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NONLINEAR SYSTEMS

An Overview of Semidefinite Relaxations for Optimal Power Flow Problem

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Abstract—The AC optimal power flow (AC OPF) problem is considered and five convex relaxations for solving this problem—the semidefinite, chordal, conic, and moment-based ones as well as the QC relaxation—are overviewed. The specifics of the AC formulation and also the nonconvexity of the problem are described in detail. Each of the relaxations for OPF is written in explicit form. The semidefinite, chordal and conic relaxations are of major interest. They are implemented on a test example of four nodes.

Keywords: power systems, semidefinite programming, convex relaxations, power flows

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1. INTRODUCTION

Electrical power is a major resource in the modern world. Furthermore, industrial processes face with an increasing power demand. For instance, in the recent years the battery vehicles have been actively developed and the growing population of the Earth has been consuming more and more power. Power generation is very costly itself; in addition, there are many "dirty generators," e.g., with fuel oil burning. Such power plants have a severe impact on the environment and also depend on the mineral resources and their prices. The classical power generators are gradually substituted by the ones with renewable power sources—windmills and solar panel farms. Unfortunately, such sources have limited capacity and heavily depend on weather conditions. All these factors lead to a whole host of problems for researchers and engineers in different fields of science and technology.

One of the problems is to determine an optimal power production mode for a given network. There exist different formulations of this problem but the most widespread and accurate one is the optimal power flow problem, which rests on the physical Kirchhoff and Ohm laws. Common optimality criteria are to minimize the total generation cost or loss subject to engineering constraints. Also note a separate approach focused on the stable operation of a power network, which is known as the anti-blackout approach. This problem has higher complexity due to additional constraints connected with its physical nature. As a rule, the stable mode is not optimal in the classical formulation. Thus, the integration of the two approaches seems promising for the industry.

A distinctive feature of the optimal power flow (OPF) problem in the classical formulation is its nonconvexity, which makes convex optimization tools directly inapplicable. The system operators adopt the linearized formulation of the problem—DC OPF. After the linearization procedure the problem can be solved in a faster and simpler way, but at the price of the resulting accuracy. Therefore, the methods for solving the original nonconvex problem (AC OPF) are of major interest. (Hereinafter, OPF will refer to AC OPF.) Relaxations are a rather popular technique for managing the problem's nonconvexity. Relaxations can be used to considerably reduce the problem's complexity and to solve it in acceptable time with sufficient accuracy. Unfortunately, relaxations do not guarantee the exact solution, and still there are no general formulas or theorems that would describe the existence conditions of the exact solution in some general formulation of the problem.

This paper is dedicated to the classical optimal power flow problem, in particular, to different convex relaxations for it. The primary goal is to study in detail some relaxations on a text example of four nodes. The remainder of this paper is organized as follows. The basic formulation of the problem is introduced in Section 2. An overview of the existing relaxations is given in Section 3. Test examples and numerical experiments are described in Section 4.

2. OPTIMAL POWER FLOW PROBLEM

The OPF problem was first formulated in 1962; the history of this problem and also the development of different methods for it can be traced back in [1–3]. From that time on, OPF has been an important control problem for power networks, especially because still there are no efficient algorithms for solving its general formulation in a fast and accurate way for real power systems with thousand nodes. For example, the Russian power network includes more than 9000 nodes. In the first place, the matter concerns the nonconvexity and NP-hardness of this problem. For that reason, the system operators in the industry adopt the linearized formulation of the problem—DC OPF [4]. This approach gives a fast solution for large networks, but at the price of the resulting accuracy (in terms of the distance to the global optimum). Due to a growing consumption and production of power, the industry is facing an urgent need for exact algorithms: even small improvements in the quality of solutions would save $ billions annually.

The OPF problem can be reformulated as a quadratically constrained quadratic programming (QCQP) problem in which the objective function and also all associated constraints are quadratic functions. Unfortunately, in this formulation the problem still remains nonconvex but different convex relaxations can be used. This approach has been intensively developed in the recent years. However, there is no guarantee that a given method will yield the exact solution (or even any solution at all!), which forms its major drawback. For a certain class of the problems, a given method may work for some problems of the class and fail for the other. Consider the standard mathematical formulation of the OPF problem.

A power network is a graph G in which the nodes N correspond to the generators and customers while the edges E to the power lines. The edges are drawn only between those nodes that have power lines to each other. A generator, or a customer, or both simultaneously can be located in each node.

The OPF problem is based on the Kirchhoff (1) and Ohm (2) laws, which establish the following relation of the electric current I, voltage V, conduction Y and power S:

$$I_j^g - I_j^l = \sum_{(j,k)\in E} I_{jk}, \quad \forall j \in N, \tag{1}$$

$$I_{jk} = Y_{jk}(V_j - V_k), \quad \forall (j,k) \in E, \tag{2}$$

where I, V, Y and S are complex values. Hereinafter, the superscripts g and l will indicate power generation and load, respectively.[1]

The electrical power is calculated as

$$S_{jk} = V_j I_{jk}^{\mathsf{H}} = V_j Y_{jk}^{\mathsf{H}}\left(V_j^{\mathsf{H}} - V_k^{\mathsf{H}}\right), \quad \forall (j,k) \in E, \tag{3}$$

[1] Unless otherwise specified, a single index j will indicate a network node (a graph node) while a pair (j,k) a network line (a graph edge).

where the superscript $^{\mathsf{H}}$ indicates the Hermitian conjugation of a matrix (the complex conjunction with transposition).

Expressions (1)–(3) considered together give the following formulas of the power flows and the net power injection (NPI) of node j (the difference between the amounts of power generated and consumed by this node):

$$S_{jk} = Y_{jk}^{\mathsf{H}} V_j V_j^{\mathsf{H}} - Y_{jk}^{\mathsf{H}} V_j V_k^{\mathsf{H}}, \quad (j,k) \in E, \tag{4}$$

$$s_j = S_j^g - S_j^l = \left(P_j^g - P_j^l\right) + i\left(Q_j^g - Q_j^l\right) = \sum_{(j,k)\in E} S_{jk}, \quad \forall j \in N. \tag{5}$$

Using (4) and (5) the node's NPI can be written a function of the network voltages:

$$s_j = \sum_{(j,k)\in E} V_j \left(V_j^{\mathsf{H}} - V_k^{\mathsf{H}}\right) Y_{jk}^{\mathsf{H}}, \quad \forall j \in N. \tag{6}$$

Formula (6) is quadratic in voltage, which makes the OPF problem "quadratic" as well.

Network nodes may have constraints on s_j that are associated with the generator's consumption and capacity,

$$\underline{s}_j \leqslant s_j \leqslant \overline{s}_j, \quad \forall j \in N, \tag{7}$$

and also constraints on the voltage at each node,

$$\underline{V}_j \leqslant |V_j| \leqslant \overline{V}_j, \quad \forall j \in N. \tag{8}$$

In addition, there are constraints for power lines (9) that limit the maximum admissible flows of each line:

$$|S_{jk}| \leqslant S_{jk}^{\max}, \quad \forall (j,k) \in E. \tag{9}$$

If this constraint is violated for some line, the latter may fail, causing a collapse of the entire network.

Formula (6) determines the amounts of power generated at different nodes through voltages, i.e., only the voltage variables can be used for further analysis. The set of conditions (7)–(9) defines the set of admissible operating modes for a given network. Different objective functions can be introduced depending on the needs arising for a specific network. The two most widespread functionals are to minimize the total active power generation losses

$$f_1(V) = \sum_{j\in N} \Re(s_j) + P_j^l = \sum_{j\in N} \Re\left(\sum_{k:(j,k)\in E} V_j \left(V_j^{\mathsf{H}} - V_k^{\mathsf{H}}\right) Y_{jk}^{\mathsf{H}} + P_j^l\right),$$

and to minimize the total active power generation cost

$$f_2(V) = \sum_{j\in N} c_j \Re(s_j + P_j^l) = \sum_{j\in N} c_j \Re\left(\sum_{k:(j,k)\in E} V_j \left(V_j^{\mathsf{H}} - V_k^{\mathsf{H}}\right) Y_{jk}^{\mathsf{H}} + P_j^l\right),$$

where c_j specifies the generation cost for generator j; $\Re(\,)$ and $\Im(\,)$ denote the real and imaginary parts of a complex value, respectively. Note that the generation cost may be some function that depends on the amounts generated (as a rule, a quadratic function). In the elementary case, the generation of 1 kW of active power has a fixed constant cost.

Here $f_j(V) : \mathbb{C}^n \to \mathbb{R}$. Both functionals are quadratic functions of the network voltage vector $V = (V_1, \cdots, V_n)$.

Now, the problem can be completely written as follows:

$$
\begin{gathered}
f(V) \to \min_V, \\
\underline{s}_j \leqslant \sum_{k:(j,k)\in E} V_j \left(V_j^{\mathsf{H}} - V_k^{\mathsf{H}}\right) Y_{jk}^{\mathsf{H}} \leqslant \overline{s}_j, \quad \forall j \in N, \\
\underline{V}_j \leqslant |V_j| \leqslant \overline{V}_j, \quad \forall j \in N, \\
|S_{jk}| \leqslant S_{jk}^{\max}, \quad \forall (j,k) \in E.
\end{gathered}
$$

In this formulation, the function $f(V)$ is any of the functions $f_1(V)$, $f_2(V)$ or any other functional of voltage as required.

2.1. DC Formulation

In the formulation described above, all complex values are written in algebraic form and will be used likewise below. The same formulas can be obtained for the trigonometric representation of complex values (which is common for engineers) and then used for deriving the linearized formulation of the problem—DC OPF.

The line's complex conduction Y_{jk} consists of the active G_{jk} and reactive B_{jk} conductions, i.e.,

$$
Y_{jk} = G_{jk} + iB_{jk}, \quad \forall (j,k) \in E. \tag{10}
$$

The complex voltage can be written in the trigonometric form:

$$
V_j = |V_j| \exp(i\delta_j), \quad \forall j \in N. \tag{11}
$$

Substituting (10), (11) into the power formula (3) yields the following expressions of the active (P) and reactive (Q) generation:

$$
S_{jk} = V_j \left(V_j^{\mathsf{H}} - V_k^{\mathsf{H}}\right) Y_{jk}^{\mathsf{H}} = |V_j| \exp(i\delta_j)(|V_j| \exp(-i\delta_j) - |V_k| \exp(-i\delta_k))(G_{jk} - iB_{jk}),
$$

$$
P_{jk} = \Re(S_{jk}) = |V_j|^2 G_{jk} + |V_j||V_k| \left(G_{jk} \cos(\delta_j - \delta_k) + B_{jk} \sin(\delta_j - \delta_k)\right), \tag{12}
$$

$$
Q_{jk} = \Im(S_{jk}) = -|V_j|^2 B_{jk} + |V_j||V_k| \left(G_{jk} \sin(\delta_j - \delta_k) - B_{jk} \cos(\delta_j - \delta_k)\right). \tag{13}
$$

These formulas describe the active and reactive power flows of line (j,k) through the voltage and conduction. From (12) and (13) we easily calculate the amounts of power consumed and generated at node j as

$$
P_j = \sum_{k:(j,k)\in E} |V_j||V_k| \left(G_{jk} \cos(\delta_j - \delta_k) + B_{jk} \sin(\delta_j - \delta_k)\right), \tag{14}
$$

$$
Q_j = \sum_{k:(j,k)\in E} |V_j||V_k| \left(G_{jk} \sin(\delta_j - \delta_k) - B_{jk} \cos(\delta_j - \delta_k)\right). \tag{15}
$$

(Note that all technicalities are omitted.)

The DC formulation of the problem can be easily obtained from (14) and (15) by making some engineering assumptions as follows. For a stationary state of a power system,

1) $G_{jk} = 0$, $\forall (j,k) \in E$;
2) $|V_j| \approx 1$, $\forall j \in N$;
3) $(\delta_j - \delta_k) \approx 0 \Rightarrow \cos(\delta_j - \delta_k) \approx 1$, $\sin(\delta_j - \delta_k) \approx \delta_j - \delta_k$.

Hence, (14) and (15) are simplified to

$$P_j = \sum_{k:(j,k)\in E} B_{jk}(\delta_j - \delta_k),$$
$$Q_j = \sum_{k:(j,k)\in E} -B_{jk} \times 1. \qquad (16)$$

In accordance with (16), in the DC formulation the reactive power Q is uniquely determined. Then the DC formulation has the complete description

$$\min_{\delta} \sum_{j\in N} f(P_j),$$
$$P_j^g = P_j^l + \sum_{k:(j,k)\in E} B_{jk}(\delta_j - \delta_k), \quad \forall j \in N : j \text{ \{is generator\}};$$

where $f(\cdot)$ is some linear function that depends on generation, e.g., $(c_j P_j)$, where c denotes the active generation price vector. In this formulation, OPF represents a linear programming problem, which can be solved very fast. We refer to the monograph [5] for all technical details of DC OPF.

3. WAYS TO SOLVE OPTIMAL POWER FLOW PROBLEM

In the recent time, many different approaches to solve the optimal power flow problem have been developed in view of its complexity and crucial importance. Among them note sequential quadratic programming, genetic algorithms, inner-point methods and relaxations [6]. As a matter of fact, semidefinite (SD) relaxations are gaining more and more attention. The key idea of these relaxations consists in the following: the original (nonconvex) problem is reformulated into a convex one by eliminating a single nonconvex condition. And the relaxed problem is solved using any method for convex optimization problems. If the resulting solution satisfies the eliminated condition, then the exact solution is obtained; otherwise the relaxation is inaccurate but gives a lower bound of the optimal value of the objective function.

Here the main obstacle is that the conditions under which the relaxation yields the exact solution of the general problem have not been established so far. But such conditions have been derived for radial networks, i.e., the networks whose graphs represent trees [7]. In some situations, the same method well works for one problem and "breaks" for its light modification with small changes in the initial data, constraints or functional (i.e., the modified problem belongs to the same class as the original one) [8].

Some popular and/or new relaxations will be discussed in the next subsection. Before that, consider the general structure of SD relaxations.

3.1. Semidefinite Programming

The semidefinite relaxation [9] is a rather widespread and also simple technique for solving nonconvex problems. The SD relaxation demonstrates the best results for the quadratically constrained quadratic programming (QCQP) problems, and OPF is among them. For the problems of this class, the SD relaxation is constructed in a natural way. Consider an illustrative example. Let the original problem from the QCQP class have the form

$$x^{\mathsf{T}} C x \to \min_{x\in\mathbb{R}^n}, \qquad (17)$$
$$x^{\mathsf{T}} H x \geqslant a,$$
$$x^{\mathsf{T}} G x = b,$$

where C, G, and H are symmetric matrices.

For any symmetric matrix A,

$$x^{\mathsf{T}}Ax = Tr(x^{\mathsf{T}}Ax) = Tr(Axx^{\mathsf{T}}) = Tr(AX).$$

This chain of equalities can be proved in the following way. First, $x^{\mathsf{T}}Ax$ is a scalar and hence the trace operator can be applied without any effect. Next, the formula in the right-hand side follows from the cyclicity of the trace operator and is a scalar as well, where $xx^{\mathsf{T}} = X$ is a matrix of dimensions $(n \times n)$. Hence, the left- and right-hand sides are equal to each other. Using such changes for the functional and each constraint of problem (17), we may write the problem

$$\begin{aligned} Tr(CX) &\to \min_{X \in \mathbb{S}^n}, \\ Tr(HX) &\geqslant a, \\ Tr(GX) &= b, \\ X &\succeq 0, \\ \operatorname{rank}(X) &= 1. \end{aligned} \tag{18}$$

If the matrix X is nonnegative definite ($X \succeq 0$) and has rank 1, then the original vector x can be uniquely restored from X using the eigenvalue decomposition. In other words, the solution of problem (18) uniquely determines the solution of problem (17). Hence, these problems are equivalent.

Problem (18) is also nonconvex due to the rank condition, which forms another difficulty. Eliminating this condition, we obtain the semidefinite relaxation of the original problem (17):

$$\begin{aligned} Tr(CX) &\to \min_{X \in \mathbb{S}^n}, \\ Tr(HX) &\geqslant a, \\ Tr(GX) &= b, \\ X &\succeq 0. \end{aligned} \tag{19}$$

Unfortunately, numerical methods may converge to the solutions of very large ranks ($\operatorname{rank}(X^*) \gg 1$) even if there exists the solution of rank 1. In this case, x^* cannot be restored in explicit form. There are different heuristics to find an admissible solution of the original problem (two of them will be considered below) but the resulting admissible solution is generally not optimal.

If the resulting solution X^* has rank 1, then only one of its eigenvalues is nonzero (actually, nonnegative). Therefore, the optimal vector x^* can be restored using the formula

$$x^* = \sqrt{\lambda}u,$$

where λ and u denote the eigenvalue and the corresponding eigenvector of the matrix X^*.[2] If the rank of the solution exceeds 1, then the maximum eigenvalue λ_1 and the corresponding eigenvector u_1 can be employed to approximate the optimal solution as follows:

$$\widetilde{x} = \sqrt{\lambda_1}u_1.$$

Another method is randomization. Instead of problem (19), consider the stochastic problem

$$\begin{aligned} \mathbb{E}_{\xi \sim N(0,X)}[\xi^{\mathsf{T}}C\xi] &\to \min_{X \in \mathbb{S}^n,\, X \succeq 0}, \\ \mathbb{E}_{\xi \sim N(0,X)}[\xi^{\mathsf{T}}H\xi] &\geqslant a, \\ \mathbb{E}_{\xi \sim N(0,X)}[\xi^{\mathsf{T}}G\xi] &= b. \end{aligned} \tag{20}$$

[2] $xx^{\mathsf{T}} = X = u\lambda u^{\mathsf{T}}$.

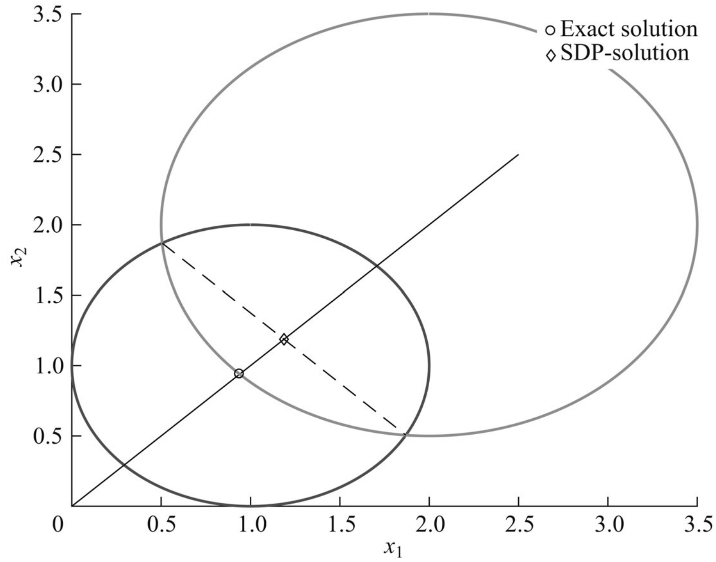

Fig. 1. Admissible domain and convex hull.

Since $\mathbb{E}[\xi^{\mathsf{T}}\xi] = X$, problem (20) is equivalent to problem (19) (up to the set of constraints).

Unfortunately, the approximate solution constructed in this way is often inadmissible for the original problem. Hence, the approximate solution has to be projected into the admissible domain.

An example of inexact relaxation

As it has been mentioned, the SD relaxation is not always exact. The next example in space $\mathbb{R}^2$ well illustrates this fact. Consider the problem

$$\min_x x_1,$$
$$(x_1 - a_1)^2 + (x_2 - a_2)^2 \leqslant r_1^2,$$
$$(x_1 - b_1)^2 + (x_2 - b_2)^2 \geqslant r_2^2,$$
$$x_1 = x_2,$$

where $a = (a_1, a_2)$, $b = (b_1, b_2)$ and r_1, r_2 are the centers and radii of two circles.

Introduce the vector $\widehat{x} = (x_1, x_2, 1)^{\mathsf{T}}$ and write the SD relaxation of this problem. For expressing the objective function and constraints through $\widehat{x}$, construct special matrices and use the trace operator as follows:

$$C = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = Tr(C\widehat{x}\widehat{x}^{\mathsf{T}}),$$

$$A = \begin{bmatrix} I & -a \\ -a^{\mathsf{T}} & a^{\mathsf{T}}a - r_1^2 \end{bmatrix} \Rightarrow (x-a)^{\mathsf{T}}(x-a) \leqslant 0 \to Tr(A\widehat{x}\widehat{x}^{\mathsf{T}}) \leqslant 0,$$

$$B = \begin{bmatrix} I & -b \\ -b^{\mathsf{T}} & b^{\mathsf{T}}b - r_2^2 \end{bmatrix} \Rightarrow (x-b)^{\mathsf{T}}(x-b) \geqslant 0 \to Tr(B\widehat{x}\widehat{x}^{\mathsf{T}}) \geqslant 0,$$

$$D = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & -0.5 \\ 0.5 & -0.5 & 0 \end{bmatrix} \Rightarrow x_1 - x_2 = 0 \to Tr(D\widehat{x}\widehat{x}^{\mathsf{T}}) = 0.$$

Let $X = \widehat{x}\widehat{x}^{\mathsf{T}}$; then rank $(X) = 1$. Write the SD relaxation

$$\begin{gathered}\min_{X \in \mathbb{S}^3} Tr(CX),\\ Tr(AX) \leqslant 0,\\ Tr(BX) \geqslant 0,\\ Tr(DX) = 0,\\ X \succeq 0.\end{gathered}$$

The example in Fig. 1 corresponds to the parameter values

$$a = (1;1), \quad b = (2;2), \quad r_1 = 1, \quad r_2 = 1.5.$$

The optimal solution can be easily found without solving the problem. The solution of the SD relaxation will differ from the analytical counterpart and its rank will exceed 1.

3.2. Equivalent Relaxations

In this section, three equivalent relaxations—the semidefinite, chordal and conic ones—will be briefly considered. Their equivalence conditions and the corresponding theorems were established in [10]. For a deeper understanding of the AC formulation and its solution using convex relations, we refer to [11–13]. First, discuss the key idea that underlies the three relaxations.

In problem (2) the goal variables are the voltages at each node $V \in \mathbb{C}^N$. Consider an illustrative example. Let a network be specified by a graph in Fig. 2. Make the change of variable $W = VV^{\mathsf{H}}$:

$$\begin{gathered}W_{jj} = |V_j|^2, \quad \forall j \in N,\\ W_{jk} = V_j V_k^{\mathsf{H}}, \quad \forall (j,k) \in E.\end{gathered}$$

Thus, the matrix W is a partially filled Hermitian matrix, and its pattern (the filled part) corresponds to the network graph G. The matrix of this graph has the form

$$W = \begin{pmatrix} W_{11} & W_{12} & - & - & - & - & W_{17}\\ W_{21} & W_{22} & W_{23} & - & - & W_{26} & -\\ - & W_{32} & W_{33} & W_{34} & W_{35} & - & -\\ - & - & W_{43} & W_{44} & W_{45} & - & -\\ - & - & W_{53} & W_{54} & W_{55} & W_{56} & -\\ - & W_{62} & - & - & W_{65} & W_{66} & W_{67}\\ W_{71} & - & - & - & - & W_{76} & W_{77} \end{pmatrix}.$$

Here dash indicates that an element is not defined, i.e., the graph does not contain a corresponding edge. A method for augmenting a partially filled matrix to a complete matrix of the same definiteness and rank was suggested in [14, 15].

Now, the original problem can be written in terms of the new variables—the elements of the matrix W. If rank $(W) = 1$, then the elements of the vector V can be uniquely restored. Problem (2)

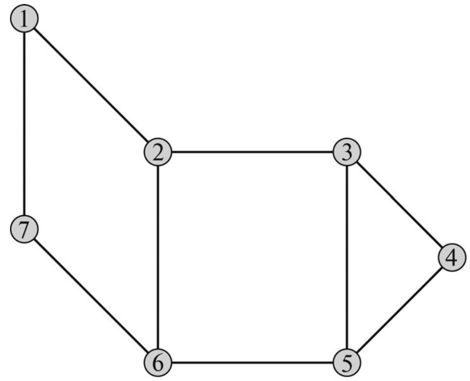

Fig. 2. Network graph.

takes the form

$$\sum_{j \in N} \sum_{k:(j,k) \in E} \Re \left(W_{jj} - W_{jk}\right) Y_{jk}^{\mathsf{H}} \to \min_{W},$$

$$\underline{s}_j \leqslant \sum_{k:(j,k) \in E} \left(W_{jj} - W_{jk}\right) Y_{jk}^{\mathsf{H}} \leqslant \overline{s}_j, \quad \forall j \in N,$$

$$\underline{V}_j^2 \leqslant W_{jj} \leqslant \overline{V}_j^2, \quad \forall j \in N,$$

$$W \succeq 0,$$

$$\operatorname{rank}(W) = 1.$$

3.2.1. Semidefinite relaxation. This relaxation seems elementary by its idea. The nonconvex constraint rank (W) is simply eliminated, and the problem is solved without it. If the resulting solution W^* has rank 1, then the optimal mode V^* can be restored, which guarantees the exactness of this relaxation. Otherwise the semidefinite relaxation is inexact. Hereinafter, the exactness of any relaxation will be comprehended as the possibility of restoring the original solution: if the resulting solution is a nonnegative definite matrix of rank 1, then the solution can be restored and the relaxation is exact. If at least one of the two conditions breaks, then the solution cannot be restored and the relaxation is inexact. A possible formulation of the relaxed problem is as follows:

$$f(W) \to \min_{W},$$

$$\underline{s}_j \leqslant \sum_{k:(j,k) \in E} (W_{jj} - W_{jk}) Y_{jk}^{\mathsf{H}} \leqslant \overline{s}_j,$$

$$(\underline{V}_j)^2 \leqslant W_{jj} \leqslant (\overline{V}_j)^2,$$

$$W \succeq 0.$$

3.2.2. Chordal relaxation. This relaxation, like the next one, is less intuitive but has an obvious advantage for large sparse networks. The key idea is that the original graph of a power network is replaced by its chordal extension [16]. A chordal graph is a graph in which any cycle of length 4 and greater has a chord connecting nonadjacent vertices. Hence, the chordal extension of an original graph is obtained by supplementing it with additional edges. (An example of the chordal extension of a graph can be seen in Fig. 3.) Then, instead of checking the definiteness of the entire matrix, it suffices to perform such checks only for some nonsparse submatrices of considerably smaller dimensions that correspond to the maximal cliques of the chordal graph. A maximal clique is a

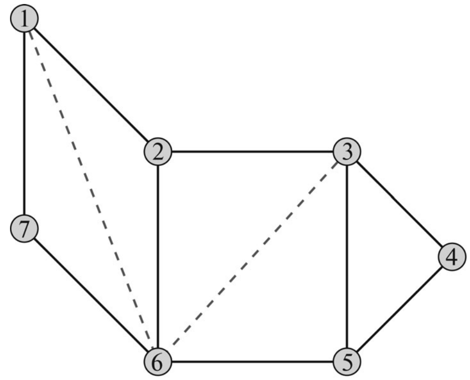

Fig. 3. Chordal extension of graph.

complete subgraph of an original graph that cannot be extended by including one more adjacent node. In other words, a maximal clique is not entirely contained in any other clique of a given graph.

The chordal extension of the graph has the matrix

$$W_{ch} = \begin{pmatrix} W_{11} & W_{12} & - & - & - & \mathbf{W_{16}} & W_{17} \\ W_{21} & W_{22} & W_{23} & - & - & W_{26} & - \\ - & W_{32} & W_{33} & W_{34} & W_{35} & \mathbf{W_{36}} & - \\ - & - & W_{43} & W_{44} & W_{45} & - & - \\ - & - & W_{53} & W_{54} & W_{55} & W_{56} & - \\ \mathbf{W_{61}} & W_{62} & \mathbf{W_{63}} & - & W_{65} & W_{66} & W_{67} \\ W_{71} & - & - & - & - & W_{76} & W_{77} \end{pmatrix}.$$

The chordal extension of the original graph includes new edges, and hence the matrix W must contain new elements corresponding to the new edges. Denote by W_{ch} the partially filled matrix based on the chordal extension.

The resulting chordal graph has five maximal cliques—$\{1,6,7\}$, $\{1,2,6\}$, $\{2,3,6\}$, $\{3,5,6\}$, and $\{3,4,5\}$. Therefore, instead of the entire matrix W_{ch} the definiteness of only five submatrices should be checked. For example, the submatrix associated with the clique $\{3,4,5\}$ is

$$W_{ch} = \begin{pmatrix} W_{33} & W_{34} & W_{35} \\ W_{43} & W_{44} & W_{45} \\ W_{53} & W_{54} & W_{55} \end{pmatrix}.$$

Consider the chordal relaxation of the original problem:

$$f(W_{ch}) \to \min_{W_{ch}},$$
$$\underline{s}_j \leqslant \sum_{k:(j,k)\in E} ([W_{ch}]_{jj} - [W_{ch}]_{jk})\, Y_{jk}^{\mathsf{H}} \leqslant \overline{s}_j,$$
$$(\underline{V}_j)^2 \leqslant [W_{ch}]_{jj} \leqslant (\overline{V}_j)^2,$$
$$W_{ch}(C) \succeq 0,\ \forall C : C\text{is maximal clique of graph } W_{ch}.$$

The conditions $W \succeq 0$ and $\operatorname{rank}(W) = 1$ are replaced by $W_{ch}(C) \succeq 0$ and $\operatorname{rank}(W_{ch}(C)) = 1$, respectively (the latter being relaxed to a nonconvex condition), over all maximal cliques of the chordal extension of the original graph G. If the resulting optimal solution of the relaxed problem satisfies these conditions, then the original partially filled matrix can be uniquely augmented to a complete nonnegative definite matrix of rank 1 too. In other words, the optimal mode V^* can be uniquely restored.

3.2.3. Conic relaxation. This relaxation is similar to the previous one. But, instead of cliques, all edges $e = (j, k) \in E$ are considered and the definiteness of the corresponding submatrices is checked. For some edge (j, k), the matrix has the general form

$$W(e) = \begin{pmatrix} W_{jj} & W_{jk} \\ W_{kj} & W_{kk} \end{pmatrix}.$$

In this relaxation, $\operatorname{rank}(W) = 1$ and $W \succeq 0$ are replaced by $\operatorname{rank}(W(e)) = 1$ and $W(e) \succeq 0$, where $e = (j, k) \in E$, since $[W_{jj}][W_{kk}] \geqslant |W_{jk}|^2$, $\forall e = (j, k) \in E$, in accordance with the following considerations:

$$\begin{gathered}
W_{jk} = V_j V_k^{\mathsf{H}}, \\
W_{jk} W_{jk}^{\mathsf{H}} = V_j V_k^{\mathsf{H}} V_j^{\mathsf{H}} V_k, \\
|W_{jk}|^2 = W_{jj} W_{kk}, \\
|W_{jk}|^2 \leqslant W_{jj} W_{kk}, \\
f(W) \to \min_{W}, \\
\underline{s}_j \leqslant \sum_{k:(j,k)\in E} (W_{jj} - W_{jk}) Y_{jk}^{\mathsf{H}} \leqslant \overline{s}_j, \\
(\underline{V}_j)^2 \leqslant W_{jj} \leqslant (\overline{V}_j)^2, \\
W(e) \succeq 0, \quad \forall e \in E.
\end{gathered}$$

The condition $\operatorname{rank}(W) = 1$ is replaced by the conditions $\operatorname{rank}(W(e)) = 1, \forall e = (j, k) \in E$. The resulting solution must satisfy the nonnegative definiteness and rank conditions for each submatrix corresponding to some edge of the graph. In addition, the cyclic condition

$$\measuredangle W_{n_1,n_2} + \cdots + \measuredangle W_{n_k,n_1} = 0 \bmod 2\pi$$

must hold for any cycle $(n_1, \cdots, n_k)$ in the graph G.

If the three conditions are valid, then the partially filled matrix can be uniquely augmented to a complete matrix with the same properties (the same definiteness and rank), and V^* can be uniquely restored accordingly.

The three relaxations described in paragraphs 3.2.1–3.2.3 were presented in detail in [17, 18].

3.3. Moment-Based Relaxation

This approach was considered in [19, 20]. The method consists in an appropriate reformulation of the original OPF problem (2) using specially designed matrices for obtaining the so-called generalized moment problem. In this case, the OPF problem is transformed into the minimization problem of some convex functional subject to a set of conditions involving nonnegative definite matrices.

Formula (6) can be rewritten in a slightly different way by separating the real and imaginary parts of s_j. Denote by V_j^d and V_j^q the real and imaginary parts of the voltage V, respectively, and partition the conduction matrix $Y_{adm} = G + iB$ by analogy. Then

$$\Re(s_j) = V_j^d \sum_{k=1}^{n} \left(G_{jk}V_k^d - B_{jk}V_k^q\right) + V_j^q \sum_{k=1}^{n} \left(B_{jk}V_k^d + G_{jk}V_k^q\right), \tag{21}$$

$$\Im(s_j) = V_j^d \sum_{k=1}^{n} \left(-B_{jk}V_k^d - G_{jk}V_k^q\right) + V_j^q \sum_{k=1}^{n} \left(G_{jk}V_k^d - B_{jk}V_k^q\right). \tag{22}$$

The amounts of active and reactive generation at the nodes can be easily obtained from (21) and (22):

$$P_j^g = f_{P_j^g}(V^d, V^q) = V_j^d \sum_{k=1}^{n} \left(G_{jk}V_k^d - B_{jk}V_k^q\right) + V_j^q \sum_{k=1}^{n} \left(B_{jk}V_k^d + G_{jk}V_k^q\right) + P_j^l, \tag{23}$$

$$Q_j^g = f_{Q_j^g}(V^d, V^q) = V_j^d \sum_{k=1}^{n} \left(-B_{jk}V_k^d - G_{jk}V_k^q\right) + V_j^q \sum_{k=1}^{n} \left(G_{jk}V_k^d - B_{jk}V_k^q\right) + P_j^l. \tag{24}$$

In addition,

$$|V_j|^2 = f_{V_j}(V^d, V^q) = (V_j^d)^2 + (V_j^q)^2. \tag{25}$$

Now, reformulate the OPF problem in these notations. Consider a special case in which the amount generated at node 1 has to be minimized subject to the voltage and demand constraints for all other nodes:

$$\begin{gathered}
f_{P_1^g} \to \min_V, \\
f_{P_k^g} \geqslant P_k^l, \quad \forall k \in N, \\
f_{Q_k^g} \geqslant Q_k^l, \quad \forall k \in N, \\
(V_k^{\min})^2 \leqslant f_{V_k}(V^d, V^q) \leqslant (V_k^{\max})^2, \quad \forall k \in N, \\
V_1^q = 0.
\end{gathered}$$

Construct the moment-based relaxation from (21)–(25). Introduce a vector x that describes all voltages:

$$x = [V_1^d, \cdots, V_n^d, V_1^q, \cdots, V_n^q]^{\mathsf{T}} \in \mathbb{R}^{2n},$$

where x^α is a monomial of degree $\alpha = [\alpha_1, \cdots, \alpha_{2n}]^{\mathsf{T}}$,

$$x^\alpha = (V_1^d)^{\alpha_1} \cdots (V_n^q)^{\alpha_n}, \quad \sum_{j=1}^{n} \alpha_j = \alpha.$$

Then the polynomial $g(x)$ with the coefficients g_α has the form

$$g(x) = \sum_{\alpha \in \mathbb{N}^{2n}} g_\alpha x^\alpha.$$

The monomials x^α can be replaced by the scalars y_α, which yields the linear functional

$$L_y\{g\} = \sum_{\alpha \in \mathbb{N}^{2n}} g_\alpha y_\alpha.$$

This replacement rests on the degrees of the monomials, i.e., for two nodes the monomial

$$(V_1^d)(V_2^d)^2(V_1^q)(V_2^q)^2$$

turns into y_{1212}. The same procedure is applied to the functionals, e.g.,

$$g(x) = -1 + (V_2^d)^2 + (V_2^q)^2 \to L_y\{g\} = -y_{000} + y_{020} + y_{002}.$$

For obtaining the moment-based relaxation of order γ, introduce the vector of monomials of degree γ given by

$$x_\gamma = \left[1, V_1^d, \cdots, V_n^q, (V_1^d)^2, V_1^d V_2^d, \cdots, (V_n^q)^2, (V_1^d)^3, \cdots, (V_n^q)^\gamma\right]^\mathsf{T}.$$

Determine the moment matrix $M_\gamma(y) = L_y(x_\gamma x_\gamma^\mathsf{T})$ in the following way. First, multiply the column vector x_γ by the row vector x_γ^T. The resulting matrix consists of different monomials that will be replaced by y_α in accordance with the above rule. For example, let $x = [1, x_1, x_2]$ and $\gamma = 1$. In this case,

$$L_y(x_1 x_1^\mathsf{T}) = L_y \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} = \begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix} = M_1(y).$$

The last concept necessary for this relaxation is the localization matrix, which is constructed using the moment matrix for a given functional as follows. If the functional f has degree 2β or $2\beta - 1$, then construct the moment matrix of order $\gamma - \beta$ and multiply each element of this matrix by f, i.e.,

$$M_{\gamma-\beta}(f(y)y) = L_y\left(f(y)x_{\gamma-\beta}x_{\gamma-\beta}^\mathsf{T}\right).$$

For the example above, let $f(x) = 1 + x_1^2 + x_2^2$; then

$$M_1(f(y)y) = L_y \begin{pmatrix} 1 + x_1^2 + x_2^2 & x_1 + x_1^3 + x_1x_2^2 & x_2 + x_1^2x_2 + x_2^3 \\ x_1 + x_1^3 + x_1x_2^2 & x_1^2 + x_1^4 + x_1^2x_2^2 & x_1x_2 + x_1^3x_2 + x_1x_2^3 \\ x_2 + x_1^2x_2 + x_2^3 & x_1x_2 + x_1^3x_2 + x_1x_2^3 & x_2^2 + x_1^2x_2^2 + x_2^4 \end{pmatrix}$$

$$= \begin{pmatrix} y_{00} + y_{20} + y_{02} & y_{10} + y_{30} + y_{12} & y_{01} + y_{21} + y_{03} \\ y_{10} + y_{30} + y_{12} & y_{20} + y_{40} + y_{22} & y_{11} + y_{31} + y_{13} \\ y_{01} + y_{21} + y_{03} & y_{11} + y_{31} + y_{13} & y_{02} + y_{22} + y_{04} \end{pmatrix}.$$

The theoretical issues of the moment method were described in detail in [21, 22].

With all these tools, the moment-based relaxation for the OPF problem can be formulated as follows:

$$\begin{gathered} \min_y L_y\left(f_{P_1^g}\right), \\ M_{\gamma-1}\left(\left(f_{P_k^g} - P_k^l\right)y\right) \succeq 0, \quad \forall k \in N, \\ M_{\gamma-1}\left(\left(f_{Q_k^g} - P_Q^l\right)y\right) \succeq 0, \quad \forall k \in N, \\ M_{\gamma-1}\left(\left(f_{V_k} - V_k^{\min}\right)y\right) \succeq 0, \quad \forall k \in N, \\ M_{\gamma-1}\left((V_k^{\max} - f_{V_k})\,y\right) \succeq 0, \quad \forall k \in N, \\ M_\gamma(y) \succeq 0, \\ y_{00\cdots0} = 1, \\ y_{\cdot p \cdots} = 0, \quad \forall p \geqslant 1. \end{gathered}$$

The condition $y_{\cdot p \dots} = 0$ is equivalent to the condition $V_1^q = 0$. Generally speaking, this condition "determines a reference point" for the voltages (a point means any value from the interval $[0, \gamma]$): any monomial that contains V_1^q is equal to 0.

For higher γ, the problem becomes more complex but the quality of its solution is improved. In practice, this method turns out to be rather difficult due to a considerably increasing number of its variables for higher γ. For example, even in the case of two nodes the second-moment matrix M_2 has dimensions 10×10; in the case of ten nodes, dimensions 210×210 (without consideration of symmetric elements). Real power networks may consist of thousand nodes, which makes the moment-based relaxation extremely computationally intensive and difficult-to-use.

An example of this relaxation and its design procedure was well described in [19].

3.4. QC Relaxation

This method involves the formulations considered above, with the feature that all nonconvex constraints are replaced by their convex hull. In particular, the matter concerns the condition $W_{jk} = V_j V_k^{\mathsf{H}}$, which has been earlier transformed into the two new conditions $\operatorname{rank}(W) = 1$ and $W \succeq 0$. The authors of the QC relaxation [23] suggested using the trigonometric representation for the voltage and also the convex hulls.

In accordance with the trigonometric representation of the complex voltages $V = v(\cos(\theta) + i\sin(\theta))$, write the matrix $W = VV^{\mathsf{H}}$ in the following way:

$$
\begin{gathered}
W_{jk} = V_j V_k^{\mathsf{H}} = v_j v_k(\cos(\theta_j) + i\sin(\theta_j))(\cos(\theta_k) + i\sin(\theta_k)), \quad \forall (j,k) \in E, \\
\Re(W_{jk}) = v_j v_k \cos(\theta_j - \theta_k), \quad \forall (j,k) \in E, \\
\Im(W_{jk}) = v_j v_k \sin(\theta_j - \theta_k), \quad \forall (j,k) \in E, \\
W_{jj} = v_j^2, \quad \forall i \in E.
\end{gathered}
$$

Introduce the convex hulls of the functions x^2, xy, $\cos(x)$, and $\sin(x)$ on the intervals $[x^l, x^u]$ and $[y^l, y^u]$:

$$
conv(x^2) = \begin{cases} \check{x} \geqslant x^2 \\ \check{x} \leqslant (x^l + x^u)x - x^l x^u, \end{cases}
$$

$$
conv(xy) = \begin{cases} \check{xy} \geqslant x^l y + y^l x - x^l y^l \\ \check{xy} \geqslant x^u y + y^u x - x^u y^u \\ \check{xy} \leqslant x^l y + y^u x - x^l y^u \\ \check{xy} \leqslant x^u y + y^l x - x^u y^l, \end{cases}
$$

$$
conv(\sin(\theta)) = \begin{cases} \check{s\theta} \leqslant \cos\left(\dfrac{\theta^u}{2}\right)\left(\theta - \dfrac{\theta^u}{2}\right) + \sin\left(\dfrac{\theta^u}{2}\right) \\ \check{s\theta} \geqslant \cos\left(\dfrac{\theta^u}{2}\right)\left(\theta + \dfrac{\theta^u}{2}\right) - \sin\left(\dfrac{\theta^u}{2}\right), \end{cases}
$$

$$
conv(\cos(\theta)) = \begin{cases} \check{c\theta} \leqslant 1 - \dfrac{1 - \cos(\theta^u)}{(\theta^u)^2}\theta^2 \\ \check{c\theta} \geqslant \cos(\theta^u). \end{cases}
$$

(See the details in [24].) Using these formulas, rewrite the condition $W = V_j V_k^{\mathsf{H}}$ as

$$
\begin{gathered}
W_{jj} = v_j^2 \to conv(v_j^2)\Re(W_{jk}) = v_j v_k \cos(\theta_j - \theta_k) \to conv(conv(v_j v_k)conv(\cos(\theta_j - \theta_k)))\Im(W_{jk}) \\
= v_j v_k \sin(\theta_j - \theta_k) \to conv(conv(v_j v_k)conv(\sin(\theta_j - \theta_k))).
\end{gathered}
$$

As was claimed by the authors, the method is not dominated by the SD relaxation and, in turn, dominates the SOCP relaxation.

4. NUMERICAL IMPLEMENTATION ON TEST EXAMPLE

A simple network of four nodes (see Fig. 4) was selected as a test example, with the representation in MATPOWER [25]. The optimization problems were solved in MatLab, using the CVX library [26] in combination with the Mosek solver [27].

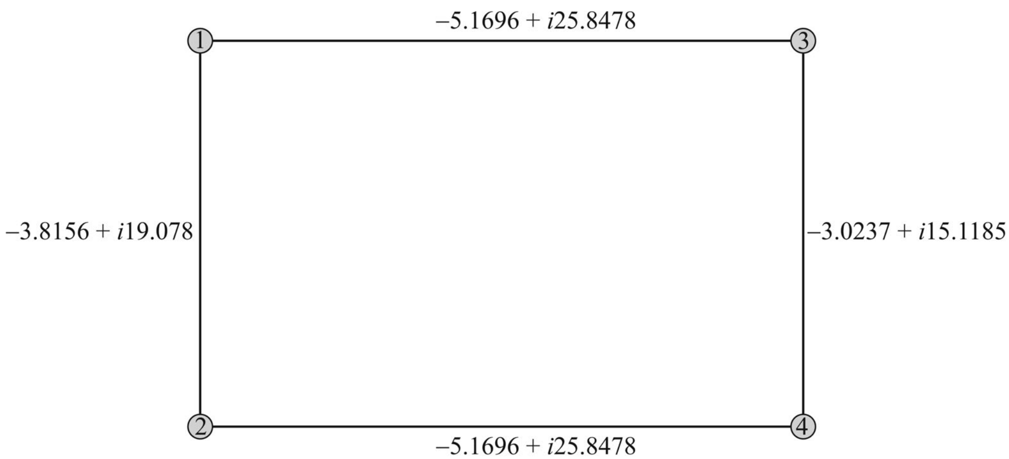

Fig. 4. Network of four nodes: graph with conduction values of power lines.

This network has two generators at nodes 1 and 4; nodes 2 and 3 are pure consumers. The graph is not directed: if there exists an edge (j, k), then the edge (k, j) is also present. The power demands at the nodes of this graph are combined in Table 1,

$$N = \{1, 2, 3, 4\} \text{ are nodes},$$

$$E = \{(1, 2), (1, 3), (2, 4), (3, 4)\} \text{ are edges (power lines)},$$

$$\mathbb{G} = \{1, 4\} \text{ are generators}.$$

The amount of active power generation at node 4 does not exceed 2.0 p.u.,[3] that is, $\overline{P}_4^g \leqslant 2.0$. The values of $\underline{V_j}$ and $\overline{V_j}$ are set equal to 0.9 and 1.1, respectively.

In the elementary formulation, the problem is to find the amounts of power generation that satisfy a given demand at each available generator under the existing voltage and capacity constraints

Table 1. Network consumption

Node	Consumption $P_j^l + iQ_j^l$
1	$0.5 + i0.31$
2	$1.7 + i1.05$
3	$2 + i1.23$
4	$0.8 + i0.5$
Total consumption	$5 + i3.09$

[3] In the per-unit analysis used for power systems, all quantities (complex voltages, powers) are measured as fractions of a defined base unit quantity. Hereinafter, the base unit quantity for power is 100. Note that the voltages are calculated directly in the per-unit system.

of the generators:

$$\min_V \sum_{j\in\mathbb{G}} \Re(s_j) + P_j^l, \tag{26}$$

$$\underline{V_j} \leqslant |V_j| \leqslant \overline{V_j}, \quad \forall j = \overline{1,n}, \tag{27}$$

$$\Re(s_j) + P_j^l = 0, \quad \forall j \notin \mathbb{G}, \tag{28}$$

$$\Im(s_j) + Q_j^l = 0, \quad \forall j \notin \mathbb{G},$$

$$\Re(s_4) + P_4^l \leqslant \overline{P}_4^g.$$

In problem (26), the objective function is to minimize the total amount of active power generated at all generators subject to the voltage constraint (27) at each node and the zero generation constraint (28) at the consumer nodes.

4.1. Semidefinite Relaxation

Using the change of variables $W = VV^{\mathsf{H}}$, write the original problem in the new variables—the elements of the Hermitian matrix W—as follows:

$$\min_W \sum_{j\in\mathbb{G}} \Re(s_j) + P_j^l,$$

$$\underline{V_j}^2 \leqslant W_{jj} \leqslant \overline{V_j}^2, \quad \forall j = \overline{1,n},$$

$$\Re(s_j) + P_j^l = 0, \quad \forall j \notin \mathbb{G},$$

$$\Im(s_j) + Q_j^l = 0, \quad \forall j \notin \mathbb{G},$$

$$\Re(s_4) + P_4^l \leqslant \overline{P}_4^g,$$

$$W \succeq 0.$$

For this example, the SD relaxation is exact. The matrix W has a single nonzero eigenvalue ($\operatorname{rank}(W) = 1$), which is positive ($W \succeq 0$). Hence, the original vector V can be restored by the

Table 2. Optimal modes of relaxation

	Voltage ($\|V\|\measuredangle V$)		
Node	SDP	Chordal	SOCP
1	$1.0488\measuredangle 1.3843°$	$1.0488\measuredangle 1.3839°$	$1.0488\measuredangle 1.378°$
2	$1.0183\measuredangle -1.1234°$	$1.0183\measuredangle -1.1236°$	$1.0183\measuredangle -1.121°$
3	$1.0094\measuredangle -1.3536°$	$1.0094\measuredangle -1.3539°$	$1.0094\measuredangle -1.3575$
4	$1.0476\measuredangle 0°$	$1.0476\measuredangle 0°$	$1.0476\measuredangle 0°$

Table 3. Power generation in optimal mode

	Amount generated $P_j^g + iQ_j^g$		
Node	SDP	Chordal	SOCP
1	$3.0447 + i1.6009$	$3.0447 + i1.6016$	$3.0447 + i1.6007$
2	0	0	0
3	0	0	0
4	$2 + i1.721$	$2 + i1.7203$	$2 + i1.7212$
Total amount generated	$5.0447 + i3.3219$	$5.0447 + i3.3219$	$5.0447 + 3.3219i$

formula $V = \sqrt{\lambda} h$, where λ denotes the nonzero eigenvalue and h is the corresponding eigenvector of the matrix W. The optimal mode and also the optimal amount generated are shown in Tables 2 and 3, respectively.

The active power losses in the network are $0.0447\,p.u.$

4.2. Chordal Relaxation

The graph of this network is not chordal. Hence, first its chordal extension must be found. One of the three possible extensions is demonstrated in Fig. 5.

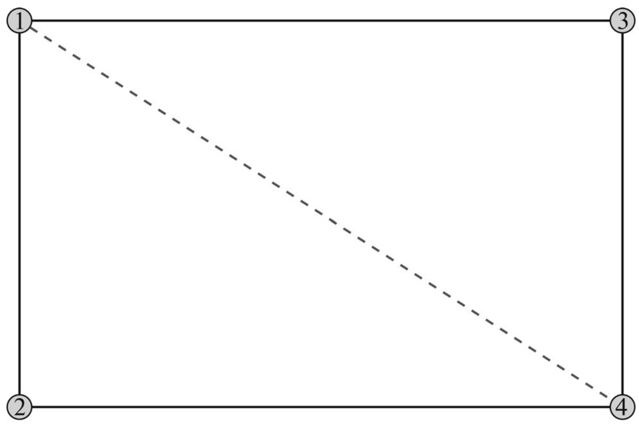

Fig. 5. Network of four nodes: chordal extension of graph.

This graph has two maximal cliques—$C_1 = \{1, 2, 4\}$ and $C_2 = \{1, 3, 4\}$. The condition $W \succeq 0$ is simplified to $W_{C_1} \succeq 0$ and $W_{C_2} \succeq 0$,

$$\begin{gathered}
\min_W \sum_{j \in \mathbb{G}} \Re(s_j) + P_j^l, \\
\underline{V_j}^2 \leqslant W_{jj} \leqslant \overline{V_j}^2, \quad \forall j = \overline{1, n}, \\
\Re(s_j) + P_j^l = 0, \quad \forall j \notin \mathbb{G}, \\
\Im(s_j) + Q_j^l = 0, \quad \forall j \notin \mathbb{G}, \\
\Re(s_4) + P_4^l \leqslant \overline{P}_4^g, \\
W_{C_1} \succeq 0, \\
W_{C_2} \succeq 0.
\end{gathered}$$

Solving this problem, we obtained a certain partially filled matrix W. Its submatrices corresponding to the cliques C_1 and C_2 are nonnegative definite and have rank 1. Hence, the chordal relaxation is exact and the partially filled matrix W can be augmented to a complete nonnegative definite matrix of rank 1. The restoration procedure yielded the mode presented in Table 2. This mode is similar to the semidefinite relaxation, but with slightly different angles. The optimal amount of power generated differs in the reactive component only. The data are combined in Table 3.

4.3. Conic Relaxation

In this case, no additional transformations of the graph are required. The condition $W \succeq 0$ is replaced by $W(e) \succeq 0$, $\forall (j, k) \in \mathbb{E}$. In other words, for each graph edge, construct a matrix of dimensions (2×2) of the form

$$W(e) = \begin{bmatrix} W_{jj} & W_{jk} \\ W_{kj} & W_{kk} \end{bmatrix}.$$

Recall that the matrix W is Hermitian, which implies $W_{jk} = W_{kj}^{\mathsf{H}}$. In other respects, the problem has a similar presentation to the case of chordal relaxation,

$$\begin{gathered} \min_W \sum_{j \in \mathbb{G}} \Re(s_j) + P_j^l, \\ \underline{V_j}^2 \leqslant W_{jj} \leqslant \overline{V_j}^2, \quad \forall j = \overline{1, n}, \\ \Re(s_j) + P_j^l = 0, \quad \forall j \notin \mathbb{G}, \\ \Im(s_j) + Q_j^l = 0, \quad \forall j \notin \mathbb{G}, \\ \Re(s_4) + P_4^l \leqslant \overline{P}_4^g, \\ W(e) \succeq 0, \forall e = (j,k) \in E. \end{gathered}$$

All submatrices corresponding to the graph edges are nonnegative definite and have rank 1. Also the cyclicity condition must be checked.

The direction of movement being neglected, the graph contains a single cycle— $((1,3),(3,4),(4,2),(2,1))$. The optimal solution of the relaxed problem is

$$\begin{pmatrix} 1.1000 + 0i & 1.0670 + 0.0468i & 1.0574 + 0.0505i & - \\ 1.0670 - 0.0468i & 1.0369 + 0.0000i & - & 1.0665 - 0.0209i \\ 1.0574 - 0.0505i & - & 1.0188 + 0{,}0000i & 1.0571 - 0.0251i \\ - & 1.0665 + 0.0209i & 1.0571 + 0.0251i & 1.0975 + 0i \end{pmatrix}.$$

As before, dash indicates that an element is not defined. Check the cyclicity condition:

$$\Im(W(1,3)) + \Im(W(3,4)) + \Im(W(4,2)) + \Im(W(2,1)) \approx 0 \bmod 2\pi.$$

Thus, the relaxation is exact and the optimal mode can be restored.

The values $|V_j|$ for all nodes are restored by the formula

$$|V_j| = \sqrt{W_{jj}}.$$

The angles are restored from the original matrix W^* using the submatrix $W(e)$ for the edges. Let the generator at node 4 be a slack bus,[4] i.e., $\measuredangle V_4 = 0$. The angle for node k can be restored from the angle for the adjacent angle j as follows:

$$\begin{gathered} W_{kj} = V_k V_j^{\mathsf{H}}, \\ |W_{kj}| = |V_k||V_j|, \\ \measuredangle W_{kj} = e^{i(\measuredangle V_k - \measuredangle V_j)}, \\ \measuredangle V_k = \measuredangle W_{kj} + \measuredangle V_j. \end{gathered}$$

[4] A slack bus is a special node used in optimal power flow problems for balancing the active and reactive components of power networks.

Restoring the voltages in this way, we obtained the same optimal mode as in the semidefinite and chordal relaxations; see Tables 2 and 3.

Thus, three equivalent relaxations of AC OPF have been considered in this section. For a simple example, it has been demonstrated how to write the relaxed versions of the original problem in a form suitable for convex solvers (e.g., CVX). Moreover, it has been shown how to restore the optimal mode from the solution of the relaxed problem of rank 1. The procedure seems rather easy and obvious for the SD relaxation; however, the things are no so trivial for the conic relaxation.

For the simple example (in which the network is not a tree), the three relaxations have turned out to be exact; the resulting solutions and restored modes have been slightly different from each other. Note that all differences have been observed in reactive generation only, more specifically, in the distribution of some optimal amount of reactive generation between two generators, causing different voltage angles in the optimal mode.

In accordance with the experimental evidence [10], for high-dimensional problems the conic relaxation is preferable to the chordal one in terms of computational time. For a network of almost 2400 nodes, its computational time was by 6.5 times less in comparison with the chordal relaxation. And the authors even did not execute the SD relaxation for that network.

5. CONCLUSIONS

The optimal operating mode of a power network is determined using different approaches but the exact mode corresponds to the solution of the AC OPF problem. This is achieved by adding different engineering constraints for an accurate consideration of all specifics of a given power network. This explains the crucial importance of the problem for the industry.

The major difficulty of the AC formulation is its nonconvexity, which creates obstacles on the way towards fast and exact solution. This difficulty can be eliminated using convex relaxations: the original set of admissible solutions is replaced by its convex hull, and the problem is solved on the latter. Note that the original physical structure of the problem is retained. Unfortunately, relaxations may turn out to be inexact. As of today, the exactness conditions have been established for the tree networks only. Moreover, real networks may have cycles.

In this paper, five different relaxations—the semidefinite (SDP), chordal, conic (SOCP), moment-based and QC relaxations—have been considered. The application of the first three relaxations has been described step-by-step on a simple example. Each of them has certain advantages and shortcomings. For instance, the semidefinite relaxation is very easy to understand and use. The chordal relaxation requires designing the chordal extension of the network graph and obtaining the maximal cliques, which is also a nontrivial problem; however, it is solved once for a given network. The transition from the complete network matrix to the submatrices of its cliques allows using the network sparsity without considerable accuracy losses in comparison with the semidefinite relaxation. The conic relaxation also utilizes the network sparsity and does not require any additional transformations of the graph but it is less accurate than the semidefinite and chordal ones. The accuracy of the moment-based is increasing with its order but a high-order relaxation introduces a huge number of new variables; in real networks, this may dramatically affect the computational time or even make the problem infeasible. The QC relaxation achieves the accuracy of the semidefinite without imposing the rank condition, i.e., the voltages can be always restored. In addition, the accuracy of this relaxation is similar to that of the SD relaxation.

In this paper, a simple formulation of the OPF problem with classical generators and a given demand has been considered. Generally speaking, the real problem is far difficult due to additional engineering constraints. First, in the recent years the share of alternative generators has been significantly increased. They supply very cheap power but suffer from high instability. Therefore, the attempts to add renewable sources into the problem cause various uncertainty. Besides renewable generation, another considerable uncertain factor is the demand, which also represents a random variable. The classical formulation with added uncertainty leads to the stochastic optimal power flow problem. The solution of this problem should avoid excessive conservatism (which often occurs in stochastic optimization), since even small improvements gain important savings. Second, many different criteria of power security or redundancy have to be considered in practice, e.g., the $(N-1)$ security criterion.[5]

Nevertheless, the main issue concerns the conditions under which the relaxations preserve their exactness for mixed networks. This issue still remains open even in the simple AC formulation without stochastics and additional engineering constraints. Small data changes can make the problem infeasible or the resulting solution can have rank above 1, meaning that the optimal mode is unrestorable. In addition to the difficulties connected with nonconvexity and inexactness of the relaxations, the problem can be infeasible due to high dimension. For example, the Russian power system includes about 9000 nodes; hence, the SD relaxation will involve a symmetric variable matrix of dimensions (9000×9000). The semidefinite problems of such a dimension will be almost unsolvable or the solution time will exceed all available limits. Of course, the chordal and conic relaxations can be used to reduce the dimension owing to network sparsity. But this will be insufficient or the chordal and conic relaxations will be inexact. Thus, numerical methods to parallel the problem are required.

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