ISSN 0005-1179, Automation and Remote Control, 2019, Vol. 80, No. 8, pp. 1419–1427. © Pleiades Publishing, Ltd., 2019.
Russian Text © The Author(s), 2019, published in Avtomatika i Telemekhanika, 2019, No. 8, pp. 64–75.

CONTROL IN TECHNICAL SYSTEMS

Optimal Control Problems with Disorder

G. I. Belyavskii[*,**,a], **N. V. Danilova**[*,**,b], **and I. A. Zemlyakova**[*,**,c]

[*] *Southern Federal University, Rostov-on-Don, Russia*
[**] *Vorovich Institute of Mathematics, Mechanics, and Computer Science, Rostov-on-Don, Russia*
e-mail: [a] *beliavsky@hotmail.com*, [b] *danilova198686@mail.ru*, [c] *eizzzie@yandex.ru*

Received October 23, 2018
Revised February 23, 2019
Accepted April 25, 2019

Abstract—We consider a generalization of processes with disorder, namely processes with a vector disorder. For these problems, we consider a class of optimal control problems that do not detect the disorder. We propose a computational method for solving control problems on a finite time interval and with an objective functional defined at the end of the interval, based on the use of the martingale technique. We consider a computational experiment for a model with two barriers and two stopping times.

Keywords: processes with disorder, vector disorder, martingale, martingale measure, Wiener process, quantile hedging

DOI: 10.1134/S0005117919080046

1. INTRODUCTION

Processes that we consider in this work have been popular for a long time; they are called processes with mode switching. Processes with disorder are an important subclass of mode switching processes. The main problem related to the disorder, which is discussed in most publications, is the fastest detection of the moment of disorder [1–3]. Let us consider a generalization of processes with disorder, namely processes with a vector disorder, and an important class of optimal control problems for such processes without disorder detection. We will consider control problems on a finite time interval with an objective functional defined at the end of the interval. For these problems, we propose a computational method of solving them. We will use the martingale technique, which has been used, e.g., in stochastic financial mathematics [4, 5]. The paper is structured as follows. Section 2 lays out the basic concepts that are used in the remaining sections, poses the optimal control problem in question and describes our approach for solving it. In Section 3, we consider the model for which we solve the control problem. Section 4 discusses the solution to the quantile hedging problem for the considered model. Section 5 shows experimental results. Section 6 summarizes the work and outlines directions for further research.

2. DEFINITIONS AND PROBLEM SETTING

Consider a stochastic basis $\left\langle \Omega, (F_t)_{t\geq 0}, F, P \right\rangle$. The elementary random event space Ω is the space of continuous trajectories on the segment $[0,T]$, the filtering $(F_t)_{t\geq 0}$ is endowed with a set of standard properties, the σ-algebra $F = \sigma\left(\bigcup_{t\in[0,T]} F_t\right) = F_T$. The main source of randomness is the standard Wiener process, canonically defined as $W_t(\omega) = \omega(t)$; moreover, we assume that this process defines a stochastic basis in the following sense: the probability measure P is the Wiener measure, the filtering $F_t = \sigma\left(W_s, s \in [0,t] \bigcup N\right)$ where N is the σ-algebra containing all sets of zero measure.

Consider the Ito process: $dX(\omega,t) = \alpha_X(\omega,t)\,dt + \beta_X(\omega,t)\,dW_t$. In the formulas that follow, the trajectory ω will be omitted where it does not lead to confusion. Suppose that the coefficient $\beta_X(t) \neq 0$ with probability one, then the Ito process X can be written as $dX(t) = \beta_X(t)\left(\frac{\alpha_X(t)}{\beta_X(t)}\,dt + dW_t\right)$. Further we will be interested in the process $d\overline{W} = \frac{\alpha_X(t)}{\beta_X(t)}dt + dW_t$, which is an Ito process if the existence condition for the integral $P\left(\int_0^T |\chi_X(s)|\,ds < \infty\right) = 1$, $\chi_X(t) = \frac{\alpha_X(t)}{\beta_X(t)}$ is fulfilled. In what follows, we need one of the statements of the Girsanov theorem, see, e.g., [4, p. 833].

Theorem 1. *Consider the process*

$$Z_t = \exp\left(-\frac{1}{2}\int_0^t \chi_X^2(s)\,ds + \int_0^t \chi_X(s)\,dW_s\right).$$

If $\mathrm{E}Z_t = \mathrm{E}\exp\left(-\frac{1}{2}\int_0^t \chi_X^2(s)\,ds + \int_0^t \chi_X(s)\,dW_s\right) = 1$, *then with respect to measure* $d\overline{P}_T = Z_T dP_T$ *the Ito process* $d\overline{W}_t = \chi_X(t)\,dt + dW_t$ *is a standard Wiener process, and the density process* Z_t *is a uniformly integrable martingale.*

P_T denotes the restriction of a measure to the σ-algebra F_T. A sufficient condition for the expectation equality to hold is the condition $P\left(\int_0^T \chi_X^2(s)\,ds < \infty\right) = 1$. Since $Z_T > 0$, the new measure is equivalent to the original measure. Regarding this new measure, the process X is expressed directly through the standard Wiener process $\overline{W}_t$ in the form of the Ito integral $X(t) = X(0) + \int_0^t \beta_X(s)\,dW_s$, for whose existence the equality $P\left(\int_0^T \beta_X^2(s)\,ds < \infty\right) = 1$ is a sufficient condition. If this condition is fulfilled, the process X is a martingale with a continuous path, and there is a single measure with respect to which the process X is a martingale. Next we use the theorem on the representation of martingales [4, p. 313].

Theorem 2. *Let* $Y(t)$ *be a martingale with a continuous trajectory. Then there exists a unique progressively measurable process* $\beta_Y(t)$, *with* $P\left(\int_0^T \beta_Y^2(s)\,ds < \infty\right) = 1$, *for which* $Y(t) = Y(0) + \int_0^t \beta_Y(s)\,dW_s$.

Corollary. *Given that* $\beta_X(t) \neq 0$, *the martingale* Y *can be expressed via the martingale* X *as follows:* $Y(t) = Y(0) + \int_0^t \beta_{Y/X}(s)dX_s$, *where* $\beta_{Y/X}(t)\,\beta_X(t) = \beta_Y(t)$. *Moreover, this representation is unique.*

Consider the optimal control problem on a finite interval $[0,T]$ of the following form:

$$\min_{\beta_{Y/X},y_0} \mathrm{E}\left(\Phi\left(\mathrm{Y_T},\xi\right)\right)$$

under constraints

$$Y(t) = y_0 + \int_0^t \beta_{Y/X}(s)\,dX_s,$$
$$X(t) = x_0 + \int_0^t \alpha_X(s)\,dX_s + \int_0^t \beta_X(s)\,dW_s, \tag{1}$$
$$y_0 \leq a.$$

Here, the random variable ξ is measurable with respect to the σ-algebra F_T, the function of two variables $\Phi(x,y)$ is a convex function with respect to the first variable for an arbitrary value of the second variable.

In what follows, we call the process X the base process, and the process $\beta_{Y/X}$ is called the control or strategy. To solve this problem, consider the martingale measure $\overline{P}$: $d\overline{P}_T = Z_T dP_T$. The density process Z_t has been defined above. With respect to this new measure, processes X_t and Y_t are martingales with continuous trajectories. Suppose that there exists a solution to the problem

$$\min_{\eta} \mathrm{E}\left(\Phi\left(\eta,\xi\right)\right) \text{ given that } \overline{\mathrm{E}}\eta \leq a. \tag{2}$$

Suppose that η^* is a solution to problem (2). Consider the process

$$V\left(t\right) = \overline{\mathrm{E}}\left(\eta^*/F_t\right) \tag{3}$$

which is a uniformly integrated martingale with respect to the previously defined filtering and measure $\overline{P}$, so the process (3) can be expressed as

$$V\left(t\right) = \overline{\mathrm{E}}\eta^* + \int\limits_0^t \beta_{V/X}\left(s\right) dX_s. \tag{4}$$

Since $\overline{\mathrm{E}}\left[\eta^*\right] < \infty$ and $V\left(T\right) = \eta^*$, the process $\beta_{Y/X} = \beta_{V/X}$ and the initial value $y_0 = \mathrm{E}\eta^*$ are the solution to problem (1). In the derivation of decomposition (4) we have used the corollary shown above. Thus, the following statement is established.

Theorem 3. *If there exists a solution of problem* (2)*, then the solution to problem* (1) *is constructed as follows.*

1. *Find the martingale measure* $\overline{P}$ *using the density process* Z*.*
2. *Solve problem* (2)*.*
3. *Calculate the martingale* V *and martingale decomposition* (4) *with respect to the process* X*.*

Let us consider the decomposition of the martingale V in more detail. An explicit way to calculate the decomposition is obtained under the assumption that there exists a function $M\left(x,t\right) \in C^{2,1}$ such that $V(\omega,t) = M(X(\omega,t),t)$, and the assumption that the integrand $\beta_X(\omega,t) = \beta_X(X(\omega,t),t)$. Under these assumptions, one can use the Ito formula to calculate the decomposition. Indeed, using the Ito formula the differential $dV(\omega,t) = \left[\frac{\partial M}{\partial t}(X(t),t) + \frac{1}{2}\beta_X^2(X(t),t)\frac{\partial^2 M(X(t),t)}{\partial x^2}\right] + \beta_X(X(t),t)\frac{\partial M(X(t),t)}{\partial x} d\overline{W}_t$ using decomposition (4) is the same differential $dV\left(\omega,t\right) = \beta_{V/X}\left(X\left(\omega,t\right),t\right)\beta_X\left(X\left(\omega,t\right),t\right) d\overline{W}_t$. By comparing the differentials, we get a fundamental equation for the function M:

$$\frac{\partial M}{\partial t} + \frac{1}{2}\beta_X^2 \frac{\partial^2 M}{\partial x^2} = 0 \tag{5}$$

with initial condition $M\left(x,0\right) = \overline{\mathrm{E}}\eta^*$ and the expression for the integrand in the decomposition (4)

$$\beta_{V/X} = \frac{\partial M}{\partial x}. \tag{6}$$

Since $\beta_{Y/X}$ may depend on the entire history of the process $\overline{W}$, we will use other relations to calculate the decomposition (4), for example, if X and Y are quadratically integrable martingales with quadratic characteristics $\left\langle X\right\rangle_t$ and $\left\langle Y\right\rangle_t$. The use of these characteristics allows us to express the integrand through them in the decomposition (4):

$$\beta_{V/X}\left(t\right) = \frac{d\left\langle XV\right\rangle_t}{d\left\langle X\right\rangle_t}. \tag{7}$$

Consider two examples.

In example 1, we need to find

$$\min_{\beta_{Y/X}, y_0} \mathrm{E}\,(\xi - Y_T)^2 \tag{8}$$

under constraints (1). We assume that the random variable ξ is quadratically integrable. Following the proposed computational scheme, we need to solve problem (2) in order to find η^*. Next, we use the result given in [6, Chapter 6], which implies that problem (2) is equivalent to the problem

$$\min_{\eta} \mathrm{E}\left(\overline{\xi} - \eta\right)^2 \quad \text{given that} \quad \mathrm{E}Z_T\eta = 0. \tag{9}$$

In (9), $\overline{\xi} = \xi - a$. Solution of problem (9) will be

$$\eta^* = \overline{\xi} - \frac{\mathrm{E}Z_T\overline{\xi}}{\mathrm{E}Z_T^2} Z_T. \tag{10}$$

Using (10), we calculate the process $Y_t = a + \mathrm{E}\,(\eta^*/F_t)$ and the optimal control using (6) or (7).

In example 2, we need to find

$$\min_{\beta_{Y/X}, y_0} (\xi - Y_T)^+ \tag{11}$$

under the same constraints. Random variable $\xi > 0$ and its expectations are $\mathrm{E}\xi < \infty$ and $\overline{\mathrm{E}}\xi < \infty$. In (11), we use the notation $(x)^+ = \max(x, 0)$. According to the proposed scheme, we need to solve problem (2), which is equivalent to the problem

$$\max \mathrm{E}\xi\varsigma \quad \text{under constraints} \quad 0 \le \varsigma \le 1 \quad \text{and} \quad \overline{\mathrm{E}}\xi\varsigma \le a. \tag{12}$$

Consider two new measures: $d\tilde{P} = \frac{\xi}{\mathrm{E}\xi}$ and $d\overline{\overline{P}} = \frac{\xi}{\overline{\mathrm{E}}\xi} d\overline{P}$, for which problem (12) will look like

$$\max \tilde{\mathrm{E}}\varsigma \quad \text{under constraints} \quad 0 \le \varsigma \le 1 \quad \text{and} \quad \overline{\overline{\mathrm{E}}}\varsigma \le \alpha, \quad \text{where } \alpha = \frac{a}{\overline{\mathrm{E}}\xi}. \tag{13}$$

Problem (13) is a randomized Neyman–Pearson problem. The solution of the randomized problem is given in [7]. Following [7], we give the solution

$$\varsigma^* = I_{\left\{d\tilde{P} - \lambda^* d\overline{\overline{P}} > 0\right\}} + \varepsilon I_{\left\{d\tilde{P} - \lambda^* d\overline{\overline{P}} = 0\right\}}, \tag{14}$$

In (14), λ^* is the smallest value of λ for which $\overline{\overline{\mathrm{E}}} I_{\left\{d\tilde{P} - \lambda^* d\overline{\overline{P}} > 0\right\}} \le \alpha$ and $\varepsilon = \frac{\alpha - \overline{\overline{\mathrm{E}}} I_{\left\{d\tilde{P} - \lambda^* d\overline{\overline{P}} > 0\right\}}}{\overline{\overline{\mathrm{E}}} I_{\left\{d\tilde{P} - \lambda^* d\overline{\overline{P}} = 0\right\}}}$. Since $0 \le \alpha - \overline{\overline{\mathrm{E}}} I_{\left\{d\tilde{P} - \lambda^* d\overline{\overline{P}} > 0\right\}} < \overline{\overline{\mathrm{E}}} I_{\left\{d\tilde{P} - \lambda^* d\overline{\overline{P}} = 0\right\}}$, $0 \le \alpha < 1$. After that we find the solution of problem (2) $\eta^* = \xi\varsigma^*$, the process Y, and the optimal control.

3. MAIN MODEL

After describing the computational scheme, we consider the main class of models we consider for the basic process X. The processes α_X and β_X involved in the definition of process X are defined by the sequence of stopping moments $0 < \tau_1 < \ldots < \tau_n < \ldots$ that go to infinity with probability one and two deterministic sequences μ and σ with non-zero elements as follows:

$$\alpha_X(t) = \sum_{i=1}^{\infty} \mu_i I_{\{\tau_i < t \le \tau_i\}}, \quad \beta_X(t) = \sum_{i=1}^{\infty} \sigma_i I_{\{\tau_i < t \le \tau_i\}}. \tag{15}$$

Since we consider a finite segment $[0, T]$, it makes sense to define the random variable $\kappa_T = \sup\{n : \tau_n \le T\}$ and use it as an upper limit of the sums in (15):

$$\begin{aligned} \alpha_X(t) &= \sum_{i=1}^{\kappa_T} \mu_i I_{\{\tau_{i-1} < t \le \tau_i\}} + \mu_{\kappa_T+1} I_{\{\tau_{\kappa_T} < t \le T\}}, \\ \beta_X(t) &= \sum_{i=1}^{\kappa_T} \sigma_i I_{\{\tau_{i-1} < t \le \tau_i\}} + \sigma_{\kappa_T+1} I_{\{\tau_{\kappa_T} < t \le T\}}. \end{aligned} \tag{16}$$

It is obvious that $P(\kappa_T < \infty) = 1$, and the trajectories of processes (16) are left semi-continuous. This fact and the fact that τ_i are the stopping moments imply the statement.

Assertion. *Processes α_X and β_X are progressively measurable, and the integrals $\int_0^T |\alpha_X(s)|\, ds$, $\int_0^T \beta_X^2(s)\, ds$ and $\int_0^T \left(\frac{\alpha_X(s)}{\beta_X(s)}\right)^2 ds$ are finite with probability one.*

Consequently, there exists a unique martingale measure with respect to which the process X is a martingale with density

$$Z_T = \exp\left(-\frac{1}{2}A_T + \sqrt{A_T}\varepsilon\right). \tag{17}$$

In (17), ε is the standard normal random variable. The random nature of the process A is determined by the stopping moments. Process A is

$$A_t = \sum \left(\frac{\mu_i}{\sigma_i}\right)^2 (t \wedge \tau_i - t \wedge \tau_{i-1}). \tag{18}$$

With respect to the martingale measure, the process

$$\overline{W}_t = \sum_{i=1}^{\infty} \frac{\mu_i}{\sigma_i} (t \wedge \tau_i - t \wedge \tau_{i-1}) + W_t \tag{19}$$

is the Wiener process.

Consider the second problem from Section 2 for this model, using (17), (18), and (19). We assume that the random variable is $\xi = f(X_T)$. Moreover, the function f is such that $f(x) > 0$, $\mathrm{E}f(X_T) < \infty$ and $\overline{\mathrm{E}}f(X_T) < \infty$. Regarding the martingale measure, the random variable X_T is determined by the following equality:

$$X_T = x_0 + \sqrt{U_T}\varepsilon. \tag{20}$$

In (20), $U_t = \sum_{i=1}^{\infty} \sigma_i^2 (t \wedge \tau_i - t \wedge \tau_{i-1})$. Relative to the original measure, the random variable

$$X_T = x_0 + C_T + \sqrt{U_T}\varepsilon, \tag{21}$$

where $C_t = \sum_{i=1}^{\infty} \mu_i (t \wedge \tau_i - t \wedge \tau_{i-1})$. We calculate the expectation $\mathrm{E}f(X_T)\eta(X_T)$ by the original measure. It follows from (21) that the expectation in question is given by the formula

$$\mathrm{E}(f(X_T)\varsigma(X_T)) = \frac{1}{\sqrt{2\pi}}\mathrm{E}\left(\frac{1}{\sqrt{U_T}}\int_{-\infty}^{\infty} f(x)\varsigma(x)\exp\left(-\frac{1}{2U_T}(x - x_0 - C_T)^2\right)dx\right). \tag{22}$$

From (22) we get that $d\tilde{P}(x) = \tilde{p}(x)\,dx$, where

$$\frac{\mathrm{E}\left(f(x)\exp\left(-\frac{1}{2U_T}(x-x_0-C_T)^2\right)\frac{1}{\sqrt{U_T}}\right)}{\int_{-\infty}^{\infty}\mathrm{E}\left(f(x)\varsigma\exp\left(-\frac{1}{2U_T}(x-x_0-C_T)^2\right)\frac{1}{\sqrt{U_T}}\right)dx}. \tag{23}$$

For the measure $\overline{\overline{P}}$, we get the same result:

$$\overline{\overline{p}}(x) = \frac{\mathrm{E}\left(f(x)\exp\left(-\frac{1}{2U_T}(x-x_0)^2\right)\frac{1}{\sqrt{U_T}}\right)}{\int_{-\infty}^{\infty}\mathrm{E}\left(f(x)\varsigma(x)\exp\left(-\frac{1}{2U_T}(x-x_0)^2\right)\frac{1}{\sqrt{U_T}}\right)dx}. \tag{24}$$

It follows from (14), (23), and (24) that the optimal value is

$$\varsigma^*(x) = I_{\left\{\tilde{p}(x)-\lambda^*\overline{\overline{p}}(x)>0\right\}} + \varepsilon I_{\left\{\tilde{p}(x)-\lambda^*\overline{\overline{p}}(x)=0\right\}}, \tag{25}$$

where

$$\varepsilon = \frac{\alpha - \overline{\overline{\mathrm{E}}}I_{\left\{\tilde{p}(X_T)-\lambda^*\overline{\overline{p}}(X_T)>0\right\}}}{\overline{\overline{\mathrm{E}}}I_{\left\{\tilde{p}(X_T)-\lambda^*\overline{\overline{p}}(X_T)=0\right\}}}.$$

It follows from (25) that the random variable η^* is a function of X_T:

$$\eta^*(X_T) = f(X_T)\left(I_{\left\{\tilde{p}(X_T)-\lambda^*\overline{\overline{p}}(X_T)>0\right\}} + \varepsilon I_{\left\{\tilde{p}(X_T)-\lambda^*\overline{\overline{p}}(X_T)=0\right\}}\right). \tag{26}$$

Further, using (26), let us consider the calculation of the conditional expectation $\overline{\mathrm{E}}(\eta^*(X_T)/F_t)$ by the martingale measure. To do this, we represent the random variable X_T as follows: $X_T = X_t + \int_t^T \sigma(s)d\overline{W}_s$. Therefore, the conditional law $Law(X_T - X_t/U_T - U_t) = Law(\sqrt{U_T - U_t}\varepsilon)$. Since X_t and U_t are measurable with respect to F_t and regularity conditions hold, it follows that the conditional expectation is a function of t, X_t, and U_t, i.e.,

$$\begin{aligned} M(X_t, U_t, t) &= \overline{\mathrm{E}}(\eta^*(X_T)/F_t) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\eta^*(x)\,\mathrm{E}_{U_T-U_t}\left(\frac{\exp\left(-(x-X_T)^2/2(U_T-U_t)\right)}{\sqrt{U_T-U_t}}\right)dx. \end{aligned} \tag{27}$$

Formula (27) is significantly simplified if the disorder affects only the shift of the process X (the classic version of the disorder), i.e., $\sigma_i = \sigma$. The corresponding formula will become

$$M(X_t, t) = \frac{1}{\sqrt{2\pi\sigma(T-t)}}\left(\int_{-\infty}^{\infty}\eta^*(x)\exp\left(-\frac{(x-X_t)^2}{2\sigma^2(T-t)}\right)dx\right). \tag{28}$$

In (28), the function M depends only on two arguments. Next, we calculate the function $M(x,t)$ by solving Eq. (5) and calculate the control β using formula (6).

4. APPLICATION TO FINANCIAL MATHEMATICS

Let us consider the financial market model as a pair of assets: risky (stock value) S and risk-free (bank deposit) B. These assets are represented by their own prices $S(t)$ and $B(t)$, $t \in [0,T]$, that is, we are talking about a (B,S)-market with continuous time.

Assets are subject to the following equations:

$$dS\left(t\right)=S\left(t\right)dX\left(t\right),\quad dB\left(t\right)=rB\left(t\right)dt \tag{29}$$

with initial values S_0 and B_0. We consider a self-financing portfolio whose capital $G(t)$ satisfies the equation $dG(t) = \gamma(t)dS(t) + \beta(t)dB(t)$. The problem is to calculate

$$\min_{\gamma,\beta}\left(f\left(S\left(T\right)\right)-G\left(T\right)\right)^{+} \tag{30}$$

in view of (29) under constraint $G_0 \le a$. In (30), the function is $f(x) \ge 0$ and is bounded above. We reduce the problem (30) to the solved problem (11); for this, we consider the discounted process $\overline{S}(t) = \frac{S(t)}{B(t)}$, whose differential is $d\overline{S}(t) = \overline{S}(t)((\alpha_X(t) - r)dt + \beta_X(t)dW_t)$. At the same time, the differential of the discounted capital $d\overline{G}(t) = d\frac{G(t)}{B(t)} = \gamma(t)d\overline{S}(t)$. We define $\overline{f}\left(\overline{S}(T)\right) = f\left(\overline{S}\left(T\right)B\left(T\right)\right)/B\left(T\right)$, which allows us to consider an equivalent and simpler task: $\min_\gamma \mathrm{E}\left(\overline{f}\left(\overline{S}_T\right) - \overline{G}\left(T\right)\right)^+$ under constraints $\overline{G}_0 \le \frac{a}{B_0}$ and $\overline{G}\left(t\right) = \overline{G}_0 + \int_0^t \gamma\left(s\right)d\overline{S}\left(s\right)$, which coincides with the task (11) if we let $X\left(t\right) = \overline{S}\left(t\right)$ and $Y\left(t\right) = \overline{G}\left(t\right)$. Further, without loss of generality, we assume that $r = 0$.

For the case when the disorder affects only the shift and $\beta_X\left(t\right) = \sigma$ is a constant, solutions of the first equation from (11) are $S\left(t\right) = S_0\exp\left(\overline{C}\left(t\right) - \sigma W_t\right)$ and $S\left(t\right) = S_0\exp\left(-\frac{\sigma^2}{2} + \sigma W_t\right)$ for the original measure and the martingale measure respectively. The process is $\overline{C}\left(t\right) = C\left(t\right) - \frac{\sigma^2}{2}t$.

5. EXAMPLE. "MODEL WITH TWO BARRIERS AND TWO STOPPING MOMENTS"

The model was presented at the Moscow symposium "Advanced Finance and Stochastics" [8]. This model of behavior for a risky asset value arises in a situation where the "regulator" wants to keep the cost within a given corridor, and the price trend at the initial moment of time is an increasing function. On the considered segment $[0, T]$, the regulator takes part in the bidding at most twice. The first time, he sells an asset when the asset reaches a top level in order to get a diminishing trend. The second time, the regulator buys an asset in order to get an increasing trend when the price reaches a lower level.

Thus, we need to consider a vector stopping moment (τ_1, τ_2), where $\tau_! = \inf(t \in [0, T] : S(t) = M_1)$, where $M_1 > S_0$ and $\tau_2 = \inf\left(t \in (\tau_1, T] :\ S\left(t\right) = M_2\right)$, where $M_2 < S_0$. We define the shift

$$\overline{C}(t)=\left(\mu_1-\frac{\sigma^2}{2}\right)(t\wedge\tau_1)+\left(\mu_2-\frac{\sigma^2}{2}\right)(t\wedge\tau_2-t\wedge\tau_1)+\left(\mu_3-\frac{\sigma^2}{2}\right)(t-t\wedge\tau_2).$$

We assume the following inequalities: $\mu_1 - \frac{\sigma^2}{2} > 0$, $\mu_2 - \frac{\sigma^2}{2} < 0$, $\mu_3 - \frac{\sigma^2}{2} > 0$. Let us find the density $p(x, y)$ for the distribution of the vector stopping moment (τ_1, τ_2) defined on the set $D = \left\{(x, y) \in R^2 :\ 0 \le x \le y\right\}$, using the equality for density: $p\left(x, y\right) = p\left(y/x\right) \times p\left(x\right)$. To find the density of the distribution of the first stopping moment $p(x)$, we use the fact that $\tau_1 = \inf\left(t : m_1 t + \sigma W_t = \overline{M}_1\right)$, where $m_1 = \mu_1 - \frac{\sigma^2}{2}$ and $\overline{M}_1 = \ln\frac{M_1}{S_0}$. To determine the density $p\left(y/x\right)$ of the conditional distribution law, we use the strictly Markov property of the Wiener process. According to this property, the conditional behavior of the second stopping moment is determined by the equality $\tau_2 = x + \inf\left(t : m_2 t + \sigma W_t = \overline{M}_2\right)$, where $m_2 = \frac{\sigma^2}{2} - \mu_2$ and $\overline{M}_2 = \ln\frac{M_1}{M_2}$. Let $p(m, M, x)$ be the density of the distribution of the stopping moment $\tau = \inf(t : mt + \sigma W_t = M)$ defined on the set $R^+ = \{x \in R : x > 0\}$. Density parameters m and M are positive numbers. Density $p(m, M, x) = \frac{M}{\sigma\sqrt{2\pi}}\exp\left(\frac{mM}{\sigma^2}\right)\frac{1}{x^{3/2}}\exp\left(-\frac{1}{2\sigma^2}\left(m^2x + \frac{M^2}{x}\right)\right)$, see [4, p. 265]. The density of the joint distribution law is expressed in terms of the density $p(m, M, x)$ as follows: $p(x, y) = p\left(m_1, \overline{M}_1, x\right) p\left(m_2, \overline{M}_2, y - x\right)$. The explicit form of the density of the joint distribution law makes it easy to calculate the function $\varsigma^*(x)$ and solve the quantile hedging problem for a model with two barriers.

5.1. Computational Experiment

Consider the function $f(x) = (x - K)^+$. The following input data is chosen for the calculation: $\mu_! = 0.1$; $\mu_2 = -0.1$; $\mu_3 = 0.1$; $\sigma = 0.05$; $S_0 = K = 6$; $M_1 = 7$; $M_2 = 5$; $\alpha = 0.35$. Function $F(\lambda) = \left(\int_{-\infty}^{\infty} \overline{\overline{p}}(s)\, I_{\left\{\tilde{p}(s) - \lambda^* \overline{\overline{p}}(s) > 0\right\}}(s)\, ds - \alpha\right)$ is a continuous function, therefore $\varepsilon = 0$ in (25).

Figure 1 shows the plot of $F(\lambda)$ for the values $\lambda \in [0.3;\, 1.5]$.

Solution of the equation $F(\lambda) = 0 - \lambda = 0.7$.

Further, in the calculations we used the Rademacher approximation $\Delta S_n = S_{n-1}\sigma/\sqrt{N}\delta_n$, $P(\delta_n = 1) = P(\delta_n = -1) = 1/2$.

Results of our calculations are shown on Fig. 2 and in the table. For $N = 10$ and $N = 11$, the calculated values of α are the same and equal to 0.3496. For $N = 10$ and a random trajectory we have calculated a portfolio which is shown in the table.

The Rademacher approximation satisfies

$$\gamma_n = \frac{\overline{\mathrm{E}}\left(\Delta Y_n \Delta S_n / F_{n-1}\right)}{\overline{\mathrm{E}}\left(\left(\Delta S_n\right)^2 / F_{n-1}\right)} = \left.\frac{\Delta Y_n}{\Delta S_n}\right|_{\delta_n = 1} = \left.\frac{\Delta Y_n}{\Delta S_n}\right|_{\delta_n = -1}.$$

The second component of the portfolio is $\beta_n = Y_{n-1} - \gamma_n S_{n-1}$.

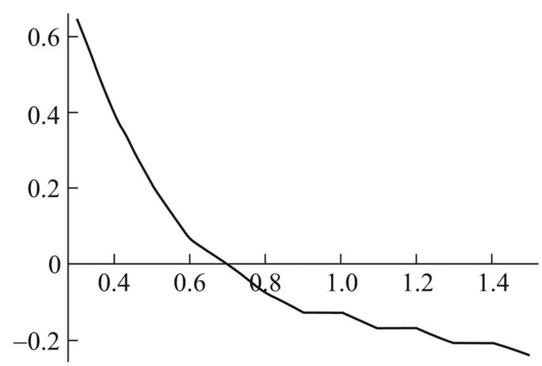

Fig. 1. Plot of $F(\lambda)$ for the values $\lambda \in [0.3;\, 1.5]$.

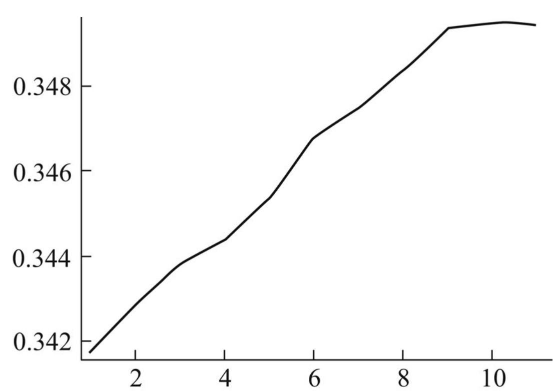

Fig. 2. Plots of the calculated α as a function of N $(N = 3, \ldots, 11)$ (exact value $\alpha = 0.35$).

Portfolio calculation

n	Arbitrarily chosen path on a tree (values of ε_n)	Values of S_n in the atoms of the chosen path	Values of Y_n in the atoms of the chosen path	Values of γ_n in the atoms of the chosen path	Values of β_n in the atoms of the chosen path
0		6	0.07966675650	0.4297220522	–2.498665556
1	1	6.094868328	0.1204337691	0.5907306195	–3.479991574
2	1	6.191236656	0.1773614912	0.7757321767	–4.625379997
3	–1	6.093344609	0.1014234805	0.5837691069	–3.455682860
4	–1	5.997000371	0.04518069073	0.3500031905	–2.053788572
5	–1	5.902179469	0.01199307250	0.1285132911	–0.7465154357
6	–1	5.808857818	0	0	0
7	–1	5.717011711	0	0	0
8	1	5.807405601	0	0	0
9	1	5.899228744	0	0	0
10	–1	5.805953748	0		

6. CONCLUSION

We have proposed a model with vector disorder and proposed for this model an algorithm for solving a wide class of stochastic optimal control problems. As an example, we have solved the quantile hedging problem for a model with a corridor. We have shown the results of applying the developed algorithm. Further studies will be directed towards obtaining an acceptable computational algorithm for solving the considered stochastic optimal control problems for other stopping times. In this case, we assume that the Wiener process is approximated with the Donsker–Prokhorov principle.

ACKNOWLEDGMENTS

This work was financially supported by the Russian Science Foundation, project no. 17-19-01038.

REFERENCES

1. Shiryaev, A., Quickest Detection Problems in the Technical Analysis of the Financial Data, in *Mathematical Finance*, Geman, H., Madan, D., Pliska, S., and Vorst, T., Eds., New York: Springer, 2000, pp. 487–521.
2. Gapeev, P.V. and Peskir, G., The Wiener Disorder Problem with Finite Horizon, *Stoch. Proc. Appl.*, 2006, vol. 116, no. 12, pp. 1770–1791.
3. Truonga, C., Oudrec, L., and Vayatisa, N., *A Review of Change Point Detection Methods*, arXiv: 1801.00718v1 [cs.CE], Jan 2, 2018.
4. Shiryaev, A.N., *Osnovy stokhasticheskoi finansovoi matematiki* (Fundamentals of Stochastic Financial Mathematics), Moscow: Fazis, 1998.
5. Fel'mer, G. and Shid, A., *Stokhasticheskie finansy* (Stochastic Finance), Moscow: MTsNMO, 2008.
6. Mel'nikov, A.V., Volkov, S.N., and Nechaev, M.L., *Matematika finansovykh obyazatel'stv* (Mathematics of Financial Obligations), Moscow: Vysshaya Shkola Ekonomiki, 2001.
7. Rudloff, B., A Generalized Neyman–Pearson Lemma for Hedge Problems in Incomplete Markets, *Proc. Workshop "Stochastic Analysis,"* 27.09.2004–29.09.2004, pp. 241–249.
8. Beliavsky, G. and Danilova, N., About (B.S)—Market Model with Stochastic Switching of Parameters, *Proc. Int. Conf. Advanced Finance and Stochastics*, Moscow, June 24–28, 2013.

This paper was recommended for publication by E.Ya. Rubinovich, a member of the Editorial Board