]\right\} \times \left[l_{k_r}^{non}, u_{k_r}^{non}\right] \times \left\{\times_{i=r+1}^{n}\left[p_{k_i}^L, p_{k_i}^U\right]\right\}.$$

Due to the necessary and sufficient conditions (2) for the optimality of a permutation $\pi_k \in \Pi$ for the instance $1|p|\sum C_i$, for each nonoptimality segment $[l_{k_r}^{non}, u_{k_r}^{non}]$ either there exists a job $J_{k_v} \in \mathcal{J}$ such that $r < v$ and

$$p_{k_v}^U = l_{k_r}^{non} < u_{k_r}^{non} = p_{k_r}^U, \tag{3}$$

or there exists a job $J_{k_w} \in \mathcal{J}$ such that $w < r$ and

$$p_{k_r}^L = l_{k_r}^{non} < u_{k_r}^{non} = p_{k_w}^L. \tag{4}$$

From Definition 1 it follows that the open nonoptimality interval $(l_{k_r}^{non}, u_{k_r}^{non})$ for a job J_{k_r} in a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ has no common points with the optimality segment $\left[l_{k_r}^{opt}, u_{k_r}^{opt}\right]$, i.e.,

$$\left(l_{k_r}^{non}, u_{k_r}^{non}\right) \bigcap \left[l_{k_r}^{opt}, u_{k_r}^{opt}\right] = \emptyset. \tag{5}$$

For illustrating these definitions, we will use Example 1 of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ with 18 jobs ($n = 18$). The segments $\left[p_i^L, p_i^U\right]$ specifying the admissible durations of jobs $J_i \in \mathcal{J} = \{J_1, \ldots, J_{18}\}$ can be found in the table. The segments $\left[p_i^L, p_i^U\right]$ are also shown in Fig. 1 in the rectangular coordinate system for a permutation $\pi_1 = (J_1, \ldots, J_{18}) \in \Pi$. The abscissa axis in Fig. 1 corresponds to the segment $[p_i^L, p_i^U]$ of the admissible durations of jobs $J_i \in \mathcal{J}$. The ordinate axis in this figure corresponds to the jobs J_i from the set $\mathcal{J}$.

The conditional optimality segment for a job J_{k_r} in a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ is an inclusion-maximal segment $\left[l_{k_r}^{copt}, u_{k_r}^{copt}\right] \subseteq \left[p_{k_r}^L, p_{k_r}^U\right]$ such that any point $p_{k_r}^* \in \left[l_{k_r}^{copt}, u_{k_r}^{copt}\right]$ does not belong to the open nonoptimality interval, $p_{k_r}^* \notin (l_{k_r}^{non}, u_{k_r}^{non})$, and also there exists at least one job $J_{k_d} \in \mathcal{J}$, $d \neq r$, for which $p_{k_r}^* \in \left[p_{k_d}^L, p_{k_d}^U\right]$.

In accordance with the definition of the conditional optimality segment, there exist points $p_{k_r}^* \in \left[l_{k_r}^{copt}, u_{k_r}^{copt}\right]$ such that a permutation $\pi_k \in \Pi$ is optimal for the instance $1|p'|\sum C_i$ with a scenario $p' = (p'_1, \ldots, p'_n) \in [p_{k_1}, p_{k_1}] \times \ldots \times [p_{k_r-1}, p_{k_r-1}] \times [p_{k_r}^*, p_{k_r}^*] \times [p_{k_r+1}, p_{k_r+1}] \times \ldots \times [p_{k_n}, p_{k_n}]$, and also there exist other points $p_{k_r}^{**} \in \left[l_{k_r}^{copt}, u_{k_r}^{copt}\right]$ such that the permutation $\pi_k \in \Pi$ is nonoptimal

Initial data for Example 1 of problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
p_i^L	1	3	2	7	2	4	11	12	11	14	7	27	30	9	36	37	38	21
p_i^U	8	5	8	9	10	6	15	15	20	18	23	34	32	40	42	40	40	41

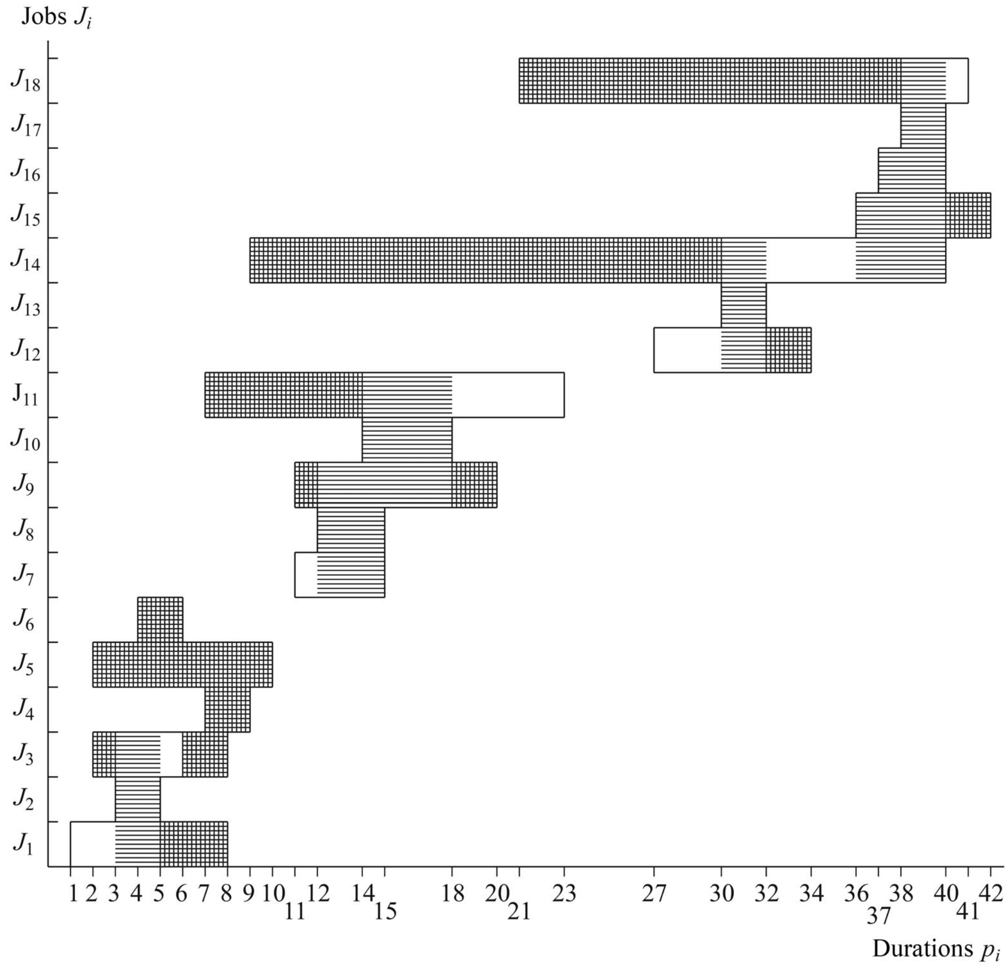

Fig. 1. Optimality segments, conditional optimality segments (horizontal hatching) and nonoptimality segments (horizontal and vertical hatching) for jobs $J_i \in \mathcal{J}$ in permutation $\pi_1 = (J_1, \ldots, J_{18})$: Example 1 of problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$.

for the instance $1|p''|\sum C_i$ with a scenario $p'' = (p''_1, \ldots, p''_n) \in [p_{k_1}, p_{k_1}] \times \ldots \times [p_{k_r-1}, p_{k_r-1}] \times [p^{**}_{k_r}, p^{**}_{k_r}] \times [p_{k_r+1}, p_{k_r+1}] \times \ldots \times [p_{k_n}, p_{k_n}]$. The conditional optimality segments for jobs $J_i \in \mathcal{J}$ in a permutation $\pi_1 = (J_1, \ldots, J_{18})$ are hatched by horizontal lines in Fig. 1.

In addition, note that in any fixed permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$, the conditional optimality segment $\left[l^{copt}_{k_r}, u^{copt}_{k_r}\right]$ for a job J_{k_r} has no common points with the open optimality interval $(l^{opt}_{k_r}, u^{opt}_{k_r})$ and also with the open nonoptimality interval $(l^{non}_{k_r}, u^{non}_{k_r})$. In other words, the following equalities hold:

$$\left[l^{copt}_{k_r}, u^{copt}_{k_r}\right] \bigcap \left(l^{opt}_{k_r}, u^{opt}_{k_r}\right) = \emptyset, \tag{6}$$

$$\left[l^{copt}_{k_r}, u^{copt}_{k_r}\right] \bigcap (l^{non}_{k_r}, u^{non}_{k_r}) = \emptyset. \tag{7}$$

If there exists no conditional optimality segment $[l^{copt}_{k_r}, u^{copt}_{k_r}]$, $l^{copt}_{k_r} < u^{copt}_{k_r}$ for a job $J_{k_r} \in \mathcal{J}$ in a permutation π_k, we will say that the job J_{k_r} has no conditional optimality in the permutation π_k.

The optimality, conditional optimality, and nonoptimality segments for all jobs $J_i \in \mathcal{J}$ in the permutation $\pi_1 = (J_1, \ldots, J_{18})$ are presented in Fig. 1. The nonoptimality segments for the jobs $J_i \in \mathcal{J}$ in the permutation π_1 are indicated by double (vertical and horizontal) hatching.

Remark 1. Due to Theorem 1, for each job $J_i \in \mathcal{J}$ in a permutation $\pi_k \in \Pi$, there may exist at most one optimality segment, at most two conditional optimality segments, and at most two nonoptimality segments.

As it has been demonstrated in Fig. 1, the job J_3 in the permutation $\pi_1 \in \Pi$ has two nonoptimality segments, $[2,3]$ and $[6,8]$, one conditional optimality segment, $[3,5]$, and also one optimality segment, $[5,6]$. For the job J_5 in the permutation π_1, there are two nonoptimality segments, $[2,7]$ and $[6,10]$, with the non-empty intersection $[6,7] = [2,7] \bigcap [6,10]$.

The following result is immediate from the definitions of the optimality, nonoptimality and conditional optimality segments for a job J_{k_r}, $r \in \{1,\dots,n\}$, in a permutation $\pi_k \in \Pi$.

Lemma 1. *The segment $[p^L_{k_r}, p^U_{k_r}]$ of all admissible durations of a job $J_{k_r} \in \mathcal{J}$ can be represented as the union of all optimality, nonoptimality and conditional optimality segments for a job J_{k_r}, $r \in \{1,\dots,n\}$, in a permutation $\pi_k = (J_{k_1},\dots J_{k_n}) \in \Pi$. For any nonoptimality segment $[l^{non}_{k_r}, u^{non}_{k_r}]$, at least one of the equalities $l^{non}_{k_r} = p^L_{k_r}$ and $u^{non}_{k_r} = p^U_{k_r}$ holds.*

The optimality region $\mathcal{OR}(\pi_k, T)$ for a permutation $\pi_k = (J_{k_1},\dots,J_{k_n}) \in \Pi$ can be obtained by constructing the optimality region for the permutation π_k for the corresponding scheduling problem $1|\widehat{p}^L_i \leqslant p_i \leqslant \widehat{p}^U_i|\sum C_i$ with the reduced segments of the admissible durations $\left[\widehat{p}^L_i, \widehat{p}^U_i\right] \subseteq \left[p^L_i, p^U_i\right]$. The reduced segments $\left[\widehat{p}^L_{k_r}, \widehat{p}^U_{k_r}\right]$, $J_{k_r} \in \mathcal{J}$, for a permutation $\pi_k = (J_{k_1},\dots,J_{k_n})$ are given by the formulas

$$\widehat{p}^L_{k_r} = \max_{1 \leqslant j \leqslant r \leqslant n} p^L_{k_j}, \quad \widehat{p}^U_{k_r} = \min_{1 \leqslant r \leqslant j \leqslant n} p^U_{k_j} \tag{8}$$

for each job $J_{k_r} \in \{J_{k_1},\dots,J_{k_n}\} = \mathcal{J}$. The set of all reduced scenarios determined by (8) will be denoted by $\widehat{T} = \left[\widehat{p}^L_1, \widehat{p}^U_1\right] \times \dots \times \left[\widehat{p}^L_n, \widehat{p}^U_n\right]$.

Theorem 2. *The optimality region $\mathcal{OR}(\pi_k, T)$ for a permutation $\pi_k = (J_{k_1},\dots,J_{k_n}) \in \Pi$ for the problem $1|p^L_i \leqslant p_i \leqslant p^U_i|\sum C_i$ coincides with the optimality region $\mathcal{OR}(\pi_k, \widehat{T})$ for the same permutation $1|\widehat{p}^L_i \leqslant p_i \leqslant \widehat{p}^U_i|\sum C_i$ with the set $\widehat{T}$ of admissible scenarios.*

Proof. In accordance with the necessary and sufficient conditions (2) for the optimality of a permutation π_k for an instance $1|p|\sum C_i$, the relations

$$p^L_{k_r} \leqslant p_{k_r} < \widehat{p}^L_{k_r} = \max_{1 \leqslant j \leqslant r \leqslant n} p^L_{k_j}$$

holding at least for one duration p_{k_r} imply that the permutation $\pi_k = (J_{k_1},\dots,J_{k_r},\dots,J_{k_n})$ cannot be optimal for an instance $1|p|\sum C_i$ with some scenario $p = (p_1,\dots,p_n) \in \left\{\times_{i=1}^{r-1}\left[p^L_{k_i}, p^U_{k_i}\right]\right\} \times \left[l^{non}_{k_r}, u^{non}_{k_r}\right] \times \left\{\times_{i=r+1}^{n}\left[p^L_{k_i}, p^U_{k_i}\right]\right\}$.

In a similar way, the relations

$$\min_{1 \leqslant r \leqslant j \leqslant n} p^U_{k_j} = \widehat{p}^U_{k_r} < p_{k_r} \leqslant p^U_{k_r}$$

holding at least for one duration p_{k_r} imply that the permutation $\pi_k = (J_{k_1},\dots,J_{k_r},\dots,J_{k_n})$ cannot be optimal for an instance $1|p|\sum C_i$ with some scenario $p = (p_1,\dots,p_n) \in \left\{\times_{i=1}^{r-1}\left[p^L_{k_i}, p^U_{k_i}\right]\right\} \times \left[l^{non}_{k_r}, u^{non}_{k_r}\right] \times \left\{\times_{i=r+1}^{n}\left[p^L_{k_i}, p^U_{k_i}\right]\right\}$.

Thus, the following result is true: *the set of all scenarios $p \in T$ for which a permutation π_k is optimal for the scheduling problem $1|p^L_i \leqslant p_i \leqslant p^U_i|\sum C_i$ is contained in the set of all scenarios $p \in T$ for which the permutation π_k is optimal for the problem $1|\widehat{p}^L_i \leqslant p_i \leqslant \widehat{p}^U_i|\sum C_i$ with the set $\widehat{T}$ of admissible scenarios.*

The converse assertion follows from the inclusion $\widehat{T} \subseteq T$.

Assertion. *The set of all scenarios $p \in T$ for which a permutation π_k is optimal for the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ is contained in the set of all scenarios $p \in T$ for which the permutation π_k is optimal for the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$.*

In view of these assertions, for the original problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ and for the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ with the set $\widehat{T} = \left[\widehat{p}_1^L, \widehat{p}_1^U\right] \times \ldots \times \left[\widehat{p}_n^L, \widehat{p}_n^U\right]$ of scenarios, the optimality regions coincide with each other for any fixed permutation $\pi_k \in \Pi$: $\mathcal{OR}(\pi_k, T) = \mathcal{OR}(\pi_k, \widehat{T})$. The proof of Theorem 2 is complete.

The next result can be easily obtained from Definition 2 and Theorem 2.

Lemma 2. *For the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ with the set $\widehat{T}$ of scenarios, the open optimality interval $\left(l_{k_r}^{opt}, u_{k_r}^{opt}\right)$ for a job J_{k_r} in a permutation $\pi_k \in \Pi$ has no common points with the segment $\left[p_{k_d}^L, p_{k_d}^U\right]$ of the admissible durations of any other job $J_{k_d} \in \mathcal{J}$, $d \neq r$, i.e., the following equality holds:*

$$\left(l_{k_r}^{opt}, u_{k_r}^{opt}\right) \bigcap \left[p_{k_d}^L, p_{k_d}^U\right] = \emptyset. \tag{9}$$

Now, we prove necessary and sufficient conditions under which the optimality region for a permutation $\pi_k \in \Pi$ is an empty set.

Theorem 3. *The optimality region $\mathcal{OR}(\pi_k, T)$ for a job $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ is an empty set, $\mathcal{OR}(\pi_k, T) = \emptyset$, if and only if there exists at least one job $J_{k_r} \in \mathcal{J}$, $r \in \{1, \ldots, n\}$, in the permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ that has no conditional optimality and simultaneously no optimality segment.*

Proof. *Sufficiency.* Assume that there exists a job $J_{k_r} \in \mathcal{J}$ in a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ that has no conditional optimality and also no optimality segment. Due to Lemma 1, $\left[l_{k_r}^{non}, u_{k_r}^{non}\right] = \left[p_{k_r}^L, p_{k_r}^U\right] \neq \emptyset$. Hence, either there exists a job $J_{k_v} \in \mathcal{J}$ such that $r < v$ and also relations (3) hold, or there exists a job $J_{k_w} \in \mathcal{J}$ such that $w < r$ and relations (4) hold. In the former case, the inequality $p_{k_v} < p_{k_r}$ is satisfied for each admissible duration $p_{k_r} \in \left[p_{k_r}^L, p_{k_r}^U\right]$ of the job J_{k_r} and for each admissible duration $p_{k_v} \in \left[p_{k_v}^L, p_{k_v}^U\right]$ of the job J_{k_v}. In the latter case, the inequality $p_{k_w} > p_{k_r}$ is satisfied for each admissible duration $p_{k_r} \in \left[p_{k_r}^L, p_{k_r}^U\right]$ of the job J_{k_r} and for each admissible duration $p_{k_w} \in \left[p_{k_w}^L, p_{k_w}^U\right]$ of the job J_{k_w}.

In accordance with Theorem 1, in both cases the permutation π_k cannot be optimal for an instance $1|p|\sum C_i$ with some scenario $p \in T$. Hence, by Definition 2, the optimality region for the permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ is an empty set, $\mathcal{OR}(\pi_k, T) = \emptyset$. Sufficiency is established.

Necessity. This part will be proved by contradiction. Let the equality $\mathcal{OR}(\pi_k, T) = \emptyset$ hold. Assume on the contrary that there exists no job $J_{k_r} \in \mathcal{J}$, $r \in \{1, \ldots, n\}$, in a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ without conditional optimality and simultaneously without an optimality segment.

Due to Definition 2, the equality $\mathcal{OR}(\pi_k, T) = \emptyset$ means the absence of any scenario $p \in T$ such that the permutation π_k is optimal for the instance $1|p|\sum C_i$ with $p \in T$. Nevertheless, we will construct a scenario $p^* \in T$ that is contained in the optimality region for the permutation π_k.

If a job J_{k_i} has the optimality segment $\left[l_{k_i}^{opt}, u_{k_i}^{opt}\right]$, $l_{k_i}^{opt} \leqslant u_{k_i}^{opt}$, in a permutation π_k, then there exists at least one point $p_{k_i}^* \in \left[l_{k_i}^{opt}, u_{k_i}^{opt}\right]$. In this case, we choose the value $p_{k_i}^*$ as the duration of the job J_{k_i}.

If a job J_{k_j} has no optimality segment in a permutation π_k, then by the assumption there exists the conditional optimality segment $\left[l_{k_j}^{copt}, u_{k_j}^{copt}\right]$ for the job J_{k_j}. In this case, we choose the value $l_{k_j}^{copt}$ as the duration of the job J_{k_j}: $p_{k_j}^* = l_{k_j}^{copt}$.

Such a choice of the durations $p_{k_j}^*$ of all jobs $J_{k_j} \in \{J_{k_1}, \dots, J_{k_n}\} = \mathcal{J}$ yields the scenario $p^* = (p_{k_1}^*, \dots, p_{k_n}^*) \in T$. From equalities (6), (7) and Lemma 2 with equality (9) it follows that the permutation π_k is optimal for the instance $1|p^*|\sum C_i$. The resulting inclusion $p^* \in \mathcal{OR}(\pi_k, T)$ obviously contradicts the equality $\mathcal{OR}(\pi_k, T) = \emptyset$. Thus, necessity is established, and the proof of Theorem 3 is complete.

From Theorem 3 it follows that, if $\mathcal{OR}(\pi_k, T) \neq \emptyset$, then in the permutation $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in \Pi$ for each job $J_{k_r} \in \mathcal{J}$, $r \in \{1, \dots, n\}$, there exists at least one optimality segment or conditional optimality segment. Therefore, the dimension of the non-empty optimality region $\mathcal{OR}(\pi_k, T)$ is $n = |\mathcal{J}|$. Because the converse assertion is also immediate from Theorem 3, we arrive in the following result.

Corollary 1. *The dimension of the optimality region* $\mathcal{OR}(\pi_k, T)$ *is* $n = |\mathcal{J}|$ *if and only if* $\mathcal{OR}(\pi_k, T) \neq \emptyset$.

In Fig. 1, the nonoptimality segment $[l_4^{non}, u_4^{non}] = [7, 9]$ for the job J_4 in the permutation $\pi_1 = (J_1, \dots, J_{18})$ satisfies the equalities $[l_4^{non}, u_4^{non}] = [7, 9] = [p_4^L, p_4^U]$, i.e., the job J_4 has no conditional optimality and simultaneously no optimality segment in the permutation π_1. From Theorem 3 it follows that the optimality region for the permutation π_1 is an empty set, $\mathcal{OR}(\pi_1, T) = \emptyset$.

4. OPTIMALITY REGION: CONSTRUCTION AND CALCULATION OF VOLUME

Theorem 3 has been adopted to develop Algorithm 1 for checking the equality $\mathcal{OR}(\pi_k, T) = \emptyset$ for a fixed permutation $\pi_k \in \Pi$. This algorithm has a complexity of $O(n)$.

If Algorithm 1 establishes the relation $\mathcal{OR}(\pi_k, T) \neq \emptyset$ for a permutation π_k, then (in accordance with Theorem 2) it constructs by formulas (8) the reduced segments $\left[\widehat{p}_i^L, \widehat{p}_i^U\right]$ of the admissible durations of jobs $J_i \in \mathcal{J}$. As a result, the initial data for the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ with the set $\widehat{T}$ of admissible scenarios are obtained.

Algorithm 1.

INPUT: Segments $[p_i^L, p_i^U]$ of durations of jobs $J_i \in \mathcal{J}$;
permutation $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in \Pi$ of jobs $\mathcal{J}$.

OUTPUT: Segments $[\widehat{p}_i^L, \widehat{p}_i^U]$ for jobs $J_i \in \mathcal{J}$ in the case of establishing $\mathcal{OR}(\pi_k, T) \neq \emptyset$.

Step 1: $\widehat{p}_{k_1}^L = p_{k_1}^L$, $t_L = p_{k_1}^L$, $r = 2$;

Step 2: **IF** $p_{k_r}^U \geqslant t_L$ **THEN GOTO** Step 3 **ELSE GOTO** Step 5;

Step 3: **IF** $p_{k_r}^L > t_L$ **THEN** $t_L = p_{k_r}^L$, $\widehat{p}_{k_r}^L = t_L$, $r := r + 1$;
ELSE $\widehat{p}_{k_r}^L = t_L$, $r := r + 1$;
IF $r \leqslant n$ **THEN GOTO** Step 2 **ELSE** $\widehat{p}_{k_n}^U = p_{k_n}^U$, $t_U = p_{k_n}^U$;

Step 4: **FOR** $r = n - 1$ **to** 1 **STEP** -1 **DO**
IF $p_{k_r}^U < t_U$ **THEN** $t_U = p_{k_r}^U$, $\widehat{p}_{k_r}^U = t_U$ **ELSE** $\widehat{p}_{k_r}^U = t_U$;
END FOR STOP.

Step 5: $\mathcal{OR}(\pi_k, T) = \emptyset$ **STOP**.

In Steps 1, 2, and 5 of Algorithm 1, the equality $\mathcal{OR}(\pi_k, T) = \emptyset$ is checked. In Steps 2–4, the reduced segments of the admissible durations of jobs from the set $\mathcal{J}$ for the problem

$1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ are constructed. Due to Theorem 2, the optimality region for a permutation $\pi_k \in \Pi$ of jobs for the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ coincides with the optimality region for the same permutation π_k of jobs for the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ with the set $\widehat{T}$ of reduced scenarios. As is easily verified, Algorithm 1 is implemented using $O(n)$ elementary operations.

4.1. Optimality Region for Permutation $\pi_k \in \Pi$ in Special Cases

In this subsection, we will construct the optimality region $\mathcal{OR}(\pi_k, T)$ for two special cases of a permutation $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in \Pi$ and calculate the volume $Vol(\pi_k, T)$ of the optimality region $\mathcal{OR}(\pi_k, T)$. In the first case, the optimality region $\mathcal{OR}(\pi_k, T)$ is determined only by the optimality segments of all jobs $J_{k_r} \in \mathcal{J}$ (Lemma 3). In the second case, the optimality region is determined only by the conditional optimality segments of all jobs $J_{k_r} \in \mathcal{J}$ (Lemma 4).

Lemma 3. *If $\mathcal{OR}(\pi_k, T) \neq \emptyset$ and each job $J_{k_r} \in \mathcal{J}$ has no conditional optimality in a permutation $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in \Pi$, then the optimality region for the permutation π_k coincides with the optimality box for the same permutation:*

$$\mathcal{OR}(\pi_k, T) = \times_{r=1}^{n} \left[l_{k_r}^{opt}, u_{k_r}^{opt}\right] = \mathcal{OB}(\pi_k, T). \tag{10}$$

The volume of this optimality region $\mathcal{OR}(\pi_k, T)$ is

$$Vol(\pi_k, T) = \prod_{J_{k_r} \in \left\{\mathcal{J} \,:\, l_{k_r}^{opt} < u_{k_r}^{opt}\right\}} \left(u_{k_r}^{opt} - l_{k_r}^{opt}\right). \tag{11}$$

Proof. Due to Theorem 2, we will consider the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ instead of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$. Since $\mathcal{OR}(\pi_k, T) \neq \emptyset$, by Theorem 3 there exists no job $J_{k_r} \in \mathcal{J}$ without conditional optimality and simultaneously without an optimality segment in the permutation π_k.

Because each job $J_{k_r} \in \mathcal{J}$ has no conditional optimality in the permutation π_k and there exists at most one optimality segment for it (see Remark 1), for each job $J_{k_r} \in \mathcal{J}$ we have the equality $\left[\widehat{p}_{k_r}^L, \widehat{p}_{k_r}^U\right] = \left[l_{k_r}^{opt}, u_{k_r}^{opt}\right]$. Hence, in accordance with Definitions 1 and 2, the optimality region $\mathcal{OR}(\pi_k, T)$ for the permutation π_k coincides with the optimality box for the same permutation π_k, being the n-dimensional box $\times_{r=1}^{n} \left[l_{k_r}^{opt}, u_{k_r}^{opt}\right] = \mathcal{OB}(\pi_k, T)$ of the volume $\prod_{J_{k_r} \in \{\mathcal{J} \,:\, l_{k_r}^{opt} < u_{k_r}^{opt}\}} \left(u_{k_r}^{opt} - l_{k_r}^{opt}\right)$. The proof of Lemma 3 is complete.

The permutation $\pi_2 = (J_1, \dots, J_{10})$ of jobs from the set $\mathcal{J} = \{J_1, \dots, J_{10}\}$ for Example 2 of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ is shown in Fig. 2. As the permutation π_2 satisfies the hypothesis of Lemma 3, the optimality region $\mathcal{OR}(\pi_2, T)$ is the following 10-dimensional box:

$$\mathcal{OR}(\pi_2, T) = \mathcal{OB}(\pi_2, T) = \left[l_1^{opt}, u_1^{opt}\right] \times \ldots \times \left[l_{10}^{opt}, u_{10}^{opt}\right]$$
$$= [4,8] \times [8,8] \times [9,12] \times [13,16] \times [16,16] \times [16,18] \times [19,19] \times [19,22] \times [22,22] \times [22,24],$$

whose volume can be calculated using formula (11):

$$Vol(\pi_2, T) = \prod_{J_r \in \{\mathcal{J} \,:\, l_r^{opt} < u_r^{opt}\}} \left(u_r^{opt} - l_r^{opt}\right)$$
$$= (8-4)(12-9)(16-13)(18-16)(22-19)(24-22) = 432.$$

Following Theorem 2, we will consider the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ with the set $\widehat{T}$ of reduced scenarios instead of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$.

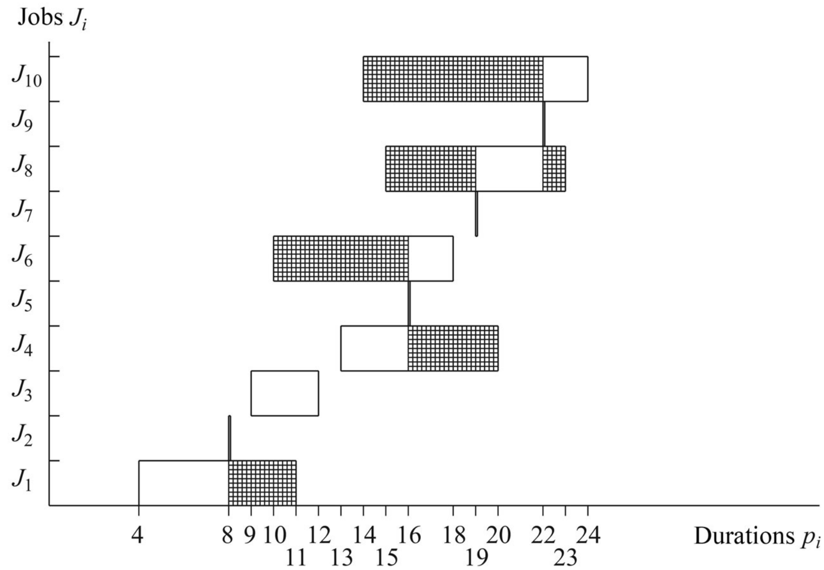

Fig. 2. Optimality segments and nonoptimality segments (hatching) for jobs $J_i \in \mathcal{J}$ in permutation $\pi_2 = (J_1, \dots, J_{20})$: Example 2 of problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$.

Definition 3. A section of a permutation $\pi_k \in \Pi$ is an inclusion-maximal permutation $s_v^{\pi_k} = (J_{k_v}, \dots, J_{k_{v+m_v}})$, $1 \leqslant v \leqslant v + m_v \leqslant n$, such that for any value $d \in \left(\widehat{p}_{k_v}^L,\, \widehat{p}_{k_{v+m_v}}^U\right)$ there exists a job $J_{k_{v+i}}$, $i \in \{0, \dots, m_v\}$, for which $d \in \left(\widehat{p}_{k_{v+i}}^L,\, \widehat{p}_{k_{v+i}}^U\right)$. The segment $\left[\widehat{p}_{k_v}^L,\, \widehat{p}_{k_{v+m_v}}^U\right]$ is called the cover of a section $s_v^{\pi_k}$.

Note that the set $S(\pi_k) = \{s_v^{\pi_k}, \dots, s_w^{\pi_k}\}$, $1 \leqslant v < \ldots < w \leqslant n$, of all sections of each permutation $\pi_k \in \Pi$ is uniquely defined.

Remark 2. From Definition 3 it follows that each job $J_{k_i} \in \mathcal{J}$ is either contained in a unique section of a permutation π_k, or is not contained in any section of π_k. If there exists at least one job $J_{k_i} \in \mathcal{J}$ not contained in any section of a permutation π_k, then $\mathcal{OR}(\pi_k, T) = \emptyset$ by Theorem 3.

The next result is immediate from Remark 2 and the proof of Theorem 3.

Corollary 2. *The optimality region for a permutation* $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in \Pi$ *is a non-empty set,* $\mathcal{OR}(\pi_k, T) \neq \emptyset$, *if and only if* $\pi_k = (s_1^{\pi_k}, \dots, s_w^{\pi_k})$.

Consider the permutation $\pi_2 = (J_1, \dots, J_{10})$ in Example 2 (Fig. 2). For this permutation, each section consists of the unique job $s_1^{\pi_2} = (J_1)$, $\dots$, $s_{10}^{\pi_2} = (J_{10})$ and also the equality $\pi_2 = (s_1^{\pi_2}, \dots, s_{10}^{\pi_2})$ holds. A section composed of a single job will be called trivial.

For proving Lemma 4, we partition the cover $\left[\widehat{p}_{k_j}^L,\, \widehat{p}_{k_{j+m_j}}^U\right]$ of each section $s_j^{\pi_k} \in S(\pi_k)$ into the inclusion-maximal conditional optimality subintervals

$$\left[\widehat{p}_{k_j}^L,\, \widehat{p}_{k_{j+m_j}}^U\right] = \left[l_1^j\left(s_j^{\pi_k}\right), u_1^j\left(s_j^{\pi_k}\right)\right) \bigcup \cdots \bigcup \left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right) \bigcup \cdots \bigcup \left[l_{n(j)}^j\left(s_j^{\pi_k}\right), u_{n(j)}^j\left(s_j^{\pi_k}\right)\right], \tag{12}$$

which are distinct from each other in the sense that, for different subsets $\mathcal{J}_i^j = \left\{J_{k_i}, \dots, J_{k_{|\mathcal{J}_i^j|}}\right\}$ of the set $\left\{J_{k_j}, \dots, J_{k_{j+m_j}}\right\}$, $j \leqslant i \leqslant j + m_j$, the inclusions $\left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right] \subseteq \left[\widehat{p}_{k_r}^L, \widehat{p}_{k_r}^U\right]$ hold

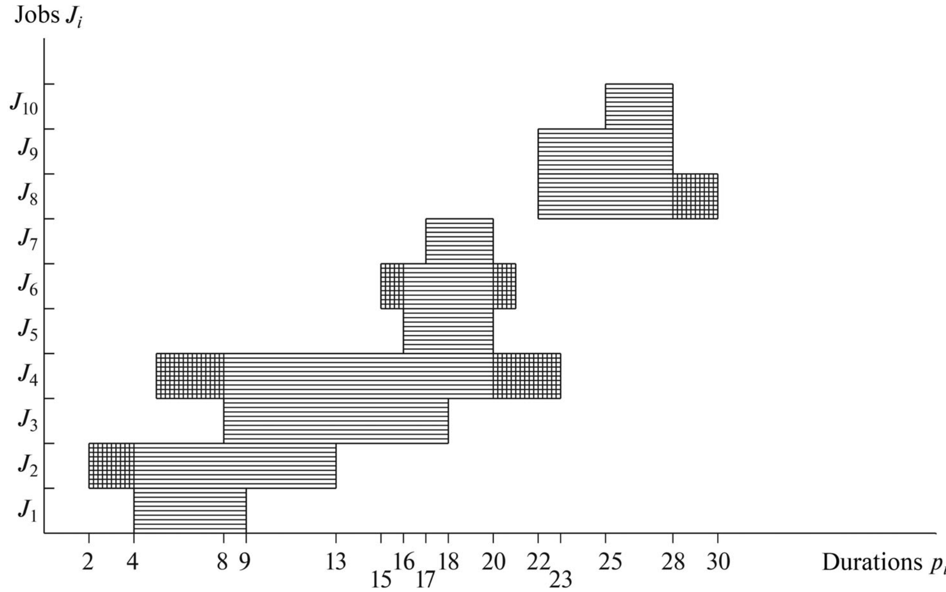

Fig. 3. Conditional optimality segments (horizontal hatching) and nonoptimality segments (horizontal and vertical hatching) for jobs $J_i \in \mathcal{J}$ in permutation $\pi_2 = (J_1, \ldots, J_{10})$: Example 3 of problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$.

for all jobs $J_{k_r} \in \mathcal{J}_i^j$. Let $\widehat{J}_i^j = \left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right)$ denote the permutation of jobs of the set $\mathcal{J}_i^j = \left\{J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right\}$.

For illustrating these notations, we will consider the partitions of the covers of the sections $s_1^{\pi_2}$ and $s_8^{\pi_2}$ of the permutation $\pi_2 = (J_1, \ldots, J_{10})$ for Example 3 (Fig. 3). The section $s_1^{\pi_2}$ consists of the seven ordered jobs $s_1^{\pi_2} = (J_1, \ldots, J_7)$ and has the cover $\left[\widehat{p}_1^L, \widehat{p}_7^U\right] = [4, 20]$. As a result, we obtain the following partition (12): $[4, 20] = [4, 8) \cup [8, 9) \cup [9, 13) \cup [13, 16) \cup [16, 17) \cup [17, 18) \cup [18, 20]$ of the cover $[\widehat{p}_1^L, \widehat{p}_7^U]$ into the conditional optimality subintervals. Here

$$\left[l_1^1(J_1, J_2), u_1^1(J_1, J_2)\right) = [4, 8);$$
$$\left[l_2^1(J_1, \ldots, J_4), u_2^1(J_1, \ldots, J_4)\right) = [8, 9);$$
$$\left[l_3^1(J_2, J_3, J_4), u_3^1(J_2, J_3, J_4)\right) = [9, 13);$$
$$\left[l_4^1(J_3, J_4), u_4^1(J_3, J_4)\right) = [13, 16);$$
$$\left[l_5^1(J_3, \ldots, J_6), u_5^1(J_3, \ldots, J_6)\right) = [16, 17);$$
$$\left[l_6^1(J_3, \ldots, J_7), u_6^1(J_3, \ldots, J_7)\right) = [17, 18];$$
$$\left[l_7^1(J_4, \ldots, J_7), u_7^1(J_4, \ldots, J_7)\right] = [18, 20].$$

Note the equalities $\widehat{J}_1^1 = (J_1, J_2)$, $\widehat{J}_2^1 = (J_1, \ldots, J_4)$, $\widehat{J}_3^1 = (J_2, J_3, J_4)$, $\widehat{J}_4^1 = (J_3, J_4)$, $\widehat{J}_5^1 = (J_3, \ldots, J_6)$, $\widehat{J}_6^1 = (J_3, \ldots, J_7)$, and $\widehat{J}_7^1 = (J_4, \ldots, J_7)$. The section $s_8^{\pi_2}$ consists of the three jobs $s_8^{\pi_2} = (J_8, J_9, J_{10})$ and has the cover $\left[\widehat{p}_8^L, \widehat{p}_{10}^U\right] = [22, 28]$. Hence, we obtain the partition $[22, 28] = [22, 25) \cup [25, 28]$ of the cover $[22, 28]$ into the conditional optimality subintervals

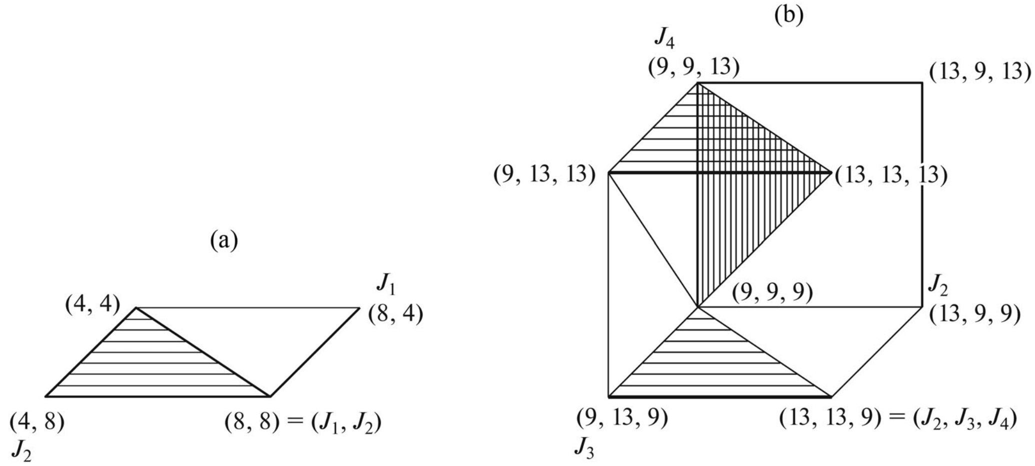

Fig. 4. (a) Optimality triangle $Pyr^{opt}(J_1, J_2)$ with base $[(4,8),(8,8)]$ and altitude $[(4,8),(4,4)]$ for conditional optimality subinterval $[4,8)$ for permutation $\pi_2 = (J_1, \ldots, J_{10})$ in Example 3. (b) Optimality pyramid $Pyr^{opt}(J_2, J_3, J_4)$ with base $[(9,9,13),(9,13,13),(13,13,13)]$ and altitude $[(9,9,13),(9,9,9)]$ for conditional optimality subinterval $[9,13)$ for permutation π_2 in Example 3.

$[l_1^2(J_8, J_9), u_1^2(J_8, J_9))$ and $[l_2^2(J_8, J_9, J_{10}), u_2^2(J_8, J_9, J_{10})]$. Note the equalities $\widehat{J}_1^2 = (J_8, J_9)$ and $\widehat{J}_2^2 = (J_8, J_9, J_{10})$.

Any section $s_j^{\pi_k} \in S(\pi_k)$ of a permutation π_k and any ordered set $\widehat{J}_i^j$ of jobs from the set $\mathcal{J}_i^j \subseteq \mathcal{J}$ are permutations of a corresponding subset of jobs from the set $\mathcal{J}$. Therefore, it is possible to consider the optimality regions for all jobs from which these permutations consist of. Such optimality regions will be denoted by $\mathcal{OR}(s_j^{\pi_k}, T)$ and $\mathcal{OR}(\widehat{J}_i^j, T)$, like the optimality regions $\mathcal{OR}(\pi_k, T)$ for a permutation $\pi_k \in \Pi$ of the entire set of jobs $\mathcal{J}$. The volumes of the optimality regions $\mathcal{OR}(s_j^{\pi_k}, T)$ and $\mathcal{OR}(\widehat{J}_i^j, T)$ will be denoted by $Vol(s_j^{\pi_k}, T)$ and $Vol(\widehat{J}_i^j, T)$, respectively.

The proof of Lemma 4, like the definition of the d-dimensional optimality pyramid $Pyr^{opt}\widehat{J}_i^j$, is postponed to the Appendix. This pyramid is the optimality region for the permutation of jobs $\mathcal{J}_i^j \subseteq \mathcal{J}$, i.e., the equalities $d = |\mathcal{J}_i^j|$ and $\mathcal{OR}(\widehat{J}_i^j, T) = Pyr^{opt}\widehat{J}_i^j$ hold.

Lemma 4. *If $\mathcal{OR}(\pi_k, T) \neq \emptyset$ and each job $J_{k_r} \in \mathcal{J}$ has no optimality segment in a permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$, then the optimality region $\mathcal{OR}(\pi_k, T)$ for the permutation π_k is the Cartesian product of the $|S(\pi_k)|$ optimality regions $\mathcal{OR}(s_j^{\pi_k}, T)$ of all sections $S(\pi_k)$:*

$$\mathcal{OR}(\pi_k, T) = \mathcal{OR}(s_1^{\pi_k}, T) \times \ldots \times \mathcal{OR}\left(s_j^{\pi_k}, T\right) \times \ldots \times \mathcal{OR}\left(s_{|S(\pi_k)|}^{\pi_k}, T\right), \tag{13}$$

where $\mathcal{OR}(s_j^{\pi_k}, T)$ is defined as the union of the d-dimensional optimality pyramids $Pyr^{opt}\widehat{J}_i^j$ in the space R^n, $d \in \left\{|\mathcal{J}_i^j|, \ldots, |\mathcal{J}_{n(j)}^j|\right\}$:

$$\mathcal{OR}(s_j^{\pi_k}, T) = \bigcup_{i=1}^{n(j)} \mathcal{OR}(\widehat{J}_i^j, T) = \bigcup_{i=1}^{n(j)} Pyr^{opt}\widehat{J}_i^j. \tag{14}$$

The volume of the optimality region for the permutation π_k is given by

$$Vol(\pi_k, T) = \prod_{j=1}^{|S(\pi_k)|} \sum_{i=1}^{n(j)} \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!}. \tag{15}$$

The permutation $\pi_2 = (J_1, \ldots, J_{10})$ of jobs from the set $\mathcal{J} = \{J_1, \ldots, J_{10}\}$ for Example 3 of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ is shown in Fig. 3. By Theorem 3, the optimality region for the permutation π_2 is a non-empty set; because the permutation π_2 satisfies the hypothesis of Lemma 4, the volume of the optimality region $\mathcal{OR}(\pi_2, T)$ can be calculated by formula (15):

$$Vol(\pi_2, T) = \prod_{j=1}^{2} \sum_{i=1}^{n(j)} \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!}$$
$$= \left[\frac{(8-4)^2}{2!} + \frac{(9-8)^4}{4!} + \frac{(13-9)^3}{3!} + \frac{(16-13)^2}{2!} + \frac{(17-16)^4}{4!} \right.$$
$$\left. + \frac{(18-17)^5}{5!} + \frac{(20-18)^4}{4!}\right]\left[\frac{(25-22)^2}{2!} + \frac{(28-25)^3}{3!}\right]$$
$$= \left[\frac{16}{2} + \frac{1}{24} + \frac{64}{6} + \frac{9}{2} + \frac{1}{24} + \frac{1}{120} + \frac{16}{24}\right]\left[\frac{9}{2} + \frac{27}{6}\right] = 210\frac{33}{40}.$$

The triangle $Pyr^{opt}\widehat{J}_1^1 = Pyr^{opt}(J_1, J_2)$ (see equality (14)) for the conditional optimality subinterval $[4, 8)$ that belongs to the optimality region $\mathcal{OR}(\pi_2, T)$ for the permutation $\pi_2 = (J_1, \ldots, J_{10})$ in Example 3 is demonstrated in Fig. 4a. Also, the three-dimensional pyramid $Pyr^{opt}\widehat{J}_3^1 = Pyr^{opt}(J_2, J_3, J_4)$ (see (14)) for the conditional optimality subinterval $[9, 13)$ that belongs to the same optimality region $\mathcal{OR}(\pi_2, T)$ is presented in Fig. 4b.

4.2. Volume of Optimality Region for Permutation $\pi_k \in \Pi$ in General Case

Let $S^*(\pi_k)$ denote the subset of trivial sections of the set $S(\pi_k)$.

Note that the permutation $\pi_2 = (J_1, \ldots, J_{10})$ in Example 2 satisfies the hypotheses of Lemma 3 and all sections of the set $S(\pi_2)$ are trivial: $S(\pi_2) = S^*(\pi_2)$.

The permutation $\pi_2 = (J_1, \ldots, J_{10})$ in Example 3 satisfies the hypothesis of Lemma 4, and the set $S^*(\pi_2)$ of its trivial sections is empty: $S^*(\pi_2) = \emptyset$.

Theorem 4. *If $\mathcal{OR}(\pi_k, T) \neq \emptyset$, then the optimality region $\mathcal{OR}(\pi_k, T)$ is the Cartesian product* (13) *of the optimality regions of the sections $S(\pi_k)$ such that*

$$\mathcal{OR}\left(s_j^{\pi_k}, T\right) = \mathcal{OB}\left(s_j^{\pi_k}, T\right) = \left[l_{k_r}^{opt}, u_{k_r}^{opt}\right]$$

for each trivial section $s_j^{\pi_k} = (J_{k_r})$ and

$$\mathcal{OR}(s_j^{\pi_k}, T) = \bigcup_{i=1}^{n(j)} \mathcal{OR}(\widehat{J}_i^j, T) = \bigcup_{i=1}^{n(j)} Pyr^{opt}\widehat{J}_i^j$$

for each nontrivial section $s_j^{\pi_k} \in S(\pi_k) \setminus S^*(\pi_k)$. The volume of the optimality region is given by

$$Vol(\pi_k, T) = \prod_{(J_{k_r}) \in \left\{S^*(\pi_k)\,:\, l_{k_r}^{opt} < u_{k_r}^{opt}\right\}} \left(u_{k_r}^{opt} - l_{k_r}^{opt}\right) \prod_{s_j^{\pi_k} \in S(\pi_k) \setminus S^*(\pi_k)} \sum_{i=1}^{n(j)} \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!}. \quad (16)$$

Proof. Due to Theorem 2, we will consider the problem $1|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U|\sum C_i$ instead of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$. Because $\mathcal{OR}(\pi_k, T) \neq \emptyset$, the dimension of the optimality region $\mathcal{OR}(\pi_k, T)$ is equal to n (Corollary 1). From Theorem 3 it follows that there exists no job $J_{k_r} \in \mathcal{J}$ without conditional optimality and simultaneously without an optimality segment in the permutation π_k.

Since $\mathcal{OR}(\pi_k, T) \neq \emptyset$, Remark 2 implies that each job $J_{k_i} \in \mathcal{J}$ is contained in a unique section of the permutation π_k. Moreover, in accordance with Corollary 2, the equality $\pi_k = (s_1^{\pi_k}, \ldots, s_w^{\pi_k})$ holds.

Using the above-mentioned properties of the permutation π_k, we will establish equalities (13) and (16) by the sequential application of Lemma 3 or Lemma 4 in special cases where a successive permutation consists of a unique section $s_v^{\pi_k} \in S(\pi_k)$ of the permutation π_k.

We begin with the first section $s_1^{\pi_k} = (J_{k_1}, \ldots, J_{k_{1+m_1}})$ of the permutation π_k. If the section $s_1^{\pi_k}$ is trivial, i.e., $s_1^{\pi_k} = (J_{k_1})$, then by Lemma 3 we obtain the first factor $\left[l_{k_1}^{opt}, u_{k_1}^{opt}\right] = \mathcal{OR}(s_1^{\pi_k}, T)$ in the requisite Cartesian product (13) and the first factor $\left(u_{k_1}^{opt} - l_{k_1}^{opt}\right)$ in the first product of equality (16) given the strict inequality $l_{k_1}^{opt} < u_{k_1}^{opt}$. (If $l_{k_1}^{opt} = u_{k_1}^{opt}$, then the factor $\left(u_{k_1}^{opt} - l_{k_1}^{opt}\right) = 0$ is not added into equality (16) due to Lemma 3.)

If the section $s_1^{\pi_k}$ is nontrivial, i.e.,

$$s_1^{\pi_k} \in S(\pi_k) \backslash S^*(\pi_k),$$

then by Lemma 4 we obtain the first factor

$$\mathcal{OR}(s_1^{\pi_k}, T) = \bigcup_{i=1}^{n(1)} \mathcal{OR}(\widehat{J}_i^1, T) = \bigcup_{i=1}^{n(1)} Pyr^{opt} \widehat{J}_i^1$$

in the Cartesian product (13) and the first factor

$$\sum_{i=1}^{n(1)} \frac{(u_i^1(s_1^{\pi_k}) - l_i^1(s_1^{\pi_k}))^{|\mathcal{J}_i^1|}}{|\mathcal{J}_i^1|!}$$

in the second product of equality (16). Here, note that the partition (12) of the section $s_1^{\pi_k}$ into the conditional optimality subintervals may also include the subintervals $[l_i^1(s_1^{\pi_k}), u_i^1(s_1^{\pi_k}))$ for which $|\mathcal{J}_i^1| = 1$. (In fact, Lemma 4 makes no provision for such a possibility.) However, we will demonstrate that the equality

$$\mathcal{OR}(\widehat{J}_i^j, T) = \frac{(u_i^j(s_j^{\pi_k}) - l_i^j(s_j^{\pi_k}))^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!},$$

which appears in (16), is valid in the case $|\mathcal{J}_i^j| = 1$ as well. Really, if $|\mathcal{J}_i^j| = 1$, then

$$\frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!} = \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^1}{1!} = \left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right),$$

which is required for satisfying equality (16).

In a similar manner, we will consider the second section $s_2^{\pi_k} = (J_{k_{m_1+1}}, \ldots, J_{k_{m_1+1+m_2}})$ of the permutation π_k. Applying Lemma 3 (if the section $s_2^{\pi_k}$ is trivial) or Lemma 4 (otherwise), we will supplement the constructed part of the Cartesian product (13) with the second factor and also supplement a corresponding product of the two ones figuring in equality (16) with the second factor.

Proceeding in this fashion up to the last section $s_w^{\pi_k}$ of the permutation π_k (and adding the corresponding factors), we will finally arrive in both equalities (13) and (16). The proof of Theorem 4 is complete.

The algorithm for calculating the volume $Vol(\pi_k, T)$ of the optimality region $\mathcal{OR}(\pi_k, T) \neq \emptyset$ for a permutation $\pi_k \in \Pi$ (see below) is based on Theorems 2, 3, and 4.

Algorithm 2.

INPUT: Permutation $\pi_k = (J_{k_1}, \ldots, J_{k_n}) \in \Pi$ for which $\mathcal{OR}(\pi_k, T) \neq \emptyset$;
segments $[\widehat{p}_i^L, \widehat{p}_i^U]$ of durations of jobs $J_i \in \mathcal{J}$.

OUTPUT: Volume of optimality region $\mathcal{OR}(\pi_k, T)$ for permutation π_k.

Step 1: Determine set of sections
$S(\pi_k) = \left\{ s_1^{\pi_k}, s_{1+m_1}^{\pi_k}, \ldots, s_{j+m_j}^{\pi_k}, \ldots, s_w^{\pi_k} \right\}$;

Step 2: $j = 1,\ Vol = 1,\ Vol^* = 1,\ Sum = 0$;

Step 3: **IF** section $s_j^{\pi_k} = (J_{k_j}, \ldots, J_{k_{j+m_j}})$ is trivial $s_j^{\pi_k} = (J_{k_j})$ **THEN**
GOTO Step 7;

Step 4: **ELSE** for section $s_j^{\pi_k} = (J_{k_j}, \ldots, J_{k_{j+m_j}})$ construct partition (12)
of cover $\left[\widehat{p}_{k_j}^L, \widehat{p}_{k_{j+m_j}}^U\right]$ into conditional optimality subintervals:

$$\left[l_1^j\left(s_j^{\pi_k}\right), u_1^j\left(s_j^{\pi_k}\right)\right) \bigcup \ldots \bigcup \left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right) \bigcup \ldots \bigcup \left[l_{n(j)}^j\left(s_j^{\pi_k}\right), u_{n(j)}^j\left(s_j^{\pi_k}\right)\right];$$

Step 5: **FOR** $i = 1$ **to** $n(j)$ **DO** calculate $OS = \frac{\left(u_i^j(s_j^{\pi_k}) - l_i^j(s_j^{\pi_k})\right)^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!}$;
$Sum := Sum + OS$ **END FOR**

Step 6: $Vol := Vol \times Sum,\ j := j + m_j$ **IF** $j \leqslant w$ **THEN GOTO** Step 3;
ELSE GOTO Step 10;

Step 7: $j := j + m_j,\ OS^* = u_{k_j}^{opt} - l_{k_j}^{opt}$ **IF** $u_{k_j}^{opt} > l_{k_j}^{opt}$ **THEN GOTO** Step 9;
ELSE IF $j \leqslant w$ **THEN GOTO** Step 3;

Step 8: **ELSE GOTO** Step 10;

Step 9: $Vol^* := Vol^* \times OS^*$ **IF** $j \leqslant w$ **THEN GOTO** Step 3 **ELSE**

Step 10: $Vol(\pi_k, T) = Vol \times Vol^*$ **STOP**.

Step 1 of this algorithm is implemented using $O(n)$ operations. Next, Steps 3–6 are implemented using $O(n^2)$ operations. Finally, Steps 7–9 are implemented using $O(n)$ operations. Hence, Algorithm 2 requires $O(n^2)$ operations for calculating the volume $Vol(\pi_k, T)$ of the optimality region $\mathcal{OR}(\pi_k, T)$ for a fixed permutation $\pi_k \in \Pi$.

The permutation $\pi_3 = (J_1, \ldots, J_3, J_6, J_5, J_4, J_7, \ldots, J_{18})$ of jobs from the set $\mathcal{J} = \{J_1, \ldots, J_{18}\}$ for Example 3 of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ is shown in Fig. 5. Due to Theorem 3, the optimality region for the permutation π_3 is non-empty; therefore, we calculate its volume by formula (16) of Theorem 4, taking into account the equality $S^*(\pi_3) = \emptyset$:

$$Vol(\pi_3, T) = \prod_{s_j^{\pi_3} \in S(\pi_3)} \sum_{i=1}^{n(j)} \frac{(u_i^j(s_j^{\pi_3}) - l_i^j(s_j^{\pi_3}))^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!}$$

$$= \left[\frac{(3-1)^1}{1!} + \frac{(4-3)^3}{3!} + \frac{(5-4)^5}{5!} + \frac{(6-5)^3}{3!} + \frac{(7-6)^1}{1!} + \frac{(9-7)^2}{2!}\right]$$

$$\times \left[\frac{(12-11)^1}{1!} + \frac{(14-12)^3}{3!} + \frac{(15-14)^5}{5!} + \frac{(18-15)^3}{3!} + \frac{(23-18)^1}{1!}\right]$$

$$\times \left[\frac{(30-27)^1}{1!} + \frac{(32-30)^3}{3!} + \frac{(36-32)^1}{1!} + \frac{(37-36)^2}{2!} + \frac{(38-37)^3}{3!}\right.$$

$$\left. + \frac{(40-38)^5}{5!} + \frac{(41-40)^1}{1!}\right] = \left[\frac{2}{1} + \frac{1}{6} + \frac{1}{120} + \frac{1}{6} + \frac{1}{1} + \frac{4}{2}\right]$$

$$\times \left[\frac{1}{1} + \frac{8}{6} + \frac{1}{120} + \frac{27}{6} + \frac{5}{1}\right] \times \left[\frac{3}{1} + \frac{8}{6} + \frac{4}{1} + \frac{1}{2} + \frac{1}{6} + \frac{32}{120} + \frac{1}{1}\right] = 657.85.$$

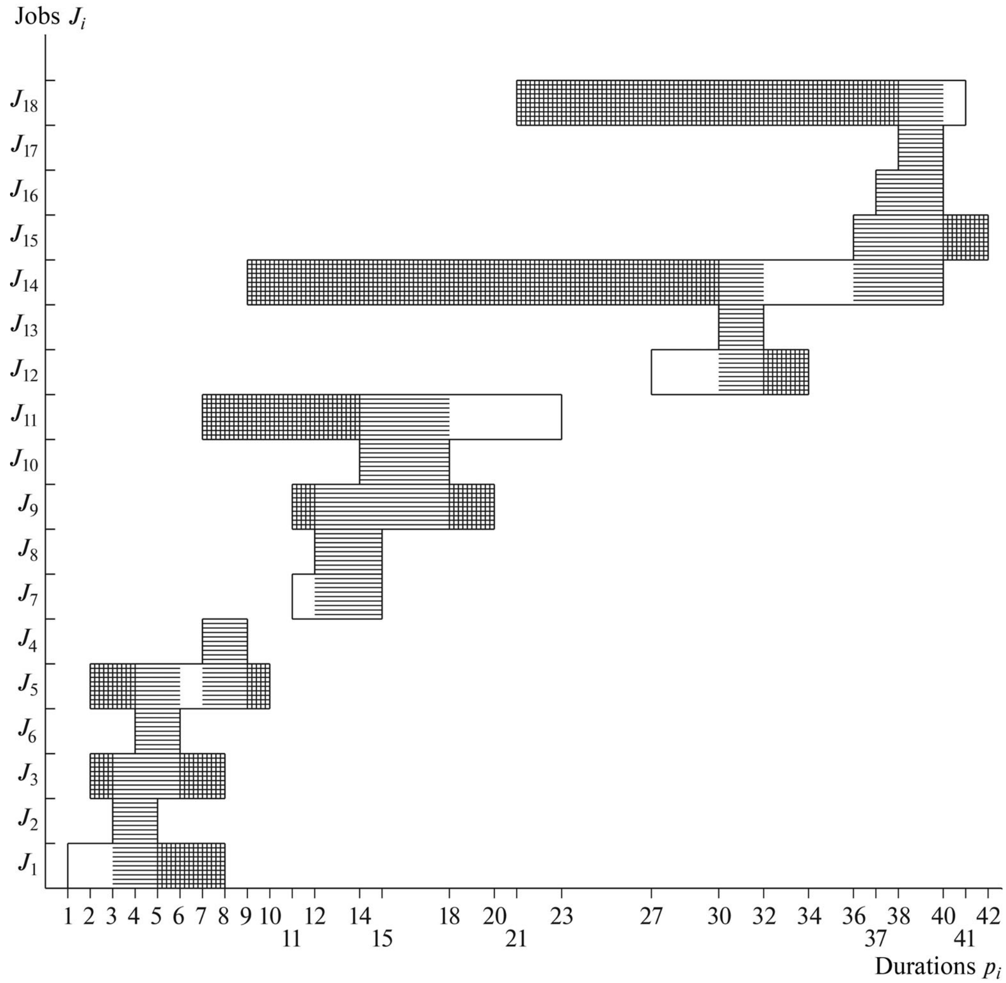

Fig. 5. Optimality segments, conditional optimality segments (horizontal hatching) and nonoptimality segments (horizontal and vertical hatching) for jobs $J_i \in \mathcal{J} = \{J_1, \dots, J_{18}\}$ in permutation $\pi_3 = (J_1, \dots, J_3, J_6, J_5, J_4, J_7, \dots, J_{18})$: Instance 1 of problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$.

5. PERMUTATION π_k WITH MAXIMAL OPTIMALITY REGION

If there exists a dominant singleton $\{\pi_k\}$ for the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$, then the permutation π_k of jobs from the set $\mathcal{J}$ is optimal for the problem $1|p|\sum C_i$ for any scenario $p \in T$. In accordance with Definition 2, such a permutation π_k satisfies the equality $\mathcal{OR}(\pi_k, T) = T$.

5.1. Maximum Possible Optimality Region for Given Scenarios

We will prove necessary and sufficient conditions for the existence of a permutation $\pi_k \in \Pi$ with the maximum possible optimality region for a given set of scenarios T, i.e., a criterion under which there exists a permutation π_k such that $\mathcal{OR}(\pi_k, T) = T$.

Theorem 5. *The optimality region for a permutation $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in \Pi$ is maximum possible for a given set of scenarios T, i.e., $\mathcal{OR}(\pi_k, T) = T$, if and only if the equality*

$$\left[l_{k_s}^{opt}, u_{k_s}^{opt}\right] = \left[p_{k_s}^L, p_{k_s}^U\right] \tag{17}$$

holds for each job $J_{k_r} \in \mathcal{J}$ in the permutation π_k.

Proof. *Sufficiency.* Let equality (17) be satisfied for each job $J_{k_s} \in \mathcal{J}$ in a permutation $\pi_k = (J_{k_1}, \dots, J_{k_n})$. Then, in accordance with Definition 1, we also have the equalities $\mathcal{OB}(\pi_k, T) =$

$\times_{k_{i_r}\in M}\left[l^{opt}_{k_{i_r}}, u^{opt}_{k_{i_r}}\right] = \times_{k_{i_r}\in M}\left[p^L_{k_{i_r}}, p^U_{k_{i_r}}\right] = T$, where $M = (k_{i_1}, \dots, k_{i_{|M|}})$, $k_{i_1} < \dots < k_{i_{|M|}}$, is an ordered set $\{k_1, \dots, k_n\} = \{1, \dots, n\}$ for which $n = |M|$. From Definition 1 it follows that the permutation π_k is optimal for the instance $1|p'|\sum C_i$ for any scenario $p' \in \mathcal{OB}(\pi_k, T) = T$. Then by Definition 2 we obtain the equality $\mathcal{OR}(\pi_k, T) = T$. Sufficiency is established.

Necessity. Let the equality $\mathcal{OR}(\pi_k, T) = T$ hold. Assume on the contrary that there exists a job $J_{k_r} \in \mathcal{J}$ in the permutation $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in \Pi$ such that equality (17) is not satisfied.

Due to Lemma 1, then there exists a non-empty nonoptimality segment $[l^{non}_{k_r}, u^{non}_{k_r}]$ or/and there exists a non-empty conditional optimality segment $[l^{copt}_{k_r}, u^{copt}_{k_r}]$ for the job $J_{k_r} \in \mathcal{J}$ in the permutation $\pi_k = (J_{k_1}, \dots, J_{k_n})$.

In the former case (the existence of $[l^{non}_{k_r}, u^{non}_{k_r}]$), equality (5) holds. In the latter case (the existence of $[l^{copt}_{k_r}, u^{copt}_{k_r}]$), equality (7) holds. In both cases, there exists a scenario $p^* \in (l^{non}_{k_r}, u^{non}_{k_r}) \bigcup (l^{copt}_{k_r}, u^{copt}_{k_r}) \subseteq T$ such that the permutation π_k is not optimal for the instance $1|p^*|\sum C_i$ with the scenario $p^* \in T$. Hence, in accordance with Definition 2, we arrive in the relation $\mathcal{OR}(\pi_k, T) \neq T$, which obviously contradicts the equality $\mathcal{OR}(\pi_k, T) = T$. Thus, necessity is established, and the proof of Theorem 5 is complete.

Theorem 5 in combination with Corollary 1 gives the following result.

Corollary 3. *If for each job $J_{k_r}\in\mathcal{J}$ in a permutation $\pi_k = (J_{k_1}, \dots, J_{k_n})$ equality* (17) *holds, then the optimality region $\mathcal{OR}(\pi_k, T)$ is the n-dimensional box $T \subset R^n_+$ with the volume $Vol(\pi_k, T) = \prod_{J_i\in\{\mathcal{J}\,:\,p^L_i<p^U_i\}}(p^U_i - p^L_i)$.*

A permutation $\pi_k = (J_{k_1}, \dots, J_{k_n})$ for which equalities (17) hold for all jobs $J_{k_r}\in\mathcal{J}$ is optimal for the instance $1|p|\sum C_i$ with any admissible scenario $p \in T$. Consequently, the set $\{\pi_k\}$ is the minimal dominant set for the problem $1|p^L_i \leqslant p_i \leqslant p^U_i|\sum C_i$.

5.2. How Should Permutation with Maximal Optimality Region Be Used?

The optimal permutation π_k for the problem $1|p^L_i \leqslant p_i \leqslant p^U_i|\sum C_i$ (see the existence criterion in Theorem 5 and Corollary 3) is quite rare in practice. However, for a specific problem $1|p^L_i \leqslant p_i \leqslant p^U_i|\sum C_i$ arising in applications, as a rule, a single permutation has to be chosen from the set Π and then implemented for processing all jobs from the set $\mathcal{J}$.

In view of the results established in Sections 3–5.1, for implementation it can be recommended to choose a permutation π_t of jobs from the set $\mathcal{J}$ for which the volume of the optimality region $\mathcal{OR}(\pi_t, T)$ achieves maximum over all permutations from the set Π. If the factual scenario $p \in T$ of processing of all jobs from the set $\mathcal{J}$ belongs to the optimality region $\mathcal{OR}(\pi_t, T)$, then the implemented permutation π_t will be optimal for the factual scenario of processing of all jobs from the set $\mathcal{J}$. Generally speaking, the greater the volume of the optimality region $\mathcal{OR}(\pi_k, T)$ is, the higher the probability that the permutation π_k is optimal for the factual scenario of processing of all jobs from the set $\mathcal{J}$ will be. Therefore, important problems for further research are the development of efficient algorithms to construct a permutation π_t with the maximal volume $Vol^{\max}(\pi_t, T) = \max\{Vol^{\max}(\pi_k, T) \,:\, \pi_k \in \Pi\}$ of the optimality region $\mathcal{OR}(\pi_t, T)$ and the testing of such algorithms on the problems $1|p^L_i \leqslant p_i \leqslant p^U_i|\sum C_i$ of practically relevant dimensions.

For the general problem $1|p^L_i \leqslant p_i \leqslant p^U_i|\sum C_i$, the difference

$$1 - \frac{Vol^{\max}(\pi_k, T)}{\prod_{J_i\in\{\mathcal{J}:\,l^{opt}_i<u^{opt}_i\}}(p^U_i - p^L_i)} =: \mu$$

can be treated as its measure of uncertainty (or measure of complexity). In particular, if $\mu = 0$, then the permutation π_t with the maximal volume $Vol^{\max}(\pi_t, T)$ of the optimality region $\mathcal{OR}(\pi_t, T)$ will surely be optimal for the factual scenario of processing of all jobs from the set $\mathcal{J}$, even despite the uncertainty of the given scenarios T. Conversely, if the value μ is equal to 1 (or very close to 1), then the probability that the permutation π_t is optimal for the factual scenario of processing of all jobs from the set $\mathcal{J}$ will be 0 (almost 0, respectively). In such cases, the uncertain problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ can be solved using approximate algorithms, like Algorithm U2 described in [1], or Algorithm 3 introduced in [10]. The latter is oriented towards achieving the minimum error of the resulting solution.

6. CONCLUSIONS

Single-machine scheduling problems for jobs with uncertain numerical parameters arise, e.g., in the course of employee's working time planning for a definite period (day, week, or month). As a rule, the ranges of admissible durations of jobs can be estimated in advance. It can be assumed that the set of planned jobs will not considerably change when implementing a schedule. The minimum total completion time of jobs (the average processing time of jobs) can be treated as an aggregate index of efficiency for an employee performing a given set of jobs.

Another example of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$, which arises in the single-truck optimal scheduling of product supply service logistics in a city retail network, was described in [27]. The product supply time to a retail outlet depends on numerous factors, such as traffic jamming, weather, the condition of supply trucks and the road.

The problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ may also arise in some multistage serving systems if one server is a bottleneck of a corresponding process and only the ranges of admissible durations of jobs on this server are known.

The results obtained in Section 3–5 as well as Algorithms 1 and 2 can be used to construct a permutation $\pi_t \in \Pi$ of given jobs with the maximal volume $Vol^{\max}(\pi_t, T)$ of the optimality region $\mathcal{OR}(\pi_t, T)$. Choosing the permutation π_t to process given jobs, we increase the probability of obtaining an almost optimal schedule, despite the fact that the probability distributions of the uncertain durations of jobs are unavailable in the scheduling problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$.

APPENDIX

For proving Lemma 4, we will consider the problem

$$1\left|\widehat{p}_i^L \leqslant p_i \leqslant \widehat{p}_i^U\right|\sum C_i$$

instead of the problem $1|p_i^L \leqslant p_i \leqslant p_i^U|\sum C_i$ (Theorem 2). Since $\mathcal{OR}(\pi_k, T) \neq \emptyset$, by Theorem 3 there exists no job $J_{k_r} \in \mathcal{J}$ without conditional optimality and simultaneously without an optimality segment in the permutation π_k. In accordance with the hypothesis of Lemma 4, each job $J_{k_r} \in \mathcal{J}$ has no optimality segment in the permutation π_k.

In view of Remark 1 and Lemma 1, we obtain the equality

$$\left[\widehat{p}_{k_r}^L,\ \widehat{p}_{k_r}^U\right] = \left[l_{k_r}^{copt}, u_{k_r}^{copt}\right],$$

which holds for each job $J_{k_r} \in \mathcal{J}$. From this equality it follows that the set $S(\pi_k)$ of all sections of the permutation π_k contains no trivial sections. We construct the partition (12) of covers $\left[\widehat{p}_{k_j}^L,\ \widehat{p}_{k_{j+m_j}}^U\right]$ of all sections $s_j^{\pi_k} \in S(\pi_k)$ into the following conditional optimality subintervals:

$$\left[l_1^j\left(s_j^{\pi_k}\right), u_1^j\left(s_j^{\pi_k}\right)\right) \bigcup \ldots \bigcup \left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right) \bigcup \ldots \bigcup \left[l_{n(j)}^j\left(s_j^{\pi_k}\right), u_{n(j)}^j\left(s_j^{\pi_k}\right)\right] = \left[\widehat{p}_{k_j}^L, \widehat{p}_{k_{j+m_j}}^U\right].$$

Using mathematical induction in the cardinality $|\mathcal{J}_i^j|$ of the set $\mathcal{J}_i^j$, we will demonstrate that, for each conditional optimality subinterval $\left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right)$ in partition (12),

$$Vol\left(\widehat{J}_i^j, T\right) = \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^{|\mathcal{J}_i^j|}}{|\mathcal{J}_i^j|!}, \tag{A.1}$$

$$\mathcal{OR}\left(\widehat{J}_i^j, T\right) = Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right), \tag{A.2}$$

where the base of the $|\mathcal{J}_i^j|$-dimensional pyramid $Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right)$ is the $(|\mathcal{J}_i^j| - 1)$-dimensional pyramid and the altitude of all pyramids is equal to $\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)$.

First, we show that for $|\mathcal{J}_i^j| = 2$, the pyramid $Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}(J_{k_i}, J_{k_{i+1}})$ turns into a triangle (a degenerate case of a pyramid) with the base $\left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right]$ and the same altitude $\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)$ as its base. This triangle $Pyr^{opt}\widehat{J}_1^1 = Pyr^{opt}(J_1, J_2)$ is presented in Fig. 4a for the conditional optimality subinterval

$$[4, 8) = \left[l_1^1(J_1, J_2),\ u_1^1(J_1, J_2)\right)$$

for the permutation $\pi_2 = (J_1, \ldots, J_{10}) \in \Pi$ in Example 3. From Theorem 1 it follows that the order $(J_{k_i}, J_{k_{i+1}}) = \widehat{J}_i^j$ of processing of two jobs is optimal in a permutation $\pi_k \in \Pi$ if and only if the durations p_{k_i} and $p_{k_{i+1}}$ of the jobs J_{k_i} and $J_{k_{i+1}}$ satisfy the inequality $p_{k_i} \leqslant p_{k_{i+1}}$. In view of the belonging of the admissible scenario $(p_{k_1}, \ldots, p_{k_n})$ to the given set T, the admissible durations p_{k_i} and $p_{k_{i+1}}$ must satisfy the final system of inequalities

$$\begin{cases} p_{k_i} \leqslant p_{k_{i+1}} \\ p_{k_i}^L \leqslant p_{k_i} \leqslant p_{k_i}^U \\ p_{k_{i+1}}^L \leqslant p_{k_{i+1}} \leqslant p_{k_{i+1}}^U. \end{cases} \tag{A.3}$$

System (A.3) determines the triangle $Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}(J_{k_i}, J_{k_{i+1}})$. In other words, all points belonging to the triangle $Pyr^{opt}\widehat{J}_i^j$ satisfy system (A.3), and no other points do so. Thus, equality (A.2) is proved for the case $|\mathcal{J}_i^j| = 2$. Since the area of the triangle $Pyr^{opt}\widehat{J}_i^j$ is the product of its base and altitude with a factor of 1/2, $\frac{(u_i^j(s_j^{\pi_k}) - l_i^j(s_j^{\pi_k}))^2}{2}$, equality (A.1) is also established for the case $|\mathcal{J}_i^j| = 2$.

Consider the next set $\mathcal{J}_i^j$ by cardinality, i.e., $|\mathcal{J}_i^j| = 3$. We will demonstrate that, in this case, the optimality region

$$\mathcal{OR}(\widehat{J}_i^j, T) = \mathcal{OR}\Big((J_{k_i}, J_{k_{i+1}}, J_{k_{i+2}}), T\Big)$$

is the 3-dimensional pyramid $Pyr^{opt}\widehat{J}_i^j$ with the altitude $\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)$ and the base in the form of a triangle with the same altitude $\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)$ and the base $\left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right]$. Note that this optimality pyramid $Pyr^{opt}\widehat{J}_1^1 = Pyr^{opt}(J_2, J_3, J_4)$ is shown in Fig. 4b for the conditional optimality subinterval

$$[9, 13) = \left[l_3^1(J_2, J_3, J_4),\ u_3^1(J_2, J_3, J_4)\right)$$

for the permutation $\pi_2 = (J_1, \ldots, J_{10})$ in Example 3. From Theorem 1 it follows that the order $(J_{k_i}, J_{k_{i+1}}, J_{k_{i+2}}) = \widehat{J}_i^j$ of processing of three jobs is optimal in a permutation $\pi_k \in \Pi$ if and only if the durations p_{k_i}, $p_{k_{i+1}}$, and $p_{k_{i+2}}$ of the jobs J_{k_i}, $J_{k_{i+1}}$, and $J_{k_{i+2}}$, respectively, satisfy the inequalities $p_{k_i} \leqslant p_{k_{i+1}} \leqslant p_{k_{i+2}}$. In view of the belonging of the admissible scenario $(p_{k_1}, \ldots, p_{k_n})$ to the given set T, the admissible durations p_{k_i}, $p_{k_{i+1}}$, and $p_{k_{i+2}}$ must satisfy the final system of inequalities

$$\begin{cases} p_{k_i} \leqslant p_{k_{i+1}} \leqslant p_{k_{i+2}} \\ p^L_{k_i} \leqslant p_{k_i} \leqslant p^U_{k_i} \\ p^L_{k_{i+1}} \leqslant p_{k_{i+1}} \leqslant p^U_{k_{i+1}} \\ p^L_{k_{i+2}} \leqslant p_{k_{i+2}} \leqslant p^U_{k_{i+2}}. \end{cases} \tag{A.4}$$

System (A.4) determines the 3-dimensional pyramid

$$Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}(J_{k_i},\ J_{k_{i+1}},\ J_{k_{i+2}})$$

with the triangle

$$\left[\left(p^L_{k_i},\ p^L_{k_{i+1}},\ p^U_{k_{i+2}}\right),\ \left(p^L_{k_i},\ p^U_{k_{i+1}},\ p^U_{k_{i+2}}\right),\ \left(p^U_{k_i},\ p^U_{k_{i+1}},\ p^U_{k_{i+2}}\right)\right]$$

as its base and the segment

$$\left[\left(p^L_{k_i},\ p^L_{k_{i+1}},\ p^U_{k_{i+2}}\right),\ \left(p^L_{k_i},\ p^L_{k_{i+1}},\ p^L_{k_{i+2}}\right)\right]$$

as its altitude. Consequently, all points of the pyramid $Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}(J_{k_i}, J_{k_{i+1}}, J_{k_{i+2}})$ satisfy system (A.4), and no other points do so. Thus, equality (A.2) is proved for the case $|\mathcal{J}_i^j| = 3$. The volume of the 3-dimensional pyramid $Pyr^{opt}\widehat{J}_i^j$ is the product of the area of its base (triangle)

$$\left[\left(p^L_{k_i}, p^L_{k_{i+1}}, p^U_{k_{i+2}}\right), \left(p^L_{k_i}, p^U_{k_{i+1}}, p^U_{k_{i+2}}\right), \left(p^U_{k_i}, p^U_{k_{i+1}}, p^U_{k_{i+2}}\right)\right],$$

and its altitude with a factor of 1/3,

$$\frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^2}{2} \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)}{3} = \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^3}{3!} = Vol\left(\widehat{J}_i^j, T\right).$$

Hence, equality (A.1) is also established for the case $|\mathcal{J}_i^j| = 3$.

Now, we make an inductive hypothesis, assuming that both equalities (A.1) and (A.2) hold in the case $|\mathcal{J}_i^j| = d$. This means that the optimality region $\mathcal{OR}(\widehat{J}_i^j, T)$ is the d-dimensional pyramid $Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}(J_{k_i}, \ldots, J_{k_t})$, with a $(d-1)$-dimensional pyramid as its base and $(u_i^j(s_j^{\pi_k}) - l_i^j(s_j^{\pi_k}))$ as the altitude of each of these pyramids. Proceeding from the inductive hypothesis, we will obtain equalities (A.1) and (A.2) for the case $|\mathcal{J}_i^j| = d+1$.

Consider the conditional optimality subinterval $\left[l_i^j\left(s_j^{\pi_k}\right), u_i^j\left(s_j^{\pi_k}\right)\right)$ for which

$$\widehat{J}_i^j = \left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}, J_{k_{|\mathcal{J}_i^j|}}\right) \quad \text{and} \quad |\mathcal{J}_i^j| = d+1.$$

Due to the inductive hypothesis, both equalities (A.1) and (A.2) hold for the set of jobs $\mathcal{J}_i^j \setminus \left\{J_{k_{|\mathcal{J}_i^j|}}\right\}$, and the optimality region $\mathcal{OR}\left(\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}\right), T\right)$ is the d-dimensional pyramid

$$Pyr^{opt}\left(J_{k_i}, \ldots, J_{k_t}\right) = \mathcal{OR}\left(\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}\right), T\right).$$

Hence, by Theorem 1 the order $\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}\right)$ of job processing is optimal in a permutation $\pi_k \in \Pi$ if and only if the durations $p_{k_i}, \ldots, p_{k_{|\mathcal{J}_i^j|-1}}$ of the jobs $J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}$ satisfy the system of inequalities

$$\begin{cases} p_{k_i} \leqslant \ldots \leqslant p_{k_{|\mathcal{J}_i^j|-1}} \\ p_{k_i}^L \leqslant p_{k_i} \leqslant p_{k_i}^U \\ \ldots \\ p_{k_{|\mathcal{J}_i^j|-1}}^L \leqslant p_{k_{|\mathcal{J}_i^j|-1}} \leqslant p_{k_{|\mathcal{J}_i^j|-1}}^U. \end{cases} \tag{A.5}$$

Adding the inequalities

$$p_{k_{|\mathcal{J}_i^j|-1}} \leqslant p_{k_{|\mathcal{J}_i^j|}} \quad \text{and} \quad p_{k_{|\mathcal{J}_i^j|}}^L \leqslant p_{k_{|\mathcal{J}_i^j|}} \leqslant p_{k_{|\mathcal{J}_i^j|}}^U$$

into system (A.5), we obtain the final system of inequalities

$$\begin{cases} p_{k_i} \leqslant \ldots \leqslant p_{k_{|\mathcal{J}_i^j|-1}} \\ p_{k_{|\mathcal{J}_i^j|-1}} \leqslant p_{k_{|\mathcal{J}_i^j|}} \\ p_{k_i}^L \leqslant p_{k_i} \leqslant p_{k_i}^U \\ \ldots \\ p_{k_{|\mathcal{J}_i^j|-1}}^L \leqslant p_{k_{|\mathcal{J}_i^j|-1}} \leqslant p_{k_{|\mathcal{J}_i^j|-1}}^U \\ p_{k_{|\mathcal{J}_i^j|}}^L \leqslant p_{k_{|\mathcal{J}_i^j|}} \leqslant p_{k_{|\mathcal{J}_i^j|}}^U. \end{cases} \tag{A.6}$$

This system determines the $(d+1)$-dimensional pyramid

$$Pyr^{opt}\widehat{J}_i^j = Pyr^{opt}\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right)$$

with the d-dimensional pyramid $Pyr^{opt}\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}\right)$ as its base and the difference $\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)$ as the altitude of each of the two pyramids.

Due to the inductive hypothesis and Theorem 1, the optimality region $\mathcal{OR}\left(\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right), T\right)$ is the $(d+1)$-dimensional pyramid $Pyr^{opt}\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right) = \mathcal{OR}\left(\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|}}\right), T\right)$. Thus, equality (A.2) is proved for the case $|\mathcal{J}_i^j| = d+1$. The volume of the $(d+1)$-dimensional pyramid $Pyr^{opt}\widehat{J}_i^j$ is the product of the area of its base (d-dimensional pyramid) $Pyr^{opt}\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}\right) = \mathcal{OR}\left(\left(J_{k_i}, \ldots, J_{k_{|\mathcal{J}_i^j|-1}}\right), T\right)$ and its altitude with a factor of $1/(d+1)$,

$$\frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)^d}{d!} \frac{\left(u_i^j\left(s_j^{\pi_k}\right) - l_i^j\left(s_j^{\pi_k}\right)\right)}{d+1} = \frac{(u_i^j(s_j^{\pi_k}) - l_i^j(s_j^{\pi_k}))^{d+1}}{(d+1)!} = Vol(\widehat{J}_i^j, T).$$

Hence, equality (A.1) is also established for the case $|\mathcal{J}_i^j| = d+1$. Thus, equalities (A.1) and (A.2) have been proved by mathematical induction.

Equality (14) follows from equality (A.2) and the fact that, for any scenario $p \in T$, each job $J_i \in \mathcal{J}$ has a unique duration p_i. Next, equality (13) follows from Remark 2 and equality (14). Finally, equality (15) follows from equalities (13), (14), and (A.1). The proof of Lemma 4 is complete.

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