ISSN 0005-1179, Automation and Remote Control, 2020, Vol. 81, No. 3, pp. 430–444. © Pleiades Publishing, Ltd., 2020.
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TOPICAL ISSUE

Analysis and Optimization of a Controlled Model for a Closed Queueing Network

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Received June 20, 2019
Revised August 19, 2019
Accepted September 26, 2019

Abstract—We consider a closed network consisting of two queuing systems: the main system simulates a packet transmission queue over an unreliable communication channel, and the auxiliary multiserver system contains lost packets for resending. The service rate in the main system is controllable and is supposed to be optimized with the aim of minimizing the time of successful transmission, taking into account the cost of using network resources. We obtain optimality conditions in two cases: 1) in the model based on fluid approximation in the presence of heavy load; 2) in the steady state using stationary strategies.

Keywords: closed queuing network, fluid approximation, Markov process, optimal control

DOI: 10.1134/S0005117920030042

1. INTRODUCTION

This work continues the general theme of the studies [1–8] devoted to the optimization of controllable infocommunication systems and networks, and is largely based on the work [9] published by R.Sh. Liptser and coauthors in connection with the problem of fluid and diffusion approximation of a closed Markov queuing network in heavy traffic conditions.

The first studies of closed queuing networks were conducted in [10]. Mathematical foundations of the theory of queuing networks are presented in [11]. The basic results in the field of queuing systems (QS) control are shown in [12, 13], while the works [14, 15] are devoted to diffusion and fluid approximations of QS and networks.

Optimization methods for Markov processes describing QS operation in stationary mode are described in [16]. The methodology of constrained optimization for discrete Markov models has been developed in [17]. In a nonstationary formulation, the problem of controlling a Markov process under constraints was studied in [18].

The work [19] proved the threshold structure of the optimal decentralized strategy for controlling service rate at each node of a closed network by the criterion of minimum total cost for holding and operating a job. In [20], the same result was extended to the more general case of an affine loss function.

In this work, we study the Markov model of a closed network [9] that models the process of transmitting data over an unreliable communication channel. The network in question consists of two QS: the main system models the queue for packet transmission, and the auxiliary multiserver system receives lost packets for repeated sending. Just like in [6], the transmission rate is controllable, but now the model has a mechanism for resending lost packets. QS models with retrials are relevant when modeling the work of contact centers, where customer calls can be repeated due to busy operators or due to the end of waiting time [21]. In the model under consideration, we distinguish two performance indices for optimizing the system operation: the time of successfully sending out a packet and the amount of resources used. To develop an approximate control strategy, we apply a fluid approximation, which allows one to study the controlled system on average in the presence of heavy traffic.

2. MODEL DESCRIPTION AND PROBLEM SETTING

Consider a queuing network consisting of two systems: main and auxiliary. The main system contains a queue and describes the sequential transmission of unified packets over an unreliable communication channel in the presence of constant input traffic. The auxiliary system is multiserver and is used to simulate the re-sending of packets in the event of their loss.

We assume that a constant number of packets N circulate in this network, which reflects a stable load level of the data transmission network.

Let $\ell(t)$ be the probability of packet loss, where $0 \leqslant \ell(t) < 1$, let $\mu(t,x)$ be the transmission rate chosen from the range

$$m_{\min} \leqslant \mu(t,x) \leqslant m_{\max}, \quad \text{where} \quad 0 < m_{\min} < m_{\max} < \infty, \tag{1}$$

and let $\alpha(t)(N-x)$ be the rate of sending out a packet from the auxiliary system to the main one for retrying transmission, where x indicates the number of packets in the main system at time t. Thus, $\mu(t,x)$ and $\alpha(t)$ are service rates in the main and auxiliary systems respectively. The function $\alpha(t) > 0$ defines the resources that the network can allocate to process requests for resending packets taking into account the load of computational resources which is changing over time. The variable nature of this load and the nonstationarity of the flow of packet losses can be used to describe the process of transmitting data from an unmanned aerial vehicle, which should provide stable control of its movement and transfer useful information during autonomous mission in a changing environment.

Figure 1 shows a diagram of the considered data transmission network. The loop corresponds to a successful transmission that occurs with probability $1-\ell(t)$. If a packet has been successfully transmitted, it leaves the main system and at the same moment a new packet subject to transmission enters it. Thus, in case of successful transmission the number of jobs in the main system does not change.

Transmission loss occurs with intensity $\ell(t)\mu(t,x)$ if $x > 0$. In this case, the number of jobs in the main system decreases, and the number of jobs in the auxiliary system increases by one. The external packet is blocked in this case since in order to enter the system it must wait for the first successful transmission.

In the zero state of the main system $x = 0$, incoming traffic is also blocked. But this happens with the purpose that a new packet has the opportunity to get into the auxiliary system in the

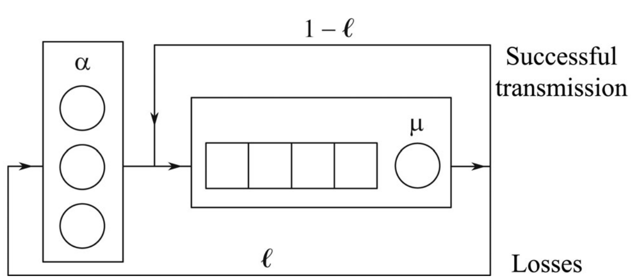

Fig. 1. Network model with two systems: on the left—auxiliary system, on the right—main system.

event of an unsuccessful transmission without changing the total number of packets circulating in the network.

The return of a packet from the auxiliary system to the main system occurs with intensity $\alpha(t)$ and in a way independent of the network state. In this case, the number of packets in the main system increases by one, and in the auxiliary system, decreases by one.

It is important to note that we mention incoming traffic here only to interpret a closed network as a model of packet data transmission: the input stream does not directly affect the state of both systems. Therefore, the number of packets N should not be interpreted as the size of the space available for storing packets but rather as a fixed load level.

Over the time interval $[0, T]$, we will describe the operation of the data transmission network with the help of an inhomogeneous Markov birth and death process $X(t)$ with values in the set $E = \{0, 1, \dots, N\}$. Then the number of packets in the main and auxiliary systems is $X(t)$ and $N - X(t)$ respectively. The process $X(t)$ has generator $\Lambda(t) = \{\lambda_{x,y}(t)\}_{x,y\in E}$ uniquely determined by the two transition rates $x \to x \pm 1$: $\lambda_{x,x+1}(t) = \alpha(t)(N - x)$ is the intensity of packets coming from the auxiliary system for a resending when the main system has $x < N$ jobs; $\lambda_{x,x-1}(t) = \ell(t)\mu(t, x)$ is the intensity of packets being forwarded to the auxiliary system in case of an unsuccessful attempt to transmit it when the main system contains $x > 0$ jobs.

In what follows, we assume that functions $\ell(t)$ and $\alpha(t)$ are continuous.

The network operation quality will be described using two characteristics

$$\mathfrak{S}_T = \frac{1}{T}\int_0^T \mathsf{M}S(t)\,dt \quad \text{and} \quad \mathfrak{R}_T = \frac{1}{T}\int_0^T \mathsf{M}R(t)\,dt,$$

where $S(t)$ is the time of fully transmitting a packet arriving at time moment t, and $R(t)$ is the instantaneous power spent by the network for data transmission and resending process.

Thus, $\mathfrak{S}_T$ determines the average time of successful transmission, and $\mathfrak{R}_T$ represents the average power consumption of resources. Both criteria will be considered as functionals $\mathfrak{S}_T[\mu]$, $\mathfrak{R}_T[\mu]$ that depend on the controlled transmission rate μ from a particular class $\mathcal{M}$.

Next, we will consider the optimal control problem for the transmission rate μ with respect to the extended functional

$$\mathfrak{L}_T[\mu, \lambda] = \mathfrak{S}_T[\mu] + \lambda\mathfrak{R}_T[\mu] \to \min_{\mu}\colon \mu \in \mathcal{M} \tag{2}$$

for a given value of the factor $\lambda \geqslant 0$.

The purpose of this work is to analyze this optimization model on two levels of description: studying the dynamics "on average" using a fluid approximation and considering the network in stationary mode for the corresponding homogeneous Markov process.

3. QUALITY FUNCTIONALS

Let c be a non-negative parameter that determines the relative cost of loading one server of the auxiliary system based on the cost of loading the main system. Then the functional $\mathfrak{R}_T[\mu]$ takes the form

$$\mathfrak{R}_T[\mu] = \frac{1}{T}\int_0^T \mathsf{M}\{\mu(t, X(t))\,\mathrm{I}\{X(t) > 0\} + c\alpha(t)\,(N - X(t))\}\,dt. \tag{3}$$

To define the functional $\mathfrak{S}_T[\mu]$, consider the value $S(t)$ equal to the total transmission time of an external packet starting from the moment t. This value can be represented as

$$S(t) = W + B_1 + A_2 + B_2 + \ldots + A_\nu + B_\nu, \tag{4}$$

where W is the timeout to enter the system; A_k and B_k, the time spent by the packet in the auxiliary and main systems, respectively, on the kth attempt ($k = 1, 2, \dots$, with $A_1 = 0$); ν is the number of attempts after which the packet will be successfully transmitted.

We will consider $S(t)$ as the virtual sojourn time (including waiting for entry into the system), and therefore all probability characteristics will be calculated based on a fixed moment t.

The value W can be represented as

$$W = W_0 \,\mathrm{I}\{X(t) = 0\} + (\tau_1 + \ldots + \tau_\varkappa)\,\mathrm{I}\{X(t) > 0\},$$

where W_0 is the waiting time for when the auxiliary system is fully loaded; $\tau_1, \tau_2, \ldots$—service times for jobs located in the main system at the time t; $\varkappa$—number of the first successfully transmitted packet; $\mathrm{I}\{\ldots\}$—random event indicator.

If rates $\alpha(t)$ and $\mu(t,x)$ change slowly with time, the conditional distribution laws $\mathsf{Law}\{W_0 \mid X(t) = 0\}$ and $\mathsf{Law}\{\tau_k \mid X(t) = x\}$ with $x > 0$ can be treated as exponential $E(\alpha(t)N)$ and $E(\mu(t,x))$ respectively. In addition, $X(t), \varkappa, \tau_1, \tau_2, \ldots$ are mutually independent, and the value $\varkappa$ has the geometric distribution $G(1 - \ell(t))$. Then, using the formula of iterated expectations, we obtain the average waiting time for entering the system

$$\begin{aligned}\overline{W} &= \frac{\mathsf{P}\{X(t) = 0\}}{\alpha(t)N} + \sum_{x=1}^{N}\sum_{k=1}^{\infty} \frac{k}{\mu(t,x)}\mathsf{P}\{\tau = k\}\mathsf{P}\{X(t) = x\} \\ &= \frac{\mathsf{P}\{X(t) = 0\}}{\alpha(t)N} + \frac{1}{1 - \ell(t)}\,\mathsf{M}\left\{\frac{\mathrm{I}\{X(t) > 0\}}{\mu(t, X(t))}\right\}.\end{aligned}$$

The time spent in the main system during a single attempt has the form

$$B_k = \tau_1 + \ldots + \tau_x \quad \text{for} \quad X(t) = x > 0.$$

Moreover, $B_k = 0$ when $x = 0$, since the packet in question is already in the main system. In a similar way, we obtain the average sojourn time of a packet in the main system

$$\overline{B} = \mathsf{M}\left\{\frac{X(t)\,\mathrm{I}\{X(t) > 0\}}{\mu(t, X(t))}\right\}.$$

For the average time spent in the auxiliary system, we have

$$\overline{A} = \frac{\mathsf{P}\{X(t) < N\}}{\alpha(t)}$$

due to the fact that $\mathsf{Law}\{A_k \mid X(t) = x\} = E(\alpha(t))$ for $x < N$ and $A_k = 0$ for $x = N$.

We take into account that $\nu \sim G(1 - \ell(t))$ and the independence of random variables included in (4). Applying the formula of iterated expectations, we obtain the average time required for the successful transmission of an external packet arriving at time t:

$$\overline{S}(t) = \overline{W} + \overline{B} + (\overline{A} + \overline{B})\,\mathsf{M}\{\nu - 1\} = \overline{W} + \frac{\overline{A}\,\ell(t) + \overline{B}}{1 - \ell(t)}.$$

Now the functional $\mathfrak{S}_T[\mu]$ takes its final form:

$$\mathfrak{S}_T[\mu] = \frac{1}{T}\int_0^T \mathsf{M}\left\{\frac{\mathrm{I}\{X(t) = 0\}}{\alpha(t)N} + \frac{\mathrm{I}\{X(t) < N\}\ell(t)}{(1 - \ell(t))\alpha(t)} + \frac{(X(t) + 1)\,\mathrm{I}\{X(t) > 0\}}{(1 - \ell(t))\mu(t, X(t))}\right\}dt. \qquad (5)$$

4. FLUID APPROXIMATIONS

Let us describe the behavior of the network on average using the fluid approximation method. To do this, we represent the process $X(t)$ as

$$X(t) = X(0) + A(t) - D(t), \tag{6}$$

where $A(t)$ is the number of packets arriving in the main system from the auxiliary, and $D(t)$ is the number of packets sent in the opposite direction (over time t). Due to the independence of this pair of events, jumps in processes $A(t)$, $D(t)$ occur at different times, i.e.,

$$\Delta A(t)\Delta D(t) = 0, \tag{7}$$

and, moreover, $A(0) = D(0) = 0$.

Counting processes $A(t), D(t)$ allow martingale representations [22, Ch. 18]

$$dA(t) = \alpha(t)(N - X(t))\,dt + dM^A(t) \quad \text{and} \quad dD(t) = \ell(t)\mu(t, X(t))\,dt + dM^D(t),$$

where quadratically integrable martingales $M^A(t), M^D(t)$ are orthogonal by virtue of (7) and have the following quadratic characteristics:

$$<M^A>(t) = \int_0^t \alpha(s)(N - X(s))\,ds, \qquad <M^D>(t) = \int_0^t \ell(s)\mu(s, X(s))\,ds.$$

Due to (6), the process $X(t)$ admits the representation

$$dX(t) = \{\alpha(t)(N - X(t)) - \ell(t)\mu(t, X(t))\}\,dt + dM(t), \tag{8}$$

where $M(t)$ is a quadratic integrable martingale with quadratic characteristic

$$<M>(t) = \int_0^t \{\alpha(s)(N - X(s)) + \ell(s)\mu(s, X(s))\}\,ds.$$

Following [9], we define a fluid approximation of the discrete process $X(t)$.

The fluid approximation $x(t)$, being a deterministic process, describes the behavior of the main system on average with a large number N of packets operating in the network. Suppose that $\mu(t,x)$ is given as a function of continuous arguments $t \geqslant 0$ and $x \in [0, N]$. Then $x(t)$ satisfies the ordinary differential equation

$$\dot{x} = \alpha(t)(N - x) - \ell(t)\mu(t, x), \tag{9}$$

which is obtained from (8) by removing the martingale component.

According to Theorem 1 from [9], it suffices for the correctness of the fluid approximation (9) that $\mu(t,x)$ is Lipschitz and $x(t)$ satisfies the condition

$$\exists\, \varepsilon > 0\colon \quad \varepsilon \leqslant x(t) \leqslant N - \varepsilon \quad \forall\, t \in [0, T]. \tag{10}$$

The meaning of this condition is obvious: the average number of packets should be in the interval $(0, N)$ except for the idle mode of the main or auxiliary system.

Lemma. *If there exists $\varepsilon > 0$ such that*

$$\varepsilon/m_{\min} \leqslant \ell(t)/\alpha(t) \leqslant (N-\varepsilon)/m_{\max}, \tag{11}$$

then condition (10) *is valid for any choice of a Lipschitz function $\mu(t,x)$ satisfying the constraints* (1).

Proofs of the lemma and the statements below are given in the Appendix.

Given the approximation $X(t) \approx x(t)$, we obtain the representation of functionals (3) and (5):

$$\mathfrak{S}_T[\mu] = \frac{1}{T}\int_0^T \left\{ \frac{\ell(t)}{(1-\ell(t))\alpha(t)} + \frac{x(t)+1}{(1-\ell(t))\mu(t,x(t))} \right\} dt,$$

$$\mathfrak{R}_T[\mu] = \frac{1}{T}\int_0^T \{\mu(t,x(t)) + c\alpha(t)(N-x(t))\}\, dt.$$

Then the extended functional (2) takes the form

$$\mathfrak{L}_T[\mu,\lambda] = \frac{1}{T}\int_0^T g(t,x(t),\mu(t,x(t)),\lambda)\, dt,$$

$$g(t,x,\mu,\lambda) = \frac{\ell(t)}{(1-\ell(t))\alpha(t)} + \frac{x+1}{(1-\ell(t))\mu} + \lambda\big[\mu + c\alpha(t)(N-x)\big].$$

To synthesize the optimal data transmission rate $\hat{\mu}(t,x)$ by the minimum criterion for the extended functional $\mathfrak{L}_T[\mu,\lambda]$, we first consider the problem

$$\mathfrak{L}_T[u,\lambda] = \frac{1}{T}\int_0^T g(t,x(t),u(t),\lambda)\, dt \to \min_{u\in\mathcal{U}} \tag{12}$$

on the class $\mathcal{U}$ of open-loop controls, i.e., piecewise continuous functions $u(t)$, $t \in [0,T]$, with values in the set $U = [m_{\min}, m_{\max}]$. In this case, the state $x(t)$ satisfies the differential Eq. (9), where instead of the feedback control $\mu(t,x)$ an open-loop control is used, namely

$$\dot{x} = f(t,x,u(t)), \quad f(t,x,u) = \alpha(t)(N-x) - \ell(t)u. \tag{13}$$

We define the Hamiltonian

$$H(t,x,\psi,u,\lambda) = \psi f(t,x,u) - g(t,x,u,\lambda),$$

where ψ is the conjugate variable. Then it is easy to get the representation

$$H(t,x,\psi,u,\lambda) = \ldots - (au + b/u),$$

where a and b are abbreviated notation for the coefficients

$$a = \ell(t)\psi + \lambda, \qquad b = (x+1)/(1-\ell(t)),$$

and the ellipsis denotes terms that are independent of the variable u. If it is known that condition $x \geqslant 0$ can be guaranteed for the state of the system (13), then coefficient b will be positive and the maximum of the Hamiltonian in the variable $u \in U$ will be achieved at a unique point:

$$Q(a,b) = \begin{cases} m_{\max}, & a/b \leqslant 1/m_{\max}^2 \\ \sqrt{b/a}, & 1/m_{\max}^2 \leqslant a/b \leqslant 1/m_{\min}^2 \\ m_{\min}, & a/b \geqslant 1/m_{\min}^2. \end{cases} \tag{14}$$

Now we define the equation for the conjugate variable taking into account the terminal condition

$$\dot{\psi} = -H_x(t, x, \psi, u(t), \lambda), \quad t \in [0, T], \quad \psi(T) = 0, \tag{15}$$

where the right-hand side of the differential equation is defined as

$$-H_x(t, x, \psi, u, \lambda) = \alpha(t)\psi + \frac{1}{(1 - \ell(t))u} - \lambda c \alpha(t).$$

Let $x(t), \psi(t)$ be a solution of the system of differential Eqs. (13), (15), where as $u(t)$ we take the control

$$\hat{u}(t) = Q\big(\ell(t)\psi(t) + \lambda, (x(t) + 1)/(1 - \ell(t))\big), \tag{16}$$

where it is assumed that $x(t) \geqslant 0$ everywhere on $[0, T]$.

We formulate the result in Theorem 1.

Theorem 1. *If for some* $\varepsilon \geqslant 0$ *the right inequality in* (11) *holds, then for any initial condition* $x(0) \geqslant 0$ *the control* (16) *is a solution to the problem* (12). *If, in addition, the following inequality holds:*

$$\lambda(c/4 + 1) \leqslant \big(1/m_{\max} + 1/(4\alpha(t))\big)/m_{\max}, \quad t \in [0, T], \tag{17}$$

then the optimal control coincides with the upper bound, i.e., $\hat{u}(t) \equiv m_{\max}$.

Theorem 1 allows us to describe a scheme for finding the optimal feedback control $\hat{\mu}(t, x)$. For each $x_T \in (0, N)$, it is necessary to integrate the system (13), (15) in reverse time, taking into account $u(t) = \hat{u}(t)$ and terminal conditions $x(T) = x_T$, $\psi(T) = 0$ up to the moment t_0, at which one of the following conditions will be satisfied: $x(t_0) = 0$, $x(t_0) = N$, or $t_0 = 0$. Now the control in question is determined along each obtained trajectory according to the rule $\hat{\mu}(t, x(t)) = \hat{u}(t)$.

The structure of the optimal control $\hat{u}$ from (16) can be explained as follows. The choice between the two boundaries $m_{\min}$ and $m_{\max}$ is determined by the relation $r = (1 - \ell)(\ell\psi + \lambda)/(x + 1)$. If energy costs expressed by the parameter λ are insignificant compared to the system load level x, then r is small and in this case it is more preferable to transmit at maximum rate, i.e., $\hat{u} = m_{\max}$. Inequality (17) describes a sufficient condition under which such a conclusion can be drawn. If, on the contrary, resource conservation is critical, which is expressed by a large value of r, then the optimal transmission rate is set to a minimum level: $\hat{u} = m_{\min}$.

Remark. The expression (16) allows us to propose a constant strategy that can be used to get a preliminary solution for the problem in a model with constant parameters α and ℓ.

Assume that the value of the conjugate variable $\psi(t)$ is negligible. This is true at least at the end of the interval, since $\psi(T) = 0$. If we assume that by this moment the state $x(t)$ is established at some equilibrium value $\bar{x}$, then in the case of a fairly wide range $[1/m_{\max}^2, 1/m_{\min}^2]$ by virtue of (13) and (14) and due to $1 \ll \bar{x}$ we get the following system of equations:

$$\alpha(N - \bar{x}) - \ell\hat{u} = 0, \quad \hat{u} = \sqrt{\bar{x}/(\lambda(1 - \ell))}.$$

Then the corresponding solution $\bar{x}$ has the form

$$\bar{x} = N - \frac{\varkappa}{2}\left(\sqrt{1 + \frac{4N}{\varkappa}} - 1\right), \quad \varkappa = \frac{\ell^2}{\lambda(1 - \ell)\alpha^2},$$

and obviously $0 < \bar{x} < N$.

5. STATIONARY MODE

Now suppose that the network parameters ℓ, α are constant, and the desired control is described by a stationary strategy

$$\mu = \{\mu_x\colon x = 1, \ldots, N\} \in U^N, \quad U = [m_{\min}, m_{\max}], \tag{18}$$

where μ_x is the transmission rate from the main system in case when it contains x packets. When the main system is idle, its server is idle, so the corresponding transmission rate is zero: $\mu_0 = 0$. Strategies (18) will be called *admissible*.

Under these assumptions, the Markov process $X(t)$ will be homogeneous, and its transition rates will take the form $\lambda_{x-1,x} = \alpha(N+1-x)$ and $\lambda_{x,x-1} = \ell\mu_x$ for $x = 1, \ldots, N$. The process $X(t)$, as a finite birth and death process, is ergodic. This means that regardless of the choice of the initial distribution $X(0)$ for any $x \in E$, the limit probabilities $\pi_x = \lim_{t\to\infty} \mathsf{P}\{X(t) = x\}$ are defined. They satisfy a known recurrent representation:

$$\pi_x = \frac{\alpha(N+1-x)}{\ell\mu_x}\pi_{x-1}, \quad x = 1, \ldots, N. \tag{19}$$

If π_0 is found from the normalization condition, then the stationary distribution $\pi = \{\pi_0, \pi_1, \ldots, \pi_N\}$ will thus be determined.

Let us show the expressions for the performance indices. The average time for successful transmission (5) takes the form

$$\mathfrak{S}[\mu] = \frac{\pi_0}{\alpha N} + \frac{(1-\pi_N)\ell}{(1-\ell)\alpha} + \sum_{x=1}^{N} \frac{(x+1)\,\pi_x}{(1-\ell)\mu_x}.$$

From (3) we get an expression for the functional that determines the average power of resource consumption:

$$\mathfrak{R}[\mu] = \sum_{x=0}^{N} \{\mu_x + c\alpha(N-x)\}\pi_x.$$

Then, taking into account the equality $1 - \pi_N = \pi_0 + \ldots + \pi_{N-1}$, the extended functional (2) is equal to

$$\mathfrak{L}[\mu, \lambda] = h_0\pi_0 + \sum_{x=1}^{N} (h_x + b_x/\mu_x + \lambda\mu_x)\pi_x, \tag{20}$$

$$h_x = \frac{1}{\alpha N}\,\mathrm{I}\{x = 0\} + \frac{\ell}{(1-\ell)\alpha}\,\mathrm{I}\{x < N\} + \lambda c\alpha(N-x), \quad b_x = \frac{x+1}{(1-\ell)}.$$

We solve the problem of minimizing the extended functional in the stationary mode

$$\mathfrak{L}[\mu, \lambda] \to \min_{\mu} \tag{21}$$

on the class of strategies (18).

To do this, we introduce the functional

$$\mathfrak{J}[\pi, \mu, \nu] = \mathfrak{L}[\mu, \lambda] - \nu(\pi_0 + \pi_1 + \ldots + \pi_N - 1),$$

including the normalization condition in the form of a term with factor $\nu \in \mathbb{R}$. The sequence $\pi = \{\pi_x\colon x = 0, 1, \ldots, N\}$ will be considered as the trajectory of the discrete system (19) with an arbitrary initial condition $\pi_0 \geqslant 0$ and control $\mu = \{\mu_x\}$.

Functional $\mathfrak{J}$ is separable, so the dynamic programming method can be applied to minimize it. To do this, we rewrite $\mathfrak{J}$ due to (19) and (20):

$$\mathfrak{J}[\pi,\mu,\nu] = \nu + \pi_0(h_0-\nu) + \sum_{x=1}^{N} \pi_{x-1} g_x(\mu_x,\nu),$$

$$g_x(u,\nu) = \big(h_x + b_x/u + \lambda u - \nu\big)\,\gamma_x/u, \qquad \gamma_x = \alpha(N+1-x)/\ell.$$

Next we define the Bellman function:

$$\mathfrak{B}_n(p,\nu) = \inf_{(\pi,\mu)\in\mathcal{S}_n(p)} \sum_{x=n}^{N} \pi_{x-1} g_x(\mu_x,\nu), \quad n = N, N-1,\ldots,1, \tag{22}$$

where $\mathcal{S}_n(p)$ is the set of pairs (π,μ) such that $\pi = \{\pi_x\colon\ \ x = n,\ldots,N\}$ satisfies the recurrent equation $\pi_x = \pi_{x-1}\gamma_x/\mu_x$ with initial condition $\pi_{n-1} = p$ and admissible strategy $\mu = \{\mu_x : x = n,\ldots,N\}$.

The dynamic programming equation takes the form

$$\mathfrak{B}_n(p,\nu) = \inf_{u\in U}\big\{p\,g_n(u,\nu) + \mathfrak{B}_{n+1}\big(p\gamma_n/u,\,\nu\big)\big\}, \quad n = N, N-1,\ldots,1,$$

where $\mathfrak{B}_{N+1}\equiv 0$. Moreover,

$$\inf_{(\pi,\mu)\in\mathcal{S}_0} \mathfrak{J}[\pi,\mu,\nu] = \inf_{\pi_0\geqslant 0}\big\{\nu + \pi_0(h_0-\nu) + \mathfrak{B}_1(\pi_0,\nu)\big\}, \tag{23}$$

where $\mathcal{S}_0$ contains $(\pi,\mu)\in\mathcal{S}_1(\pi_0)$ for an arbitrary choice of the initial condition $\pi_0\geqslant 0$.

It is easy to verify that the Bellman function has the form $\mathfrak{B}_n(p,\nu) = p\,\beta_n(\nu)$, where the sequence $\{\beta_n(\nu)\}$ is determined from the recurrence relation

$$\beta_n(\nu) = \inf_{u\in U}\{g_n(u,\nu) + \gamma_n\beta_{n+1}(\nu)/u\}, \quad n = N, N-1,\ldots,1, \quad \beta_{N+1}\equiv 0.$$

After a change of variables $w = 1/u$, the latter equation takes the form

$$\beta_n(\nu) = \inf_{1/m_{\max}\leqslant w\leqslant 1/m_{\min}} \big\{\lambda + (h_n - \nu + \beta_{n+1}(\nu))\,w + b_n w^2\big\}\gamma_n.$$

If we find the absolute minimum point of this quadratic function

$$w_n^* = -\frac{h_n - \nu + \beta_{n+1}(\nu)}{2\,b_n},$$

then it can be argued that the strategy

$$\mu_x^* = \begin{cases} m_{\max}, & w_x^*\leqslant 1/m_{\max}\\ 1/w_x^*, & 1/m_{\max}\leqslant w_x^*\leqslant 1/m_{\min}\\ m_{\min}, & w_x^*\geqslant 1/m_{\min}, \end{cases} \qquad x = 1,\ldots,N, \tag{24}$$

is a minimizer of (22) for a fixed $\nu\geqslant 0$.

We choose the parameter ν so that the minimized expression on the right-hand side of (23) does not depend on π_0. Since $\mathfrak{B}_1(\pi_0,\nu) = \pi_0\beta_1(\nu)$, this condition is equivalent to the equation

$$h_0 - \nu + \beta_1(\nu) = 0. \tag{25}$$

Now we can describe a way to solve the problem (21).

Table 1. Value of the extended functional depending on the choice of the factor

i	λ_0^i	λ^i	$\mathfrak{L}[\mu^*, \lambda^i]$	$\mathfrak{L}[\hat{u}, \lambda^i]$	$\mathfrak{L}[m_{\min}, \lambda^i]$	$\mathfrak{L}[m_{\text{med}}, \lambda^i]$	$\mathfrak{L}[m_{\max}, \lambda^i]$
1	0.2	0.324	8.616	8.755	36.215	11.204	8.616
2	0.8	1.298	16.686	16.906	37.676	16.805	18.358
3	1.4	2.271	22.079	22.249	39.137	22.407	28.100
4	2.0	3.244	26.393	26.537	40.599	28.009	37.843
5	2.6	4.217	30.094	30.222	42.060	33.611	47.585
6	3.2	5.191	33.388	33.504	43.521	39.213	57.327
7	3.8	6.164	36.385	36.491	44.983	44.815	67.070
8	4.4	7.137	39.152	39.251	46.444	50.416	76.812
9	5.0	8.110	41.736	41.829	47.905	56.018	86.554

Table 2. Time of successful transmission and volume of resources at the strategies μ^* and $\hat{u}$

i	λ_0^i	λ^i	$\mathfrak{S}[\mu^*]$	$\mathfrak{S}[\hat{u}]$	$\mathfrak{R}[\mu^*]$	$\mathfrak{R}[\hat{u}]$
1	0.2	0.324	5.3684	5.6916	10.0099	9.4416
2	0.8	1.298	8.3332	9.7654	6.4365	5.5029
3	1.4	2.271	11.0310	12.4353	4.8649	4.3214
4	2.0	3.244	13.1885	14.5780	4.0701	3.6862
5	2.6	4.217	15.0402	16.4194	3.5695	3.2727
6	3.2	5.191	16.6888	18.0591	3.2171	2.9754
7	3.8	6.164	18.1890	19.5514	2.9519	2.7482
8	4.4	7.137	19.5757	20.9300	2.7429	2.5670
9	5.0	8.110	20.8711	22.2173	2.5726	2.4181

Theorem 2. *A solution ν^* of the Eq.* (25) *exists and is uniquely defined, it is positive and does not exceed the value of the functional $\mathfrak{L}[\mu, \lambda]$ at any admissible strategy μ.*

The strategy μ^*, defined in (24) *at $\nu = \nu^*$ is the steady-state optimal control in the problem of minimizing the extended functional* (21), *with $\mathfrak{L}[\mu^*, \lambda] = \nu^*$.*

If follows from the proof of Theorem 2 that the function on the right-hand side of Eq. (25) is continuous and does not increase monotonically. Therefore, the solution ν^* can be found by the interval bisection method or by the golden ratio method on the segment $[0, \mathfrak{L}[\mu, \lambda]]$, where μ is any admissible strategy, for example $\mu_x = m_{\min}$ or $\mu_x = m_{\max}$.

Let us give an example of numerically finding the optimal control of a closed data transmission network considered in a stationary mode.

Example. We make the following assumptions: the network contains $N = 50$ packets; the range of admissible values of the transmission rate is defined by the boundaries $m_{\min} = 1.5$ and $m_{\max} = 10$; $\alpha = 0.02$ is the servicing intensity in the auxiliary system; $\ell = 0.05$ is the probability of packet loss; $c = 0.02$ is the relative cost of using the auxiliary system server.

Table 1 shows the values of the extended functional depending on the choice of the factor λ given several controls: μ^* is the optimal control described in Theorem 2; $\hat{u}$ is a constant control whose value is chosen according to the Remark; $m_{\min}$, m_{med}, $m_{\max}$ are constant controls equal to the lower bound, middle point, and upper bound of the range of admissible values respectively. For the factor λ, which determines the weight of the energy criterion, we take several values of $\lambda^i = \lambda_0^i \mathfrak{S}[m_{\text{med}}]/\mathfrak{R}[m_{\text{med}}]$, where λ_0^i are dimensionless coefficients, $i = 1, \ldots, 9$.

As can be seen from Table 1, values of the extended functional at the optimal control μ^* and the constant strategy $\hat{u}$ obtained on the basis of the fluid approximation differ little. Nevertheless, a comparison of the functionals of the successful transmission time $\mathfrak{S}$ and the amount of resources $\mathfrak{R}$ allows us to see quite serious differences in the quality characteristics (see Table 2).

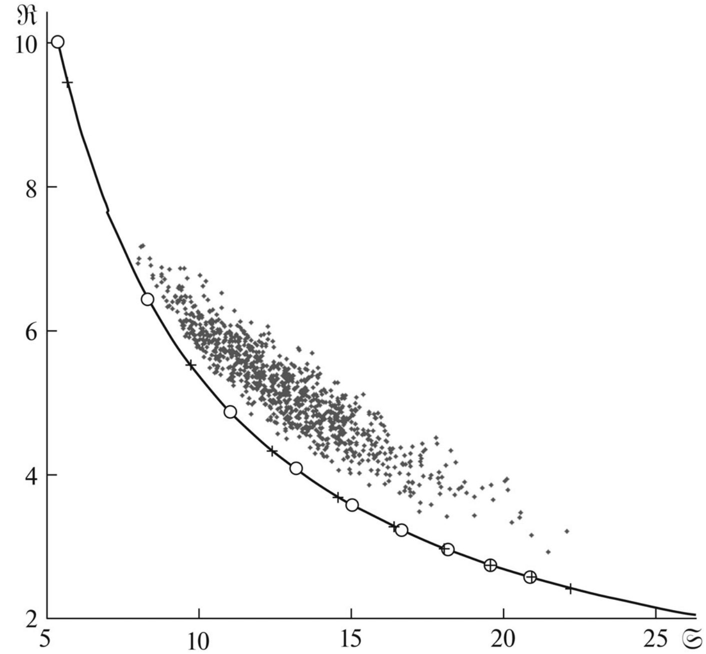

Fig. 2. Boundary of the attainable set (solid curve) on the plane of criteria $(\mathfrak{S}, \mathfrak{R})$ with optimal control μ^* (circles), constant control $\hat{u}$ (crosses), and randomly generated strategies (dots).

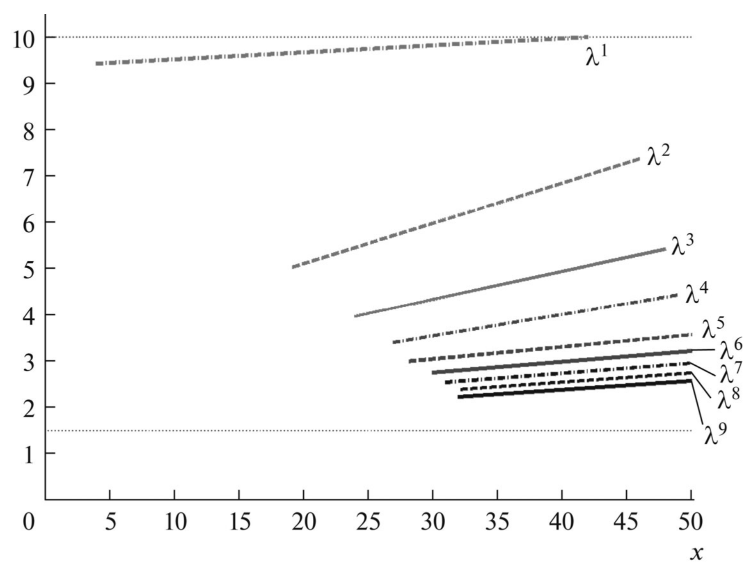

Fig. 3. Optimal control $\{\mu_x^*\}$ as a function of the state $x = 1, 2, \ldots, N$ for several values of the factor $\lambda = \lambda^i$.

On the plane of criteria $(\mathfrak{S}, \mathfrak{R})$, the optimal control μ^* must lie on the boundary of the attainable set

$$\mathcal{A} = \{(s, r)\colon \exists\, \mu \in U^N\colon\ s \geqslant \mathfrak{S}[\mu],\ r \geqslant \mathfrak{R}[\mu]\}.$$

This fact is confirmed by Fig. 2, where points $(\mathfrak{S}[\mu^*], \mathfrak{R}[\mu^*])$ are shown by circles, and the boundary $\mathcal{A}$ is a solid curve.

For a constant strategy $\hat{u}$, due to $\mathfrak{L}[\hat{u}] \approx \mathfrak{L}[\mu^*]$ the points $(\mathfrak{S}[\hat{u}], \mathfrak{R}[\hat{u}])$, indicated by crosses, also visually lie on the border. However, the points $(\mathfrak{S}[\mu^*], \mathfrak{R}[\mu^*])$ and $(\mathfrak{S}[\hat{u}], \mathfrak{R}[\hat{u}])$ do not coincide, since the difference is significant according to at least one of the two criteria (see Table 2).

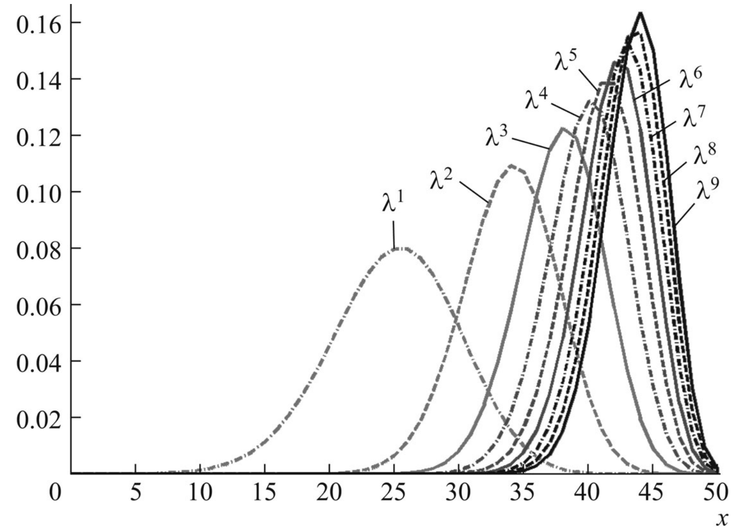

Fig. 4. Probabilities $\{\pi_x^*\}$ of states $x = 0, 1, \ldots, N$ of the main system when using several optimal controls μ^* depending on the choice of the factor $\lambda = \lambda^i$.

Other strategies μ (more precisely, the corresponding values of criteria $\mathfrak{S}[\mu], \mathfrak{R}[\mu]$) are shown by dots on Fig. 2. These strategies $\mu = \{\mu_x\}$, 1000 in total, were obtained as independent samples $\{\mu_1, \ldots, \mu_N\}$ from the uniform distribution $\mathcal{R}(m_{\min}, m_{\max})$.

Figure 3 shows the optimal controls $\{\mu_x^*\}$ for the same values of the factor λ as listed in Tables 1 and 2. The corresponding stationary distributions $\pi^* = \{\pi_x^*\}$ are presented in Fig. 4. For unlikely states x, an instability effect was observed when numerically finding the optimal strategies μ_x^*. Therefore, Fig. 3 shows for every control μ^* only the part of the plot that remained unchanged while increasing the precision of solving Eq. (25). The corresponding values of the optimal control were found with acceptable precision for the interval where the state $X \sim \pi^*$ falls with probability 0.999.

6. CONCLUSION

In this work, we have studied the optimal control problem for the data transmission rate in the model with an unreliable communication channel and a resending mechanism according to the criterion of the minimum total transmission time, taking into account the transmitter's energy consumption. The data transmission network is modeled by a closed queuing network with the main transmitter node (in the form of a finite single-server system) and an auxiliary node for resending the packets (in the form of a multiserver QS). We have obtained optimality conditions for the desired control in an approximate model based on fluid approximation, as well as in the steady state mode using stationary strategies.

APPENDIX

Proof of Lemma. If $\mu(t, x)$ is Lipschitz, then for any initial condition $x(0)$ Eq. (9) has a solution $x(t)$ defined over the entire interval $[0, T]$. Taking into account the notation $\beta(t) = N - \ell(t)\mu(t, x(t))/\alpha(t)$, we get $\dot{x} = -\alpha(t)x + \alpha(t)\beta(t)$, which implies the representation

$$x(t) = \varphi(t)x(0) + \int\limits_0^t \frac{\varphi(t)}{\varphi(s)}\alpha(s)\beta(s)\,ds, \qquad \varphi(t) = \exp\left\{-\int\limits_0^t \alpha(\tau)\,d\tau\right\}. \tag{A.1}$$

Due to (11), the inequality $\varepsilon \leqslant \beta(t) \leqslant N - \varepsilon$ holds. If in (A.1) instead of β we take $\beta_1 = \varepsilon$ and $\beta_2 = N - \varepsilon$, then the corresponding solutions $x_k(t) = \varphi(t)x(0) + (1 - \varphi(t))\beta_k$, $k = 1, 2$, will satisfy the inequalities $x_1(t) \leqslant x(t) \leqslant x_2(t)$. Since $0 < \varphi(t) < 1$, we get that $x(t)$ is a convex combination of the numbers $x(0)$, β_1 and β_2, each of which belongs to the segment $[\varepsilon, N-\varepsilon]$. This entails $x(t) \in [\varepsilon, N-\varepsilon]$, as required. This completes the proof of the lemma.

Proof of Theorem 1. First, let us establish the existence of an optimal control in problem (12). To do this, it suffices to check the conditions of Theorem III.4.1 from [23], namely the Lipschitz property in (x,u) for the function $f(t,x,u)$ on the right-hand side of the differential Eq. (13), continuity of the integrand $g(t,x,u,\lambda)$, compactness of the set of admissible control values U, and convexity of the sets

$$F(t,x) = \{(p,z)\colon \exists\, u \in U\colon p = f(t,x,u),\ z \geqslant g(t,x,u,\lambda)\}.$$

The last condition is fulfilled if $f(t,x,u)$ is linear, and $g(t,x,u,\lambda)$ is convex in u. The convexity of the second function in $u > 0$ follows from the lemma, and therefore it can be guaranteed that $x \geqslant 0$.

A formulation of the necessary optimality conditions in the form of the maximum principle can be found, for example, in Sections 2.4.1 and 2.4.2 in [24]. Due to the existence of an optimal control and the fact that the maximum of the Hamiltonian is reached at a single point (14), it can be argued that the control (16) is a solution for problem (12).

Now we derive a sufficient condition for the optimal control $\hat{u}(t)$ to coincide with the upper bound $m_{\max}$. Since $\hat{u}(t)$ is uniquely determined by the expressions (14) and (16), we find out in what case it holds that $a/b \leqslant 1/m_{\max}^2$. Omitting the time dependence, this inequality can be rewritten in the form

$$\ell(1-\ell)\psi + \lambda(1-\ell) \leqslant (x+1)/m_{\max}^2.$$

Taking into account that $0 \leqslant \ell < 1$ and $x \geqslant 0$, the above inequality is obviously satisfied for

$$\psi/4 + \lambda \leqslant 1/m_{\max}^2. \tag{A.2}$$

Next, let us bound the conjugate variable $\psi(t)$ from above. To do this, we write Eq. (15) in the form $\dot{\psi} = \alpha(t)\psi - \alpha(t)\gamma(t)$, where $\gamma(t) = \lambda c - ((1-\ell(t))\alpha(t)\hat{u}(t))^{-1}$. We use the inequality

$$\gamma(t) \leqslant \gamma_{\max} = \lambda c - \frac{1}{\alpha_{\max} m_{\max}}, \qquad \alpha_{\max} = \max_{t\in[0,T]} \alpha(t),$$

and a representation similar to (A.1). Then by virtue of $\psi(T) = 0$ we have that

$$\psi(t) = \int_t^T \frac{\varphi(s)}{\varphi(t)} \alpha(s)\gamma(s)\, ds = \frac{\gamma_{\max}}{\varphi(t)}(\varphi(t) - \varphi(T)) \leqslant \gamma_{\max}.$$

Now, substituting $\psi = \gamma_{\max}$ into (A.2), we obtain the required inequality (17). This completes the proof of Theorem 1.

Proof of Theorem 2. Let us prove that the solution (25) exists, is unique and positive. By definition, β_n is a minimum with respect to w for a function that continuously depends on w, β_{n+1}, ν, and w spans a segment. Then, by the continuity theorem of the marginal function [25, Section 3.1.23], β_n is a continuous function of the variables β_{n+1}, ν. Therefore, $\beta_1(\nu)$ is a continuous function as well. At $\nu = 0$, the function $h_0 - \nu + \beta_1(\nu)$ from the right-hand side of Eq. (25) will be positive. If one can choose $\nu_{\max} > 0$ such that $h_0 - \nu + \beta_1(\nu) < 0$ for $\nu = \nu_{\max}$, then due to the continuity of this function it can be argued that on the interval $(0, \nu_{\max})$ Eq. (25) will have a solution ν^*.

To find $\nu_{\max}$, note that for any choice of an acceptable strategy μ the following inequality holds:

$$\nu + \pi_0(h_0 - \nu + \beta_1(\nu)) \leqslant \mathfrak{J}[\pi, \mu, \nu],$$

where the right-hand side is $\mathfrak{L}[\mu, \lambda]$ if the sequence $\{\pi_x\}$ satisfies the normalization condition. Therefore, $h_0 - \nu + \beta_1(\nu) < 0$ for $\nu > \mathfrak{L}[\mu, \lambda]$. Therefore, it suffices to take the desired $\nu_{\max}$ a little larger than $\mathfrak{L}[\mu, \lambda]$.

To prove the uniqueness of ν^*, we note that $\beta_1(\nu)$ is a non-increasing and concave function as a minimum of such functions. Therefore, the function $h_0 - \nu + \beta_1(\nu)$, for which zero is an intermediate value, has the same properties. Therefore, the function takes this value at a unique point.

To establish the optimality of the strategy μ^* in problem (21), it suffices to verify that if π^* is a stationary distribution corresponding to the strategy μ^* then (π^*, μ^*) and ν^* form a saddle point of the functional $\mathfrak{J}[\pi, \mu, \nu]$:

$$\mathfrak{J}[\pi^*, \mu^*, \nu] \leqslant \mathfrak{J}[\pi^*, \mu^*, \nu^*] \leqslant \mathfrak{J}[\pi, \mu, \nu^*] \quad \forall\, (\pi, \mu) \in \mathcal{S}_0, \quad \forall\, \nu \in \mathbb{R}. \tag{A.3}$$

The left inequality in (A.3) turns into equality since both of its parts are equal to $\mathfrak{L}[\mu^*, \lambda]$ due to the normalization condition. The right inequality follows from (23) due to the fact that $\pi_0(h_0 - \nu) + \mathfrak{B}_1(\pi_0, \nu) = \pi_0(h_0 - \nu + \beta_1(\nu)) = 0$ with $\nu = \nu^*$.

In this case, $\mathfrak{J}[\pi^*, \mu^*, \nu^*] = \mathfrak{L}[\mu^*, \lambda] = \nu^*$, as required. This completes the proof of Theorem 2.

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This paper was recommended for publication by E.Ya. Rubinovich, a member of the Editorial Board