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Sufficient Relative Minimum Conditions in the Optimal Control Problem for Quasilinear Stochastic Systems

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Abstract—We consider the optimal control problem for quasilinear stochastic systems with continuous time whose coefficients have a generally non-linear dependence on the program control. We establish sufficient conditions for a strong and weak relative minimum. We give examples of using the resulting conditions for constructing optimal control in a nonlinear one-dimensional problem and in a two-dimensional linear problem with information constraints and analyze the possible results.

Keywords: stochastic optimal control; quasilinear dynamical system; nonlinear dynamic system

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1. INTRODUCTION

We are developing an extension for stochastic control systems of the Lagrange–Pontryagin method proposed by V.F. Krotov and V.I. Gurman for deterministic systems [1, 2]. This second-order method is based on the extension principle [3] and the procedure for constructing a Krotov function in the linear-quadratic form. These approaches were also significantly developed in the works of V.A. Baturin [4, 5], where, among other things, effective numerical algorithms for strong and weak improvement were constructed.

In this work, using the Lagrange–Pontryagin method we formulate sufficient conditions for a strong and weak relative minimum in the optimization problem for quasilinear stochastic systems with continuous time, whose coefficients have, in the general case, nonlinear dependencies on the program control. A special case of this problem is the optimization problem for control strategies with incomplete feedback for linear systems with multiplicative noise [6, Section 6]. This work as a whole continues the studies initiated in [6], where necessary optimality conditions for the class of problems in question were obtained.

2. PROBLEM SETTING

The control process is described by the Ito equation

$$dx(t) = [A(t,u(t))x(t) + B(t,u(t))]\,dt + \sum_{l=1}^{\nu}\left[G^{(l)}(t,u(t))x(t) + C^{(l)}(t,u(t))\right]dw_l(t), \qquad x(t_0) = x_0, \tag{1}$$

where $t \in T = [t_0; t_1]$ is time; $x(t) \in R^n$, $u(t) \in U \subset R^m$ are the state and control vectors of the system at time t; $w(\cdot)$ is the ν-dimensional standard Wiener process; $t \to u(t) : T \to R^m$ is a bounded Borel function on T, the set U is open. Here $(t,u) \to A(t,u) : T \times R^m \to R^{n\times n}$, $(t,u) \to$

$B(t,u) : T \times R^m \to R^n$, $(t,u) \to G^{(l)}(t,u) : T \times R^m \to R^{n\times n}$, $(t,u) \to C^{(l)}(t,u) : T \times R^m \to R^n$, $l = \overline{1,\nu}$, are functions on $T \times R^m$ that are continuous with respect to t and twice continuously differentiable in u. The random vector x_0 has a given distribution, its expectation $m_0 \in R^n$ and the covariance matrix $K_0 \in R^{n\times n}$ are known.

We denote by $\mathcal{D}_x$ the set of admissible control processes $z = (x(\cdot), u(\cdot))$ satisfying the following condition: for a given control u the random process x is a solution of Eq. (1).

For the process $z \in \mathcal{D}_x$ we define the control performance functional

$$z \to J_x(z) = \mathbb{E}\int_{t_0}^{t_1} \Big(x^{\mathrm{T}}(t)D(t,u(t))x(t) + S^{\mathrm{T}}(t,u(t))x(t) + E(t,u(t))\Big)\,dt + \mathbb{E}\Big[x^{\mathrm{T}}(t_1)Qx(t_1)\Big] : \mathcal{D}_x \to R^1, \tag{2}$$

where $Q \in R^{n\times n}$, and $(t,u) \to D(t,u) : T \times R^m \to R^{n\times n}$, $(t,u) \to S(t,u) : T \times R^m \to R^n$, $(t,u) \to E(t,u) : T \times R^m \to R^1$ are continuous in t and twice continuously differentiable in u functions on $T \times R^m$, and for each $(t,u) \in T \times R^m$ conditions $D(t,u) \geqslant 0$, $Q \geqslant 0$ are hold. Hereinafter, matrices of quadratic forms are assumed to be symmetric. The control goal is to minimize functional (2) on the set $\mathcal{D}_x$.

3. DETERMINISTIC OPTIMAL CONTROL PROBLEM

There are various approaches to the study of the formulated linear-quadratic optimization problem for a stochastic system. One of them is the transition from the stochastic optimal control problem for a random process $x(t)$ to the deterministic optimal control problem for its moments. This approach is as follows.

We write the equations for the functions $t \to m(t) : T \to R^n$, $t \to K(t) : T \to R^{n\times n}$ that have the meaning of the expectation and covariance matrix of the random process $x(t)$ [6, 7],

$$\frac{dm(t)}{dt} = A(t,u(t))m(t) + B(t,u(t)), \tag{3}$$

$$\begin{aligned}\frac{dK(t)}{dt} &= A(t,u(t))K(t) + K(t)A^{\mathrm{T}}(t,u(t)) \\ &+ \sum_{l=1}^{\nu}\Big(G^{(l)}(t,u(t))K(t)G^{(l)^{\mathrm{T}}}(t,u(t)) \\ &+ \Big[C^{(l)}(t,u(t)) + G^{(l)}(t,u(t))m(t)\Big]\Big[C^{(l)}(t,u(t)) + G^{(l)}(t,u(t))m(t)\Big]^{\mathrm{T}}\Big),\end{aligned} \tag{4}$$

integrable with initial conditions

$$m(t_0) = m_0, \quad K(t_0) = K_0. \tag{5}$$

Using the functions m, K, we rewrite linear-quadratic performance functional (2) as

$$J = \int_{t_0}^{t_1} \Big[\mathrm{tr}(D(t,u(t))K(t)) + m^{\mathrm{T}}(t)D(t,u(t))m(t) + S^{\mathrm{T}}(t,u(t))m(t) + E(t,u(t))\Big]\,dt + \mathrm{tr}(QK(t_1)) + m^{\mathrm{T}}(t_1)Qm(t_1). \tag{6}$$

To simplify further calculations, we introduce the notation $N = K + mm^{\mathrm{T}}$ for the matrix of second raw moments and will instead of (4) use the equation

$$\frac{dN(t)}{dt} = A(t,u(t))N(t) + N(t)A^{\mathrm{T}}(t,u(t)) + B(t,u(t))m^{\mathrm{T}}(t) + m(t)B^{\mathrm{T}}(t,u(t)) + \sum_{l=1}^{\nu}\Big(G^{(l)}(t,u(t))N(t)G^{(l)^{\mathrm{T}}}(t,u(t)) + C^{(l)}(t,u(t))m^{\mathrm{T}}(t)G^{(l)^{\mathrm{T}}}(t,u(t)) + G^{(l)}(t,u(t))m(t)C^{(l)^{\mathrm{T}}}(t,u(t)) + C^{(l)}(t,u(t))C^{(l)^{\mathrm{T}}}(t,u(t))\Big)$$

with condition

$$N(t_0) = K_0 + m_0 m_0^{\mathrm{T}}.$$

In this case, formula (6) can be rewritten as

$$J = \int_{t_0}^{t_1}\Big[\mathrm{tr}(D(t,u(t))N(t)) + S^{\mathrm{T}}(t,u(t))m(t) + E(t,u(t))\Big]dt + \mathrm{tr}(QN(t_1)).$$

Thus, we arrive at the following deterministic optimal control problem:

$$\frac{dm(t)}{dt} = A(t,u(t))m(t) + B(t,u(t)), \tag{7}$$

$$\frac{dN(t)}{dt} = A(t,u(t))N(t) + N(t)A^{\mathrm{T}}(t,u(t)) + B(t,u(t))m^{\mathrm{T}}(t) + m(t)B^{\mathrm{T}}(t,u(t)) + \sum_{l=1}^{\nu}\Big(G^{(l)}(t,u(t))N(t)G^{(l)^{\mathrm{T}}}(t,u(t)) + C^{(l)}(t,u(t))m^{\mathrm{T}}(t)G^{(l)^{\mathrm{T}}}(t,u(t)) + G^{(l)}(t,u(t))m(t)C^{(l)^{\mathrm{T}}}(t,u(t)) + C^{(l)}(t,u(t))C^{(l)^{\mathrm{T}}}(t,u(t))\Big), \tag{8}$$

$$m(t_0) = m_0, \quad N(t_0) = K_0 + m_0 m_0^{\mathrm{T}}, \tag{9}$$

$$J = \int_{t_0}^{t_1}\Big[\mathrm{tr}(D(t,u(t))N(t)) + S^{\mathrm{T}}(t,u(t))m(t) + E(t,u(t))\Big]dt + \mathrm{tr}(QN(t_1)). \tag{10}$$

Note that, in this form, the optimization problem is linear in state and nonlinear in control, both in the part of differential constraints and in the part of the performance functional.

We introduce the state vector $y \in R^{n(n+1)}$ such that

$$y = \mathrm{vec}(m, N) = \begin{pmatrix} m_1 \\ \dots \\ m_n \\ N_{11} \\ \dots \\ N_{n1} \\ \dots \\ N_{1n} \\ \dots \\ N_{nn} \end{pmatrix}.$$

Hereinafter, vec denotes an operator that maps a set of its vector or matrix arguments into one vector composed of their columns.

Then problem (7)–(10) takes completely explicit form, linear with respect to state:

$$\frac{dy(t)}{dt} = \tilde{A}(t,u(t))y(t) + \tilde{B}(t,u(t)), \quad y(t_0) = y_0 = \mathrm{vec}(m_0,\ K_0 + m_0 m_0^{\mathrm{T}}), \tag{11}$$

$$J_y(v) = \int\limits_{t_0}^{t_1} \left[\tilde{D}^{\mathrm{T}}(t,u(t))y(t) + E(t,u(t))\right] dt + \tilde{Q}^{\mathrm{T}} y(t_1), \tag{12}$$

where

$$\tilde{A} = \left(\begin{array}{c|c} A & 0 \\ \hline \tilde{A}_1 & \tilde{A}_2 \end{array}\right),$$

$$\tilde{A}_1 = B \oplus B + \sum_{l=1}^{\nu} \left(G^{(l)} \otimes C^{(l)} + C^{(l)} \otimes G^{(l)}\right), \quad \tilde{A}_2 = A \oplus A + \sum_{l=1}^{\nu} \left(G^{(l)} \otimes G^{(l)}\right),$$

$$\tilde{B} = \mathrm{vec}\left(B, \sum_{l=1}^{\nu} C^{(l)} C^{(l)^{\mathrm{T}}}\right), \quad \tilde{D} = \mathrm{vec}(S, D), \quad \tilde{Q} = \mathrm{vec}\,(0, Q)\,.$$

Here, the symbols $\otimes$ and $\oplus$ denote Kronecker product and Kronecker sum respectively ($A \oplus B = A \otimes I_n + I_n \otimes B$), I_n here and below denotes Kronecker identity matrix of size $n \times n$. The symbol v in (12) denotes the pair $(y(\cdot), u(\cdot))$ from the set $\mathcal{D}_y$ of "state-control" pairs such that for a given $u(\cdot)$ function $y(t)$ is the solution of Eq. (11). Since in the latter case, the values of the functionals J_x and J_y on the corresponding pairs z and v coincide, we can reformulate the control goal as follows: we need to minimize functional (12) on the set $\mathcal{D}_y$.

Remark 1. If we have found a solution $v^* = (y^*(\cdot), u^*(\cdot))$ of the resulting minimization problem for J_y on $\mathcal{D}_y$, then the control $u^*(\cdot)$ will be optimal for the initial minimization problem J_x on $\mathcal{D}_x$. However, the corresponding solution $z^* = (x^*(\cdot), u^*(\cdot))$ will not be determined completely, which means that these two tasks, strictly speaking, are not equivalent. Therefore, to work with the solution z^* of the original stochastic problem, for example if it is necessary to simulate the control process, we need to consider Ito equation (1) again, with $u(t) = u^*(t)$.

Remark 2. Since the matrix of second raw moments and square matrices in criterion (6) are symmetric, the state vector y will contain matching components (for example, $y_{2\times n} = N_{n1} = N_{1n} = y_{n\times n+1}$). To avoid considering the same components of the vector y in advance, second raw moments the symmetric vectorization operator and the symmetric Kronecker product [8] should be used to write linear deterministic problem (11), (12). However, in this case it is difficult to write Eq. (11) in matrix form because an ordinary operation of matrix vectorization from [8] is complicated by adding vectors to the list of its arguments.

Let $\overline{v} = (\overline{y}(\cdot), \overline{u}(\cdot))$ be some pair from $\mathcal{D}_y$. We denote by $\overline{\mathcal{D}_y}(\varepsilon)$ a subset of $\mathcal{D}_y$ consisting of pairs $v = (y(\cdot), u(\cdot))$ that satisfy an additional condition

$$\max_{t\in T} |y(t) - \overline{y}(t)| < \varepsilon, \quad \varepsilon > 0,$$

and by $\overline{\mathcal{D}^*_y}(\varepsilon)$ we denote a subset of $\overline{\mathcal{D}_y}(\varepsilon)$ whose elements satisfy the condition

$$\sup_{t\in T} |u(t) - \overline{u}(t)| < \varepsilon, \quad \varepsilon > 0.$$

Definition 1. We will say that functional (12) reaches at $\overline{v} \in \mathcal{D}_y$ a strong relative minimum if there exists $\varepsilon > 0$ such that

$$J_y(\overline{v}) = \inf_{v \in \overline{\mathcal{D}_y(\varepsilon)}} J_y(v),$$

and a weak relative minimum if there exists $\varepsilon > 0$ such that

$$J_y(\overline{v}) = \inf_{v \in \overline{\mathcal{D}^*_y(\varepsilon)}} J_y(v).$$

4. OPTIMALITY CONDITIONS

According to the Lagrange–Pontryagin method [2], let us consider a function $t \to \psi(t) : T \to R^{n(n+1)}$ and write the Hamiltonian

$$H(t, y, \psi, u) = \psi^{\mathrm{T}} \left[\tilde{A}(t, u)y + \tilde{B}(t, u)\right] - \tilde{D}^{\mathrm{T}}(t, u)y(t) - E(t, u). \tag{13}$$

Now Eq. (11) can be rewritten as

$$\frac{dy(t)}{dt} = \frac{\partial H}{\partial \psi}(t, y(t), \psi(t), u(t)).$$

In turn, the adjoint equation

$$\frac{d\psi(t)}{dt} = -\frac{\partial H}{\partial y}(t, y(t), \psi(t), u(t))$$

with the final condition looks as follows:

$$\frac{d\psi(t)}{dt} = -\tilde{A}^{\mathrm{T}}(t, u(t))\psi(t) + \tilde{D}(t, u(t)), \quad \psi(t_1) = -\tilde{Q}. \tag{14}$$

Here, the condition at time t_1 is determined by terminal part (12).

Note that formulas (14) coincide with the system of equations obtained in [6] when developing the necessary optimality conditions by the method of Lyapunov–Lagrange–Krotov functions, if we assume $\psi = -\mathrm{vec}(\lambda, M)$, $\lambda(t) \in R^n$, $M(t) \in R^{n \times n}$. In addition, by augmenting Eqs. (11), (14) with the relation

$$\frac{\partial H}{\partial u}(t, y(t), \psi(t), u(t)) = 0,$$

we can also obtain necessary optimality conditions [6] themselves. We strengthen these necessary conditions with the following definition.

Definition 2. A process $\overline{v} \in \mathcal{D}_y$ is called extremal if it, together with a function $\psi(t)$, satisfies relations (11), (14) and the condition

$$H(t, \overline{y}(t), \psi(t), \overline{u}(t)) = \max_{u \in U} H(t, \overline{y}(t), \psi(t), M(t), u). \tag{15}$$

Following further the Lagrange–Pontryagin method [2], we consider a function $(t, y) \to \varphi(t, y) : T \times R^{n(n+1)} \to R^1$ of the form

$$\varphi(t, y) = \psi^{\mathrm{T}}(t)y + \frac{1}{2}\Delta y^{\mathrm{T}}\Sigma(t)\Delta y,$$

where $t \to \Sigma(t) : T \to R^{n(n+1)\times n(n+1)}$ is some symmetric matrix function consisting of elements $\sigma_{ij}(t)$, $i, j = \overline{1, n(n+1)}$, $\Delta y = y - \overline{y}$.

Using the function φ we form structures

$$R(t,y,u) = \left(\frac{\partial\varphi(t,y)}{\partial y}\right)^{\mathrm{T}} \left[\tilde{A}(t,u)y + \tilde{B}(t,u)\right] - \tilde{D}^{\mathrm{T}}(t,u)y(t) - E(t,u) + \frac{\partial\varphi(t,y)}{\partial t}, \tag{16}$$

$$F(y) = \varphi(t_1, y) + \tilde{Q}^{\mathrm{T}} y. \tag{17}$$

It has been shown in [1, 2] that the functions φ, R and F constructed in this way reduce the task of finding the minimum of functional (12) to the problem of maximizing $R(t, y, u)$ with respect to variables (y, u) for $t \in T$ and minimizing $F(y)$ at the same time.

The first differentials of functions $(y, u) \to R(t, y, u)$ and $y \to F(y)$ at the extreme point $(\overline{y}(t), \overline{u}(t))$, $t \in T$, can be written as

$$d\overline{R} = (\overline{R}'_y)^{\mathrm{T}} dy + (\overline{R}'_u)^{\mathrm{T}} du, \quad d\overline{F} = (\overline{F}'_y)^{\mathrm{T}} dy,$$

where $\overline{f}$ denotes the value of f on the extremal, and $(\cdot)'_y$ denotes the gradient with respect to the variables y. From the definition of functions R, F and extremality conditions, we have

$$\overline{R}'_y = \overline{H}'_y + \frac{d\psi}{dt} + \Sigma\left(\tilde{A}\overline{y} + \tilde{B} - \overline{y}'\right) = 0, \quad \overline{R}'_u = \overline{H}'_u = 0, \quad \overline{F}'_y = \psi(t_1) + \tilde{Q} = 0,$$

therefore, the first differentials $d\overline{R}$ and $d\overline{F}$ are zero.

The second differential d^2R at the extreme point is most conveniently represented as

$$d^2\overline{R} = -\mathrm{vec}^{\mathrm{T}}(dy, du)\overline{\Omega}\mathrm{vec}(dy, du),$$

where $\overline{\Omega}$ denotes the matrix of second mixed derivatives with the inverse sign in block form

$$\overline{\Omega} = \left(\begin{array}{c|c} -\overline{R}''_{yy} & -\overline{R}''_{yu} \\ \hline (-\overline{R}''_{yu})^{\mathrm{T}} & -\overline{R}''_{uu} \end{array}\right), \tag{18}$$

$$\overline{R}''_{yy}(t) = \Sigma'(t) + \Sigma(t)\tilde{A}(t,\overline{u}(t)) + \tilde{A}^{\mathrm{T}}(t,\overline{u}(t))\Sigma(t), \quad \overline{R}''_{yu} = ||\overline{R}''_{y_i u_j}||,$$

$$\overline{R}''_{y_i u_j}(t) = \sum_{s=1}^{n}\left[\psi_s(t)\frac{\partial\tilde{A}_{si}(t,\overline{u}(t))}{\partial u_j} + \sigma_{is}(t)\left(\sum_{l=1}^{n}\frac{\partial\tilde{A}_{sl}(t,\overline{u}(t))}{\partial u_j}\overline{y}_l(t) + \frac{\partial\tilde{B}_s(t,\overline{u}(t))}{\partial u_j}\right)\right] - \frac{\partial\tilde{D}_i(t,\overline{u}(t))}{\partial u_j},$$

$$\overline{R}''_{uu} = \overline{H}''_{uu} = ||\overline{H}''_{u_j u_k}||,$$

$$\overline{H}''_{u_j u_k}(t) = \psi^{\mathrm{T}}(t)\left[\frac{\partial^2\tilde{A}(t,\overline{u}(t))}{\partial u_j\partial u_k}\overline{y}(t) + \frac{\partial^2\tilde{B}(t,\overline{u}(t))}{\partial u_j\partial u_k}\right] - \left(\frac{\partial^2\tilde{D}(t,\overline{u}(t))}{\partial u_j\partial u_k}\right)^{\mathrm{T}}\overline{y}(t) - \frac{\partial^2 E(t,\overline{u}(t))}{\partial u_j\partial u_k}.$$

In turn, the second differential $d^2\overline{F}$ looks like

$$d^2\overline{F} = (dy)^{\mathrm{T}}\Sigma(t_1)dy.$$

Based on the relations obtained for the second differentials of the optimized functions R and F at the extreme point, we can formulate the following result.

Theorem 1. *Suppose that we have found the extremal $\overline{v} = (\overline{y}(\cdot), \overline{u}(\cdot))$ and the corresponding function $\psi(t)$. If on T there exists a continuously differentiable symmetric matrix function $\Sigma(t)$ such that the matrices $\overline{\Omega}(t)$ and $\Sigma(t_1)$ are positive definite, then functional* (12) *reaches on $\overline{v}$ at least a weak relative minimum.*

Remark 3. If the set U, which defines the geometric constraints on the control, is limited, and the function $u \to H(t, \overline{y}(t), \psi(t), u)$ has a strict maximum on the closure $\overline{U}$ of the set U, then the theorem becomes a sufficient condition for a strong relative minimum.

Remark 4. Detailed proofs of Theorem 1 and Remark 3 are given in [2] in the context of the optimization problem for nonlinear deterministic control systems.

Remark 5. The criterion of positive definiteness for the matrix $\overline{\Omega}$, formulated in Theorem 1, is the positivity of its $n(n+1)+m$ corner minors, which makes it necessary to determine the function $\Sigma(t)$ from $n(n+1)+m$ differential inequalities. If we rewrite the matrix $\overline{\Omega}(t)$ in the form

$$\overline{\Omega}^*(t) = \left(\begin{array}{c|c} -\overline{R}''_{uu} & -\overline{R}''_{yu} \\ \hline (-\overline{R}''_{yu})^{\mathrm{T}} & -\overline{R}''_{yy} \end{array}\right)$$

and supplement the conditions of the theorem with the requirement of positive definiteness of the block $-\overline{R}''_{uu} = -\overline{H}''_{uu}$, then their number can be reduced to $n(n+1)$. At the same time, the positivity of $-\overline{H}''_{uu}$ can be checked even at the stage of finding the extremal. Moreover, for a number of problems the block $-\overline{H}''_{uu}$ is obviously a positive definite matrix. For example, in linear in both state and control stochastic systems with a quadratic performance functional, the latter is ensured by the positiveness of the quadratic form $u^{\mathrm{T}}Eu$ in the integrand of the functional by the problem setting (see, e.g., [6, Section 6]). We note that, in the general case, this approach also allows one to avoid in advance considering those extremals for which the proposed method is not applicable. However, further verification of the conditions of the theorem, although it becomes easier, still remains a non-trivial task.

To substantially simplify the conditions of Theorem 1, we transform the positivity requirements for the block $-\overline{R}'_{yy}$ and matrix $\Sigma(t_1)$. We will ensure their fulfillment by satisfying the following equalities:

$$\overline{\Omega}(t) = \overline{\Omega}_\gamma(t) = \left(\begin{array}{c|c} \gamma I_{n(n+1)} & -\overline{R}''_{yu} \\ \hline (-\overline{R}''_{yu})^{\mathrm{T}} & -\overline{H}''_{uu} \end{array}\right), \quad \Sigma(t_1) = \gamma_1 I_{n(n+1)}, \qquad \gamma,\ \gamma_1 > 0.$$

Thus, the unknown function $\Sigma(t)$ can be defined by the condition in the form of a Cauchy problem

$$\begin{gathered} \frac{d\Sigma(t)}{dt} = -\Sigma(t)\tilde{A}(t, \overline{u}(t)) - \tilde{A}^{\mathrm{T}}(t, \overline{u}(t))\Sigma(t) - \gamma I_{n(n+1)}, \\ \Sigma(t_1) = \gamma_1 I_{n(n+1)}. \end{gathered} \tag{19}$$

Then further verification of the positive definiteness of the matrix $\overline{\Omega}(t) = \overline{\Omega}_\gamma(t)$ is reduced to finding the numbers γ and γ_1 satisfying for all $t \in T$ the system of m remaining inequalities for the corner minors. In view of Remark 5, Theorem 1 takes the following form.

Theorem 2. *Suppose that we have found the extremal $\overline{v} = (\overline{y}(\cdot), \overline{u}(\cdot))$ and the corresponding function $\psi(t)$, and the matrix $-\overline{H}''_{uu}(t)$ is positive definite on T. In order for the point $\overline{v}$ to be at least a point of a weak relative minimum of functional* (12)*, it is sufficient to find such positive numbers γ, γ_1 that the solution $\Sigma(t)$ of Cauchy problem* (19) *ensures positiveness of the last m corner minors of matrix $\overline{\Omega}_\gamma(t)$, $t \in T$.*

Remark 6. For problems with scalar control $u(t)$ (the case $m = 1$), the conditions of Theorem 2, in contrast to Theorem 1, require checking only one inequality corresponding to the positiveness of the determinant $\overline{\Omega}_\gamma(t)$.

Remark 7. Remark 3 remains valid in relation to Theorem 2.

Remark 8. To formulate the conditions of Theorem 2, instead of diagonal matrices $\gamma I_{n(n+1)}$, $\gamma_1 I_{n(n+1)}$, γ, $\gamma_1 > 0$, one can use in Eqs. (19) the positive definite matrices Θ, Θ_1 of an arbitrary form [3, p. 83].

We make another transformation of the conditions of Theorem 1. Following [2, p. 99], we consider the function

$$P(t,y) = R(t,y,u^*(t,y)), \quad u^*(t,y) \in \operatorname{Arg}\max_{u\in U} R(t,y,u), \tag{20}$$

then the problem of maximizing the function $R(t,y,u)$ with respect to (y,u) is reduced to the problem of maximizing the function $P(t,y)$ with respect to y, where

$$\begin{aligned} d\overline{P}(t) = dP(t,\overline{y}) &= \left(\frac{\partial R(t,\overline{y}(t),\overline{u}(t))}{\partial y} + \frac{\partial u^*(t,\overline{y}(t))}{\partial y}\frac{\partial R(t,\overline{y}(t),\overline{u}(t))}{\partial u}\right)^{\mathrm{T}} dy \\ &= \left(\overline{R}'_y + \overline{(u^*)}'_y \overline{R}'_u\right)^{\mathrm{T}} dy = 0; \end{aligned}$$

here $\overline{u}(t) = u^*(t,\overline{y}(t))$.

Therefore, for the second differential

$$d^2\overline{P}(t) = -(dy)^{\mathrm{T}}(-\overline{P}''_{yy})dy$$

we need to establish the positive definiteness of the matrix $-\overline{P}''_{yy}$ of the form

$$\overline{P}''_{yy} = \overline{R}''_{yy} + \overline{(u^*)}'_y\left(\overline{R}''_{yu}\right)^{\mathrm{T}} + \overline{R}''_{yu}\left(\overline{(u^*)}'_y\right)^{\mathrm{T}} + \overline{(u^*)}'_y\overline{R}''_{uu}\left(\overline{(u^*)}'_y\right)^{\mathrm{T}} + \left[\overline{R}'_u\overline{(u^*)}''_{yy}\right],$$

where the last (tensor) product in square brackets is zero since $\overline{R}'_u = 0$. The right-hand side of the resulting relation depends on an unknown matrix $\Sigma(t)$, which, by analogy with the above, we determine from the Cauchy problem

$$\begin{aligned} \frac{d\Sigma(t)}{dt} &= -\Sigma(t)\tilde{A}(t,\overline{u}(t)) - \tilde{A}^{\mathrm{T}}(t,\overline{u}(t))\Sigma(t) \\ &- \overline{(u^*)'_y}(t,\Sigma(t))\left(\overline{R}''_{yu}\right)^{\mathrm{T}}(t,\Sigma(t)) - \overline{R}''_{yu}(t,\Sigma(t))\left(\overline{(u^*)}'_y\right)^{\mathrm{T}}(t,\Sigma(t)) \\ &- \overline{(u^*)'_y}(t,\Sigma(t))\overline{H}''_{uu}(t)\left(\overline{(u^*)'_y}\right)^{\mathrm{T}}(t,\Sigma(t)) - \gamma I_{n(n+1)}, \quad \Sigma(t_1) = \gamma_1 I_{n(n+1)}. \end{aligned} \tag{21}$$

Here matrices $\overline{R}''_{yu}$, $\overline{H}''_{uu}$, as before, are determined by relations (18).

Theorem 3. *Suppose that we have found the extremal* $\overline{v} = (\overline{y}(\cdot),\overline{u}(\cdot))$ *and the corresponding function* $\psi(t)$, *and the matrix* $-\overline{H}''_{uu}(t)$ *is positive definite on* T. *For the point* $\overline{v}$ *to be a point of strong relative minimum of functional* (12), *it is sufficient to find positive numbers* γ, γ_1 *such that there exists a solution* $\Sigma(t)$ *of Cauchy problem* (21).

Remark 9. Cauchy problem (21) is generally not linear in the variable Σ, which means it may have no solutions in the general case.

5. MODEL EXAMPLES

Example 1. Consider the following problem [6]. Suppose that on the time interval $T = [0;1]$ the control process is defined by a scalar equation

$$dx(t) = [-x(t)+1]dt + 5u^2(t)dw(t)$$

with the initial condition $x(0) = x_0$, where x_0 is a random variable with expectation $m_0 = 0$ and variance $K_0 = 1$, $u(t) \in U = R^1$.

The problem is to find the control $\overline{u}(t)$ that minimizes the performance functional

$$J = \mathbb{E}\int_0^1 u^3(t)dt + \mathbb{E}\left[\frac{1}{2}x(1)^2\right].$$

In vector form of (11) and (12), the example can be written using matrices

$$\tilde{A}(t,u) = \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix}, \quad \tilde{B}(t,u) = \begin{pmatrix} 1 \\ 25u^4 \end{pmatrix},$$
$$\tilde{D}(t,u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \quad E(t,u) = u^3,$$

so the corresponding deterministic optimization problem takes the form

$$\frac{dy_1(t)}{dt} = -y_1(t) + 1, \quad y_1(0) = 0,$$
$$\frac{dy_2(t)}{dt} = 2y_1(t) - 2y_2(t) + 25u^4(t), \quad y_2(0) = 1,$$
$$J = \int_0^1 u^3(t)dt + \frac{1}{2}y_2(1).$$

Then the adjoint system of Eqs. (14) and the Hamiltonian (13) can be written as

$$\frac{d\psi_1(t)}{dt} = \psi_1(t) - 2\psi_2(t), \quad \psi_1(1) = 0,$$
$$\frac{d\psi_2(t)}{dt} = 2\psi_2(t), \quad \psi_2(1) = -1/2,$$
$$H(t,y,\psi,u) = \psi_1\left[-y_1 + 1\right] + \psi_2\left[2y_1 - 2y_2 + 25u^4\right] - u^3.$$

A necessary condition for maximizing the Hamiltonian with respect to u is the ratio

$$\frac{\partial H(t,y(t),\psi(t),u)}{\partial u} = 100\psi_2(t)u^3 - 3u^2 = 0,$$

from which we have

$$u_1(t) = 0, \quad u_2(t) = \frac{3}{100\psi_2(t)}.$$

Considering the fact that the second derivative of the Hamiltonian with respect to u for $u = u_1$ is zero, only the solution $u_2(t)$ requires a check for extremality. Substituting it into the direct and adjoint systems of equations and carrying out their integration, we can obtain the pair $\overline{v} = (\overline{y}(\cdot), \overline{u}(\cdot))$ and the function ψ. In particular, the second component $\psi_2(t)$ that we are interested in will have the form

$$\psi_2(t) = -\frac{1}{2}e^{2(t-1)},$$

which implies that

$$\overline{u}(t) = -\frac{3}{50}e^{2(1-t)}.$$

Moreover,

$$\frac{\partial^2 H(t,\overline{y}(t),\psi(t),\overline{u}(t))}{\partial u^2} = [300\psi_2(t)\overline{u}(t) - 6]\,\overline{u}(t) = -\frac{9}{50}e^{2(t-1)} < 0, \quad t \in T,$$

which means that $\overline{v}$ satisfies the extremality conditions. At this stage, we once again note that the result obtained is fully consistent with the results on the necessary optimality conditions of the work [6] for this example.

Now we apply sufficient optimality conditions from Theorem 2. Compose adjoint Eqs. (19) for the components of the symmetric 2×2 matrix $\Sigma(t)$

$$\frac{d\sigma_{11}(t)}{dt} = 2\sigma_{11}(t) - 4\sigma_{12}(t) - \gamma, \quad \sigma_{11}(1) = \gamma_1,$$
$$\frac{d\sigma_{12}(t)}{dt} = 3\sigma_{12}(t) - 2\sigma_{22}(t), \quad \sigma_{12}(1) = 0,$$
$$\frac{d\sigma_{22}(t)}{dt} = 4\sigma_{22}(t) - \gamma, \quad \sigma_{22}(1) = \gamma_1.$$

Integrating the system from the end, we get

$$\sigma_{11}(t,\gamma,\gamma_1) = \frac{5}{6}\gamma - \left(\frac{5}{2}\gamma - 5\gamma_1\right)e^{2(t-1)} + \left(\frac{8}{3}\gamma - 8\gamma_1\right)e^{3(t-1)} - (\gamma - 4\gamma_1)\,e^{4(t-1)},$$
$$\sigma_{12}(t,\gamma,\gamma_1) = \frac{1}{6}\gamma - \left(\frac{2}{3}\gamma - 2\gamma_1\right)e^{3(t-1)} + \left(\frac{1}{2}\gamma - 2\gamma_1\right)e^{4(t-1)},$$
$$\sigma_{22}(t,\gamma,\gamma_1) = \frac{1}{4}\gamma - \left(\frac{1}{4}\gamma - \gamma_1\right)e^{4(t-1)}.$$

We use the found components of the solution $\Sigma(t)$ to compose the matrix

$$\overline{\Omega}_\gamma(t) = \left(\begin{array}{c|c} \gamma I_2 & -\overline{R}''_{yu} \\ \hline (-\overline{R}''_{yu})^{\mathrm{T}} & -\overline{H}''_{uu} \end{array}\right),$$

where the blocks $-\overline{R}''_{yu}$, $-\overline{H}''_{uu}$ are calculated using formulas (18) and have the form

$$-\overline{R}''_{yu}(t) = -100\overline{u}^3(t)\begin{pmatrix} \sigma_{12}(t,\gamma,\gamma_1) \\ \sigma_{22}(t,\gamma,\gamma_1) \end{pmatrix}, \quad -\overline{H}''_{uu} = -300\psi_2(t)\overline{u}^2(t) + 6\overline{u}(t).$$

We get

$$\overline{\Omega}_\gamma(t) = \begin{pmatrix} \gamma & 0 & -100\overline{u}^3(t)\sigma_{12}(t,\gamma,\gamma_1) \\ 0 & \gamma & -100\overline{u}^3(t)\sigma_{22}(t,\gamma,\gamma_1) \\ -100\overline{u}^3(t)\sigma_{12}(t,\gamma,\gamma_1) & -100\overline{u}^3(t)\sigma_{22}(t,\gamma,\gamma_1) & -300\psi_2(t)\overline{u}^2(t) + 6\overline{u}(t) \end{pmatrix}.$$

Thus, taking into account Remark 6, to verify the positive definiteness of $\overline{\Omega}_\gamma(t)$ and thereby fulfill the conditions of Theorem 2, it suffices to establish for some values of $\gamma,\ \gamma_1 > 0$ the validity of a single inequality

$$f(t) > 0, \quad t \in T,$$

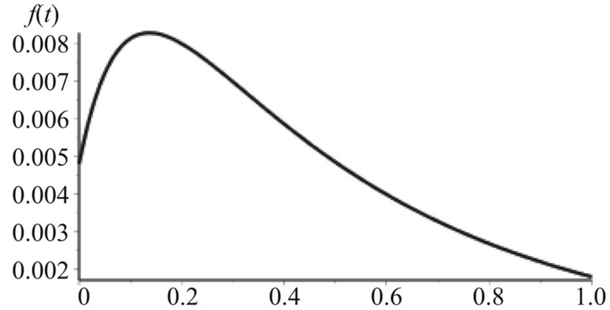

Fig. 1. Plot of function $f(t)$ for $\gamma = \gamma_1 = 0.1$.

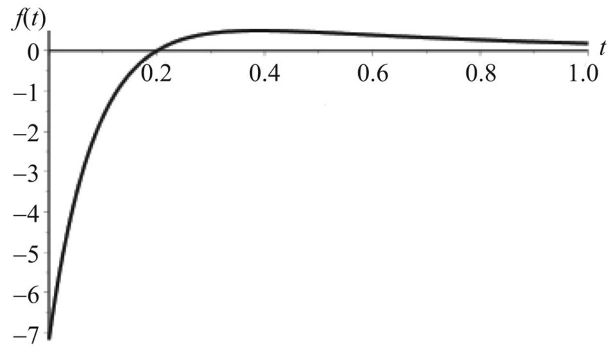

Fig. 2. Plot of function $f(t)$ for $\gamma = \gamma_1 = 1$.

where

$$f(t) = \left|\overline{\Omega}_\gamma(t)\right| = \left[6\overline{u}(t) - 300\psi_2(t)\overline{u}^2(t)\right]\gamma^2 - 10\,000\gamma\overline{u}^6(t)\left[\sigma_{12}^2(t,\gamma,\gamma_1) + \sigma_{22}^2(t,\gamma,\gamma_1)\right].$$

Such numbers γ, γ_1 exist. For example, with $\gamma = \gamma_1 = 0.1$ the plot of the function $f(t)$ has the form shown in Fig. 1.

Therefore, the resulting control $\overline{u}(t)$ is optimal. Note that already when choosing the numbers γ, γ_1 greater or equal to 1, the inequality ceases to hold at some points of the interval T (see Fig. 2).

Next, consider the approach of Theorem 3. We compose $R(t,y,u)$ of the form (16) using

$$\varphi(t,y) = \psi_1(t)y_1 + \psi_2(t)y_2 + \frac{1}{2}\left(\sigma_{11}(t)\Delta y_1^2 + 2\sigma_{12}(t)\Delta y_1\Delta y_2 + \sigma_{22}(t)\Delta y_2^2\right),$$

will get

$$R(t,y,u) = (\psi_1(t) + \sigma_{11}(t)\Delta y_1 + \sigma_{12}(t)\Delta y_2)(-y_1+1)$$
$$+ (\psi_2(t) + \sigma_{12}(t)\Delta y_1 + \sigma_{22}(t)\Delta y_2)(2y_1 - 2y_2 + 25u^4) - u^3 + \varphi_t'(t,y).$$

From condition (20), we define the function

$$u^*(t,y) = \arg\max_{u\in R^1} R(t,y,u)$$
$$= \arg\max_{u\in R^1}\left\{25(\psi_2(t) + \sigma_{12}(t)\Delta y_1 + \sigma_{22}(t)\Delta y_2)u^4 - u^3\right\}$$
$$= \frac{3}{100(\psi_2(t) + \sigma_{12}(t)[y_1 - \overline{y}_1(t)] + \sigma_{22}(t)[y_2 - \overline{y}_2(t)])},$$

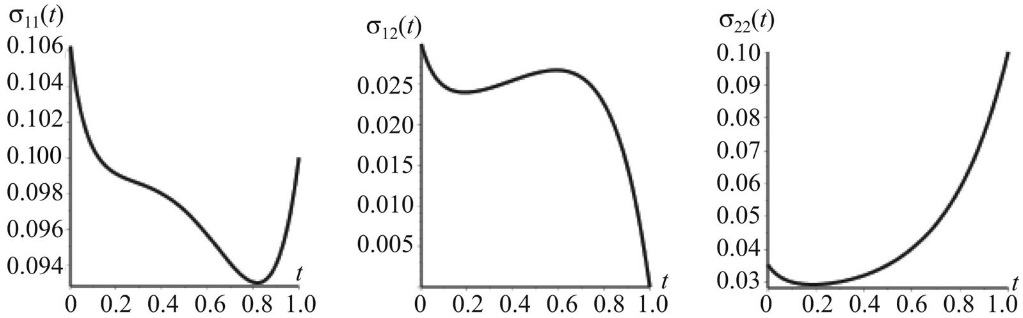

Fig. 3. Plots of components of the matrix $\Sigma(t)$.

and u^* found at the point $(t, \overline{y}(t))$, as already noted, coincides with $\overline{u}(t)$ and, therefore, satisfies the sufficient maximum condition

$$[300\psi_2(t)\overline{u}(t) - 6]\,\overline{u}(t) = -\frac{9}{50}e^{2(t-1)} < 0, \quad t \in T.$$

Then the first derivative $(u^*)'_y$ at point $(t, \overline{y}(t))$ will have the form

$$\overline{(u^*)'_y}(t, \Sigma(t)) = -\frac{3}{100\psi_2^2(t)}\begin{pmatrix}\sigma_{12}(t)\\ \sigma_{22}(t)\end{pmatrix}.$$

Let us compose Eqs. (21)

$$\frac{d\sigma_{11}(t)}{dt} = 2\sigma_{11}(t) - 4\sigma_{12}(t) + \eta(t)\sigma_{12}^2(t) - \gamma, \quad \sigma_{11}(1) = \gamma_1,$$
$$\frac{d\sigma_{12}(t)}{dt} = 3\sigma_{12}(t) - 2\sigma_{22}(t) + \eta(t)\sigma_{12}(t)\sigma_{22}(t), \quad \sigma_{12}(1) = 0,$$
$$\frac{d\sigma_{22}(t)}{dt} = 4\sigma_{22}(t) + \eta(t)\sigma_{22}^2(t) - \gamma, \quad \sigma_{22}(1) = \gamma_1,$$

where we have denoted

$$\eta(t) = 3\frac{400\overline{u}^2(t) + 450\psi_2(t)\overline{u}(t) - 9}{200\psi_2^2(t)}\overline{u}(t).$$

By numerically integrating the system from the end for the same values $\gamma = \gamma_1 = 0.1$, one can obtain a solution of the following form (see Fig. 3).

However, already with $\gamma = \gamma_1 = 1$ the solution does not exist. For example, numerical integration of the Cauchy problem for the function $\sigma_{22}(t)$ from $t_1 = 1$ to $t_0 = 0$ goes to "computational infinity" to the left of the time moment $t^* = 0.15$.

Example 2. On the same time interval $T = [0; 1]$, consider a linear two-dimensional problem

$$dx_1(t) = [-x_1(t) - x_2(t)]dt + 5dw(t),$$
$$dx_2(t) = [-x_2(t) + u(t, x(t))]dt + 5dw(t)$$

with initial condition $x(0) = x_0$, where x_0 is a random vector with expectation $m_0 = 0$ and covariance matrix $K_0 = I_2$. The equations include a control strategy $u(t, x)$, which is not subject to geometric constraints ($U = R^1$), but an additional information constraint is imposed [6]: control u should be independent of the second component x_2 of the state vector x.

The problem is to find the control strategy $\overline{u}(t,x)$ in the form of a linear controller $-P^{\mathrm{T}}(t)x$ that minimizes the quadratic performance functional

$$J = \mathbb{E}\int\limits_0^1 \Big[x_1^2(t) + x_2^2(t) + u^2(t,x(t))\Big]\,dt.$$

Based on the given information constraint, we can conclude that the desired optimal controller should be $u(t,x) = -P_1(t)x_1$, where P_1 is the first component of the vector P. Thus, we need to determine the optimal value of the coefficient $\overline{P}_1(t)$. Denoting $u(t) = P_1(t)$, we can rewrite the problem as (1), (2) using matrices

$$A(t,u) = \begin{pmatrix} -1 & -1 \\ -u & -1 \end{pmatrix}, \quad B(t,u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C(t,u) = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \quad G(t,u) = 0,$$
$$D(t,u) = \begin{pmatrix} 1+u^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad S(t,u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad E(t,u) = 0, \quad Q = 0$$

or in vector form (11), (12) using matrices

$$\tilde{A}(t,u) = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ -u & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 0 \\ 0 & 0 & -u & -2 & 0 & -1 \\ 0 & 0 & -u & 0 & -2 & -1 \\ 0 & 0 & 0 & -u & -u & -2 \end{pmatrix}, \quad \tilde{B}(t,u) = \begin{pmatrix} 0 \\ 0 \\ 25 \\ 25 \\ 25 \\ 25 \end{pmatrix},$$
$$\tilde{D}(t,u) = \begin{pmatrix} 0 \\ 0 \\ 1+u^2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad E(t,u) = 0, \quad \tilde{Q} = 0$$

with the initial condition $y_0 = (0,0,1,0,0,1)^{\mathrm{T}}$.

Note that in this example, for any values of $u(t)$ a solution of the direct system of Eqs. (11) for the first two components $y_1(t)$, $y_2(t)$ of the vector function $y(t)$ will be $y_1(t) = y_2(t) \equiv 0$ and, similarly, the solution of the adjoint system (14) for the first two components $\psi_1(t)$, $\psi_2(t)$ of the vector function $\psi(t)$ will be $\psi_1(t) = \psi_2(t) \equiv 0$. Moreover, due to the symmetry of the initial matrix of moments, we have $y_4(t) = y_5(t)$ and $\psi_4(t) = \psi_5(t)$, $t \in T$.

Then the necessary condition for the maximum of the Hamiltonian (13) takes the form

$$\psi_4(t)y_3(t) + \psi_6(t)y_4(t) + u(t)y_3(t) = 0.$$

From here, we can find the unique extreme solution

$$\overline{u}(t) = -\frac{\psi_4(t)y_3(t) + \psi_6(t)y_4(t)}{y_3(t)},$$

for which the sufficient condition $y_3(t) > 0$, $t \in T$ is also satisfied, since component y_3 corresponds to element N_{11} of the matrix of raw moments N.

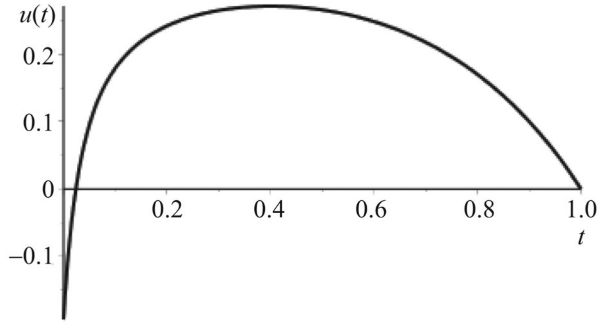

Fig. 4. Plot of the function $u(t)$.

Substituting $\overline{u}(t)$ into the direct and adjoint systems of equations, we obtain the boundary problem made of six different equations and six conditions

$$\frac{dy_3(t)}{dt} = -2\left(y_3(t) + y_4(t)\right) + 25,$$

$$\frac{dy_4(t)}{dt} = \left[\psi_4(t)y_3(t) + \psi_6(t)y_4(t)\right] - 2y_4(t) - y_6(t) + 25,$$

$$\frac{dy_6(t)}{dt} = 2\left(\frac{\psi_4(t)y_3(t) + \psi_6(t)y_4(t)}{y_3(t)}y_4(t) - y_6(t)\right) + 25,$$

$$\frac{d\psi_3(t)}{dt} = 2\left(\psi_3(t) - \frac{\psi_4(t)y_3(t) + \psi_6(t)y_4(t)}{y_3(t)}\psi_4(t)\right) + \frac{\left[\psi_4(t)y_3(t) + \psi_6(t)y_4(t)\right]^2}{y_3^2(t)} + 1,$$

$$\frac{d\psi_4(t)}{dt} = \psi_3(t) + 2\psi_4(t) - \frac{\psi_4(t)y_3(t) + \psi_6(t)y_4(t)}{y_3(t)}\psi_6(t),$$

$$\frac{d\psi_6(t)}{dt} = 2\left(\psi_4(t) + \psi_6(t)\right) + 1,$$

$$y_3(0) = 1, \quad y_4(0) = 0, \quad y_6(0) = 1, \quad \psi_3(1) = 0, \quad \psi_4(1) = 0, \quad \psi_6(1) = 0.$$

The solution of the boundary value problem can be found numerically, and as a result we find the extremal $(\overline{y}(\cdot), \overline{u}(\cdot))$ and the function $\psi(t)$. In particular, the plot of the function $\overline{u}(t)$ will have the form shown on Fig. 4.

Next we apply sufficient conditions of Theorem 2. To do this, we need to compose a symmetric matrix $\Sigma(t)$ of size 6×6 and solve Cauchy problem (19) for it. It is easy to verify that the symmetry of the matrix of raw moments in the original problem will also affect the number of different equations for the elements $\Sigma(t)$. Namely, the matrix Σ will have the following form:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{14} & \sigma_{16} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{24} & \sigma_{26} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{34} & \sigma_{34} & \sigma_{36} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44} & \sigma_{45} & \sigma_{46} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{45} & \sigma_{44} & \sigma_{46} \\ \sigma_{16} & \sigma_{26} & \sigma_{36} & \sigma_{46} & \sigma_{46} & \sigma_{66} \end{pmatrix}.$$

Thus, the problem is reduced to the integration of 16 varied differential Eqs. (19) with initial conditions. This problem can also be solved numerically for given values of γ, γ_1. Then, to verify sufficient conditions for some γ, γ_1 and the found solution $\Sigma(t, \gamma, \gamma_1)$, it remains to compose the

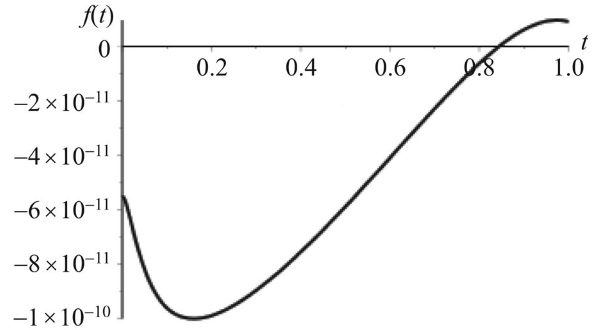

Fig. 5. Plot of function $f(t)$ for $\gamma = \gamma_1 = 0.01$.

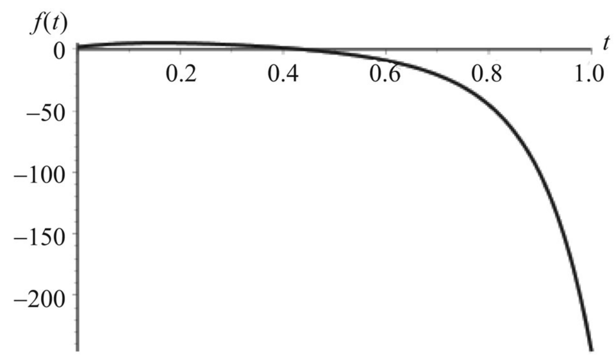

Fig. 6. Plot of function $f(t)$ for $\gamma = \gamma_1 = 1$.

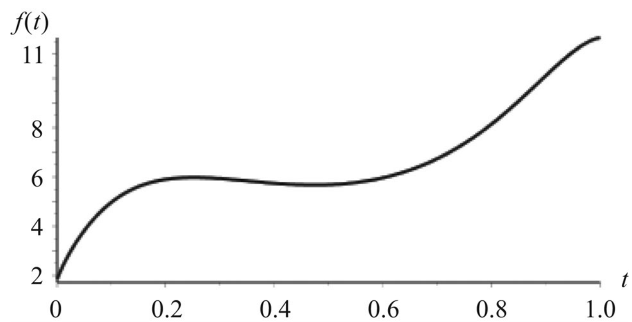

Fig. 7. Plot of function $f(t)$ for $\gamma = 1,\ \gamma_1 = 0.01$.

matrix $\overline{\Omega}_\gamma(t)$ and establish the sign of its determinant. Using (18), we get

$$\overline{\Omega}_\gamma(t) = \left(\begin{array}{c|c} \gamma I_6 & -\overline{R}''_{yu} \\ \hline (-\overline{R}''_{yu})^{\mathrm{T}} & -\overline{H}''_{uu} \end{array}\right),$$

$$R''_{yu} = \begin{pmatrix} -2\sigma_{14}(t,\gamma,\gamma_1)\overline{y}_3(t) - 2\sigma_{16}(t,\gamma,\gamma_1)\overline{y}_4(t) \\ -2\sigma_{24}(t,\gamma,\gamma_1)\overline{y}_3(t) - 2\sigma_{26}(t,\gamma,\gamma_1)\overline{y}_4(t) \\ -2\psi_4(t) - 2\sigma_{34}(t,\gamma,\gamma_1)\overline{y}_3(t) - 2\sigma_{36}(t,\gamma,\gamma_1)\overline{y}_4(t) - 2\overline{u}(t) \\ -\psi_6(t) - \sigma_{44}(t,\gamma,\gamma_1)\overline{y}_3(t) - \sigma_{45}(t,\gamma,\gamma_1)\overline{y}_3(t) - 2\sigma_{46}(t,\gamma,\gamma_1)\overline{y}_4(t) \\ -\psi_6(t) - \sigma_{44}(t,\gamma,\gamma_1)\overline{y}_3(t) - \sigma_{45}(t,\gamma,\gamma_1)\overline{y}_3(t) - 2\sigma_{46}(t,\gamma,\gamma_1)\overline{y}_4(t) \\ -2\sigma_{46}(t,\gamma,\gamma_1)\overline{y}_3(t) - 2\sigma_{66}(t,\gamma,\gamma_1)\overline{y}_4(t) \end{pmatrix},$$

$$\overline{H}''_{uu} = -2\overline{y}_3(t).$$

Figures 5–7 show the plots of the function $t \to f(t, \gamma, \gamma_1) = |\overline{\Omega}_\gamma(t)|$ for some values $\gamma,\ \gamma_1 > 0$.

As can be seen from Fig. 7, the positive numbers $\gamma,\ \gamma_1$ that ensure the fulfillment of sufficient optimality conditions from Theorem 2, exist, however, as is the case with the first example, their values need to be chosen accurately enough.

Using the conditions of Theorem 3 leads to similar results. The derivative $(u^*)'_y$ at the point $(t, \overline{y}(t))$ will be

$$(u^*)'_y(t, \Sigma(t)) = -\frac{1}{2\overline{y}_3(t)} \begin{pmatrix} 2\sigma_{14}(t)\overline{y}_3(t) + 2\sigma_{16}(t)\overline{y}_4(t) \\ 2\sigma_{24}(t)\overline{y}_3(t) + 2\sigma_{26}(t)\overline{y}_4(t) \\ 2\psi_4(t) + 2\sigma_{34}(t)\overline{y}_3(t) + 2\sigma_{36}(t)\overline{y}_4(t) + 2\overline{u}(t) \\ \psi_6(t) + \sigma_{44}(t)\overline{y}_3(t) + \sigma_{45}(t)\overline{y}_3(t) + 2\sigma_{46}(t)\overline{y}_4(t) \\ \psi_6(t) + \sigma_{44}(t)\overline{y}_3(t) + \sigma_{45}(t)\overline{y}_3(t) + 2\sigma_{46}(t)\overline{y}_4(t) \\ 2\sigma_{46}(t)\overline{y}_3(t) + 2\sigma_{66}(t)\overline{y}_4(t) \end{pmatrix},$$

and substituting it into the Cauchy problem (21) we can obtain a numerical solution $\Sigma(t)$ with $\gamma = 1,\ \gamma_1 = 0.01$. With values of $\gamma = \gamma_1 = 0.01$ or $\gamma = \gamma_1 = 1$, as in the case of Example 1, a solution cannot be found.

6. CONCLUSION

We have obtained sufficient conditions for a strong and weak relative minimum in the optimization problem for quasilinear stochastic systems with continuous time whose coefficients in the general case depend nonlinearly on the program control.

The results of solving the examples that we give in this paper show certain difficulties in applying the obtained sufficient optimality conditions even in model problems. First of all, they are related to finding the numbers $\gamma,\ \gamma_1$, which in each specific case nothing is known in advance about except for their positivity. The analysis of permissible ranges of values of these parameters is a subject of further research.

In addition, the examples clearly demonstrate a significant increase in the computational complexity of problems as the problem dimensions grow when using the proposed approaches. For instance, a two-dimensional problem with scalar control and incomplete information already requires a numerical solution of a boundary problem of order 10 and a Cauchy problem of order 16.

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