ISSN 0005-1179, Automation and Remote Control, 2018, Vol. 79, No. 5, pp. 884–896. © Pleiades Publishing, Ltd., 2018.
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INTELLECTUAL CONTROL SYSTEMS, DATA ANALYSIS

Time-Optimal Boundary Control for Systems Defined by a Fractional Order Diffusion Equation

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Received December 20, 2016

Abstract—We consider the optimal control problem for a system defined by a one-dimensional diffusion equation with a fractional time derivative. We consider the case when the controls occur only in the boundary conditions. The optimal control problem is posed as the problem of transferring an object from the initial state to a given final state in minimal possible time with a restriction on the norm of the controls. We assume that admissible controls belong to the class of functions $L_\infty[0, T]$. The optimal control problem is reduced to an infinite-dimensional problem of moments. We also consider the approximation of the problem constructed on the basis of approximating the exact solution of the diffusion equation and leading to a finite-dimensional problem of moments. We study an example of boundary control computation and dependencies of the control time and the form of how temporal dependencies in the control dependent on the fractional derivative index.

Keywords: optimal control, diffusion equation, Caputo's fractional derivative, the problem of moments

DOI: 10.1134/S0005117918050090

1. INTRODUCTION

Dynamic systems of non-integer order have attracted increasing attention of researchers in recent years [1–4]. We consider both systems with lumped parameters (defined in terms of fractional order equations with total derivatives) and systems with distributed parameters (defined in terms of fractional partial differential equations). Interest in such systems stems from a significant volume of experimental observations and models of real physical systems and processes that exhibit the signs of the so-called "fractional dynamics" [5]. In particular, the fractional-order diffusion equation investigated in this work formalizes the phenomena of anomalous diffusion in inhomogeneous and/or irregular (fractal) media, for example, the diffusion of biopolymers in living cells and charge carriers in amorphous semiconductors or electrolytes (see [6, 7] and references therein). In these publications, it is noted that the operator of fractional differentiation can be understood both in the sense of Caputo and in the sense of Riemann–Liouville (see Appendix, item 1). Such ambiguity is due to the lack of a unified definition of the fractional differentiation operation (and the existence of several dozen alternative definitions), as well as the absence of clear consequences from physical theories (for example, from the theory of "random walks," CTRW) or from experimental data that would allow to select a particular unique species of fractional derivatives.

In the last 15–20 years, researchers have been interested in the task of control of systems of fractional order [3, 4, 8]. The possibilities are studied of using fractional order systems or elements to perform fine-tuning of controllers and construct more complex control strategies that would take

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into account the previous state of the system and/or the state of the system in a rather extensive and extended area of space. Here we can note, for example, the use of non-integer order controllers for controlling microclimate and power systems. The so-called supercapacitors (capacitors with strongly irregular electrodes), viscoelastic elements in vibration protection systems, porous and microstructured media, which are carriers or transmitters of matter or heat, are considered as elements or systems of fractional order. In most publications, optimal control problems were studied mainly for fractional-order systems with lumped parameters within the framework of the variational approach [9, 10], where, as a rule, the control was assumed to be continuous in advance, and explicit constraints on its norm were not considered. The works [11–13] consider the optimal control problem for a fractional-order diffusion equation with the Riemann–Liouville derivative and quadratically-integrable distributed control. In this case, the minimization problem is solved for a functional which is quadratic in both state and control.

In this work, for systems with distributed parameters defined by the diffusion equation with a fractional order time derivative, we consider the problem of finding the optimal boundary control that ensures the shortest control time for a given constraint on the control norm (the performance problem). As follows from the above, such a problem has not been solved earlier for this class of systems: known publications do not consider boundary control and do not solve the performance problem. At the same time, these issues have both an undoubtable theoretical interest and a certain practical significance. Boundary control is very important, for example, in problems of heating structurally complex materials or creating (maintaining) a given concentration of a certain substance diffusing through a semipermeable membrane into a limited area of space. In both cases mentioned here, a given state can be achieved by applying influences to the boundary of a given region. The performance problem is important since there exist a huge number of applications where it is necessary to bring the system to a predetermined state (for example, to heat to the desired temperature or create the desired concentration of the substance) in the shortest possible time, based on technological or economic requirements.

To solve this problem, in this work we use the method of moments, known in the classical theory of systems [14]. This method has previously been used to study the optimal control problem of fractional-order systems [15, 16]. Thus, for the systems with distributed parameters considered in this work, we solve the problem of finding a quadratically-integrable boundary control of minimal norm [16]. In addition, we show that, using the method of moments, one can investigate the optimal control problem not only in the case of boundary control, but also in the case of distributed control [17].

2. PROBLEM SETTING

We consider a system whose state is defined by the following equation:

$$ {}_0^C D_t^\alpha Q(x,t) = K \frac{\partial^2 Q(x,t)}{\partial x^2}, \quad t \geqslant 0, \quad x \in [0,L], \tag{1}$$

where $Q(x,t)$ is the state of the system, K is the diffusion coefficient, ${}_0^C D_t^\alpha$ is the operator of Caputo's fractional differentiation with respect to time (see Appendix, item 1), $\alpha \in (0,1]$, $(x,t) \in \Omega = [0,L] \times [0,\infty)$. We will call Eq. (1) a diffusion equation of fractional order. In what follows we assume $K = 1$, since this simplifies the computations somewhat, and the solution of the diffusion equation for a diffusion coefficient other than one does not qualitatively differ from the solution for an arbitrary (positive) coefficient, which can be found in [7]. The optimal control problem for this case can be considered completely similarly to the considerations shown in this work. Such a choice is also related to the fact that earlier A.G. Butkovskii had considered this problem for systems of integer order, which makes it possible to compare the results given below.

We assume that the function $Q(x,t)$ is differentiable on the positive semiaxis with respect to the time variable and twice differentiable on the interval $[0,L]$ with respect to the spatial variable.

The initial and boundary conditions for Eq. (1) are given as follows:

$$Q(x,0+) = Q_0(x), \quad x \in [0,L], \tag{2}$$

$$\left[b_i \frac{\partial Q(x,t)}{\partial x} + a_i Q(x,t)\right]_{x=x^i} = h_i(t) + u_i(t), \quad t \geqslant 0, \tag{3}$$

where a_i and b_i are constant coefficients, $b_1 \leqslant 0$, $b_2 \geqslant 0$; $h_i(t)$ are some known fully regular (differentiable) functions, $i = 1, 2$, $x^1 = 0$, $x^2 = L$. Functions $u_1(t) \in L_\infty[0,T]$ and $u_2(t) \in L_\infty[0,T]$ are boundary controls. Such a choice of controls as essentially bounded functions leads to the fact that equality in expression (3) should be understood not pointwise but as equality almost everywhere (since essentially bounded functions can be undefined on a set of measure zero, and two such functions are the same if they differ only on a set of measure zero). We will also assume that when $t > 0$ the continuity conditions are fulfilled: $Q(0,t) = Q_0(0)$, $Q(L,t) = Q_0(L)$.

The final condition is defined as follows:

$$Q(x,T) = Q^*(x), \quad T > 0, \quad x \in [0,L]. \tag{4}$$

Boundary controls can be combined into a vector $U(t) = (u_1(t), u_2(t)) \in L_\infty[0,T]$, the norm of which will be expressed by the formula

$$\|U(t)\| = \operatorname{vrai} \max_{t\in[0,T]} \max_i |u_i(t)|,$$

where vrai max denotes an essential maximum of the function in question, i.e. the lower bound of the set of numbers A such that the set of values $t \in [0,T]$, for which $\max\limits_i |u_i(t)| > A$, has measure zero [14].

We pose the optimal control problem as follows.

Problem 1 (optimal control). Find controls $u_1(t) \in L_\infty[0,T]$ and $u_2(t) \in L_\infty[0,T]$ such that for system (1) with initial condition (2) and boundary conditions (3), the final state (4) is reached in minimal possible time T^* with a constraint on the control norm $\|U(t)\| \leqslant l$, $l > 0$.

3. REPRESENTATION OF THE OPTIMAL CONTROL PROBLEM IN THE FORM OF A PROBLEM OF MOMENTS

For system (1)–(3), the exact analytic solution is known [7, formula (14)]. Let us write it for the final state (4):

$$Q(x,T) = Q^*(x) = R(x,T) + v_1(x)u_1(T) + v_2(x)u_2(T)$$
$$- \sum_{n=1}^{\infty} X_n(x) \int_0^T \frac{E_{\alpha,\alpha}[-\lambda_n(T-t)^\alpha]\left[v_{1n} \times {}_0^C D_t^\alpha u_1(t) + v_{2n} \times {}_0^C D_t^\alpha u_2(t)\right] dt}{(T-t)^{1-\alpha}}, \tag{5}$$

where

$$v_1(x) = \frac{a_2(x-L) - b_2}{a_2 b_1 - a_1 b_2 - a_1 a_2 L}; \qquad v_2(x) = \frac{b_1 - a_1 x}{a_2 b_1 - a_1 b_2 - a_1 a_2 L};$$

$$R(x,T) = V(x,T) + \sum_{n=1}^{\infty} E_\alpha[-\lambda_n T^\alpha]\left[Q_{0n} - V_n(0+) - v_{1n}u_1(0+) - v_{2n}u_2(0+)\right] X_n(x)$$
$$- \sum_{n=1}^{\infty} X_n(x) \int_0^T \frac{E_{\alpha,\alpha}[-\lambda_n(T-t)^\alpha]\left[{}_0^C D_t^\alpha V_n(t)\right]}{(T-t)^{1-\alpha}} dt;$$

Q_{0n}, $V_n(t)$ and $v_{(1,2)n}$ are the coefficients of the expansion of functions $Q_0(x)$, $V(x,t)$, and $v_{1,2}(x)$ with the system of eigenfunctions $\{X_n(x)\}$; $V(x,t) = v_1(x)h_1(t) + v_2(x)h_2(t)$; $E_{\alpha,\beta}(t)$ is a two-parameter Mittag-Leffler function [2] (see Appendix, item 1); $E_\alpha(t) = E_{\alpha,1}(t)$. The eigenvalues λ_n and eigenfunctions $X_n(x)$ are the result of solving the following Sturm–Liouville problem [7]:

$$X''(x) + \lambda X(x) = 0,$$
$$\left[b_i X'(x) + a_i X(x)\right]_{x=x^i} = 0, \quad i = 1, 2.$$

Equation (5) includes the fractional derivatives of functions $u_{1,2}(t)$, which are (by definition) the convolution of the first derivative of these functions with a fractional power function. Therefore, if the controls $u_{1,2}(t)$ are differentiable, then their Caputo's fractional derivative will be defined everywhere on the considered interval. Previously we have noted that controls are assumed to be functions from the space $L_\infty[0,T]$. You can additionally require that these functions are differentiable, but such a requirement will be redundant and may narrow the class of functions for admissible controls too much. We note here that the first derivative of piecewise-constant functions will be defined in the class of generalized functions and can be represented as a linear combination of delta functions, and the Caputo derivative of the delta function is defined in the class of continuous functions (see Appendix, item 2). We will show below that the optimal controls obtained as a result of solving the problem of moments [14] are represented by piecewise-constant functions (see below, formula (12)) and in the case considered in this work these functions have at most a countable number of discontinuity points and monotonicity intervals.

Functions $Q^*(x)$ and $R(x,T)$ that occur in (5) can also be expanded in the system of functions $\{X_n(x)\}$, and below we denote the corresponding coefficients of the expansion as Q_n^* and $R_n(T)$. Since the system is complete, for the equality in (5) to hold it is necessary and sufficient that a similar equality holds for the corresponding expansion coefficients for every n. Thus, expression (5) can be written in the form of some generalized problem of moments of the form

$$\int_0^T g_n(t,T)\left[v_{1n} \times {}_0^C D_t^\alpha u_1(t) + v_{2n} \times {}_0^C D_t^\alpha u_2(t)\right] dt = \tilde{c}_n(T), \quad n = 1, 2, \ldots, \tag{6}$$

where

$$\tilde{c}_n(T) = R_n(T) + v_{1n}u_1(T) + v_{2n}u_2(T) - Q_n^*,$$
$$g_n(t,T) = \frac{E_{\alpha,\alpha}[-\lambda_n(T-t)^\alpha]}{(T-t)^{1-\alpha}}. \tag{7}$$

Using the formula of fractional integration by parts (see Appendix, item 1, formula (A.4)), we can convert expression (6) to a form that contains the moments of immediate boundary controls (see Appendix, item 3):

$$\int_0^T g_n(t,T)\left[v_{1n}u_1(t) + v_{2n}u_2(t)\right] dt = c_n(T), \tag{8}$$

where

$$c_n(T) = \frac{Q_n^* - R_n(T) - \left(v_{1n}u_1(0) + v_{2n}u_2(0)\right) E_\alpha(-\lambda_n T^\alpha)}{\lambda_n}. \tag{9}$$

We should note right here that solving the problem of moments (8) generally allows one to find only the combination $\tilde{U}(t) = v_1u_1(t) + v_2u_2(t)$ and not each individual control. Individual controls can be found in a number of special cases, for example when the relation between them is known (defined by a known function), or when one of the controls is zero, or when the two controls are equal to each other. All these special cases are non-trivial and have a completely clear physical meaning: heating (or introducing a diffusing substance) in reality can be carried out not independently on each of the boundaries, but, for example, by sources with the same temperature or known (e.g., constant) temperature difference at the boundaries.

The solution of the infinite-dimensional problem of moments (8) in the general case can not always be obtained, and it is not always possible to determine its solvability [14, 18]. In addition, in practice it is often sufficient to find an approximate solution that describes the system state with acceptable accuracy. Since the series in (5) are expansions in the complete system of (eigen) functions $\{X_n(x)\}$ and are assumed to be uniformly convergent, we can replace these series with a finite sum with any given accuracy. Then such an approximate solution can be, similar to the above calculations, reduced to a finite-dimensional problem of moments whose dimension will be determined by the number of retained terms of the series occurring (5), chosen based on the required accuracy of the approximation. Below we will consider the finite-dimensional problem of moments.

It is known that the problem of moments (8) with fixed $n = N$ and $u_{1,2}(t) \in L_\infty[0, T]$ is equivalent to the following conditional convex minimization problem [14].

Problem 2 (conditional minimization). Find

$$\min_{\xi_1,\dots,\xi_N} \left(\int_0^T \left| \sum_{i=1}^N \xi_i g_i(t) \right| dt \right) = \left(\int_0^T \left| \sum_{i=1}^N \xi_i^* g_i(t) \right| dt \right) \tag{10}$$

given that

$$\sum_{i=1}^N \xi_i c_i = 1, \tag{11}$$

where ξ_i are arbitrary numbers, and the numbers ξ_i^* correspond to the solution of the minimization problem, $i = 1, \dots, N$.

In this case, the optimal control will be determined by the formula [14]

$$\tilde{U}(t) = l \operatorname{sgn} \left[\sum_{i=1}^N \xi_i^* g_i(t) \right], \quad t \in [0, T^*], \tag{12}$$

where T^* is the smallest nonnegative real root of equation

$$\Lambda_N(T) = l, \tag{13}$$

where

$$\frac{1}{\Lambda_N(T)} = \left(\int_0^T \left| \sum_{i=1}^N \xi_i^* g_i(t) \right| dt \right).$$

Formula for the general solution of problem (5) and, therefore, expressions for the moments $c_n(T)$ (formula (9)) includes the values $u_{1,2}(0)$ and $u_{1,2}(T)$. In the general case, these values should be determined based on additional assumptions or constraints imposed on the system, for example, from the principle of matching the boundary and initial conditions at given points [16]. In the case we consider here, $U(t) \in L_\infty[0, T]$, these values can be determined based on the fact that the control is represented by a piecewise constant function (12) that takes only the values $\pm l$; for example, one can let $u_{1,2}(0) = l$, $u_{1,2}(T) = (-1)^M l$ (M is the number of switchings in the corresponding controls).

The work [16] studies the correctness and solvability of the problem of moments (8) for a fixed n in case $U(t) \in L_p[0,T]$, $p > 1$; the key condition of this problem is the boundedness of the norm of functions $g_n(t,T)$ (formula (7)). It was shown that for $U(t) \in L_\infty[0,T]$ the problem of moments is correctly posed and solvable provided that inequality $\alpha > 0$ is satisfied, i.e., for all considered values $\alpha \in (0,1]$.

4. SAMPLE COMPUTATION OF THE OPTIMAL CONTROL

Let us consider the system (1) and specify the initial and final conditions in the following form:

$$Q(x,0) = Q_0,$$
$$Q(x,T) = Q^T,$$

where Q_0, Q^T are constants, $x \in [0,L]$.

In what follows we consider an example of calculating the optimal control that occurs in the Dirichlet boundary conditions. We will consider the solution obtained by approximating the infinite-dimensional problem of moments of finite-dimensionality.

Boundary conditions (3) are assumed to be the same on both boundaries and equal to

$$Q(0,t) = u(t),$$
$$Q(L,t) = u(t).$$

Eigenfunctions $X_n(x)$ and eigenvalues λ_n in this case will look like

$$X_n(x) = \sin\frac{\pi n x}{L}, \tag{14}$$

$$\lambda_n = \left(\frac{\pi n}{L}\right)^2. \tag{15}$$

Then the problem of moments (8) can be written in the following form (see Appendix, item 4):

$$\int_0^T g_n(t,T)u(t)dt = c_n(T), \tag{16}$$

where

$$c_n(T) = \frac{Q^T - Q_0 E_\alpha[-\lambda_n T^\alpha]}{\lambda_n}.$$

Note that in the resulting formula for the moments, the quantities $u_{1,2}(0)$ and $u_{1,2}(T)$ do not occur at all, so the task of finding them does not arise in this example.

We will consider an approximate solution for the problem of moments (16) when $N = 3$. Then, using condition (11), we arrive at the following unconditional minimization problem:

$$\frac{1}{\Lambda_3(T)} = \min_{\xi_1,\xi_2} \rho_\xi(T),$$

where

$$\rho_\xi(T) = \int_0^T \left| \xi_1 g_1(t) + \xi_2 g_2(t) + \frac{1 - \xi_1 c_1 - \xi_2 c_2}{c_3} g_3(t) \right| dt.$$

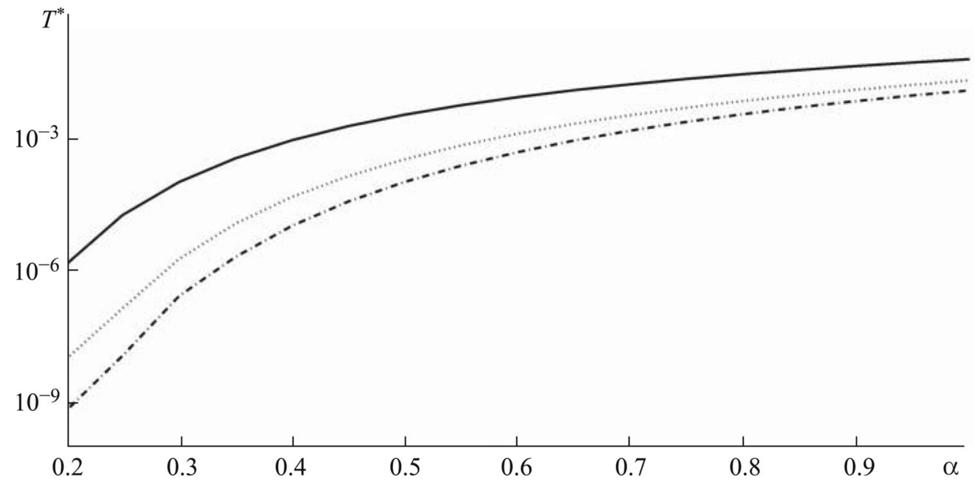

Fig. 1. Dependence of the control time on the parameter α. The Y-axis is in logarithmic scale.

This unconditional minimization problem can be solved numerically using the algorithm proposed in [14, Ch. 4, § 1]. We first set test values of $\xi^0_{1,2}$. For them, we find the point T^0 where the plot of the function $y = \rho_\xi(T)$ intersects the line $y = 1/l$. Then we minimize the function $\rho_\xi(T^0)$ with respect to $\xi_{1,2}$, finding the values $\xi^1_{1,2}$ for which there is a new intersection point T^1, and so on. The criterion for stopping this procedure at iteration $k+1$ is the condition $|\frac{T^{k+1}-T^k}{T^k}| < \epsilon$, where ϵ is the predefined estimation accuracy. This algorithm was implemented in the software suite MATLAB 7.9. The estimation accuracy of the control time was set at the level $\epsilon = 0.05$, and the point of intersection of the above-mentioned graphs was found with the same accuracy.

The control time T^* and numbers $\xi^*_{1,2}$ found using the above algorithm make it possible, according to (12), to write down an approximate solution of the optimal control problem posed in the form

$$u(t) = l\mathrm{sgn}\left(\int\limits_0^t \left[\xi_1^* g_1(\tau) + \xi_2^* g_2(\tau) + \frac{1-\xi_1^* c_1 - \xi_2^* c_2}{c_3} g_3(\tau)\right] d\tau\right).$$

Having computed the control, we can now compute an approximation for the system state at the final time moment T^*, using an approximation of the solution of (5) as a partial sum

$$Q^N(x,T^*) = \frac{2}{\pi}\sum_{n=1}^{N} \frac{\sin\sqrt{\lambda_n}x}{n}\left(1-(-1)^n\right)$$
$$\times \left[Q_0 E_\alpha(-\lambda_n (T^*)^\alpha) - u(T^*) + \lambda_n \int\limits_0^{T^*} g_n(t)u(t)dt\right] + u(T^*). \quad (17)$$

Next, we present the results of computations that illustrate the resulting solution and its properties. Calculations were carried out with the following values of the problem's parameters: $L = 1$, $Q_0 = 10$, $Q^T = 30$.

Figure 1 shows the dependencies of the control time on the parameter α for different values of the parameter l: $l = 100$ (solid line), $l = 200$ (dotted line), $l = 500$ (dot-and-dash line). We can see that the curve is characterized by rapid growth for small α and saturation in the region $\alpha > 0.8$. In addition, we see that as the value of l increases ("weakening" the constraint), the value of T^* decreases. Figure 2 shows the temporal dependencies of the controls obtained as a

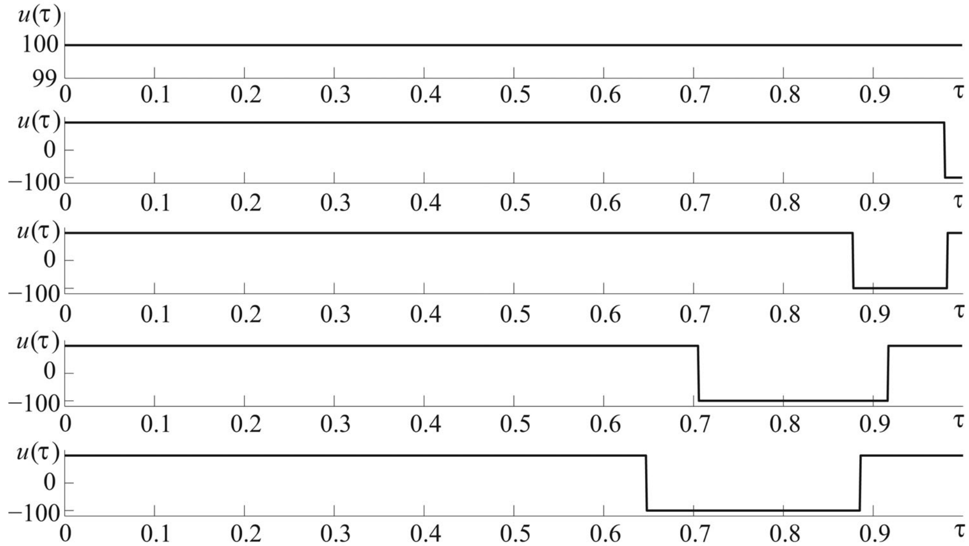

Fig. 2. Temporal dependencies of the control for different values of the parameter α. Top down: $\alpha = 0.2$, $\alpha = 0.35$, $\alpha = 0.6$, $\alpha = 0.9$, $\alpha = 1$.

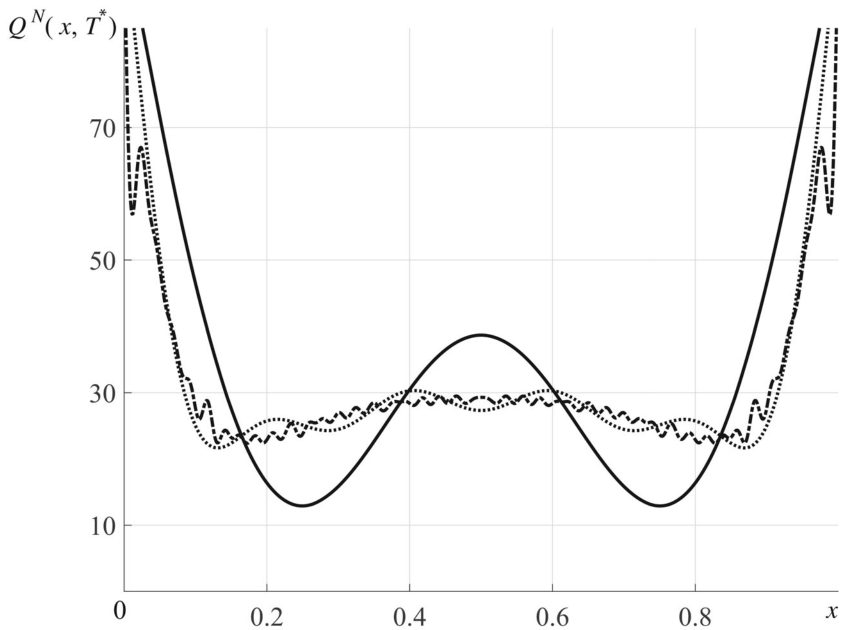

Fig. 3. The spatial distribution of the system state approximation for $t = T^*$ for different values of the parameter N: $N = 3$ (solid line), $N = 10$ (dotted line), $N = 100$ (dot-and-dash line).

result of calculation for different values of α for $l = 100$. In this case, the relative time $\tau = t/T^*$ is plotted along the X-axis. We can see that when $\alpha < 0.5$ and $\alpha > 0.5$ the control has a qualitatively different character: in the first of these areas it either does not have switching points or has only one such point (for $\alpha > 0.25$), and in the second of these areas, the control always has two switching points. As α increases, the distance between switching points also increases, and their position shifts toward smaller values of the relative time.

Figure 3 presents an evaluation of the system state at $t = T^*$ for $\alpha = 0.2$, $l = 100$ and various values of the parameter N. It can be seen that although the control was calculated with respect to only three moments, increasing the number of terms in the expression for estimating the state makes it possible to increase the accuracy of the resulting estimate and bring the state to the required constant value $Q^T = 30$. There is also a noticeable manifestation of the Gibbs effect (the presence of oscillations and the presence of regions near the boundaries of the interval where the solution is very different from the desired one), the magnitude of which decreases as N grows. We should also note that we have seen no explicit dependence of the state estimate on the index α: for the same values of the remaining parameters, the curves corresponding to different values of α practically coincide.

Finally, we note that the results obtained in this section with $\alpha = 1$ qualitatively coincide with the corresponding results obtained in [14, Chap. 5, §§ 2 and 3] and [19, Chap. 4, §§ 1 and 2]. In addition, the distribution of control switching points on the time interval is of the same nature as for systems with integer order [19, 20].

5. CONCLUSION

In this work we have studied the problem of finding a time-optimal boundary control with respect to a given constraint on the control norm for a system with distributed parameters defined by a diffusion equation with a fractional time derivative. We have considered a control that can be represented in the form of a piecewise constant function. The problem has been reduced to an infinite-dimensional problem of moments and solved with a finite-dimensional approximation. We have shown an example of solving the problem and analyzed properties of the solution. Our results can be used in the search for optimal control for systems defined by equations of quasiparabolic type with a fractional time derivative, in particular, for controlling thermal and diffusion processes in various systems.

ACKNOWLEDGMENTS

The authors are grateful to A.G. Kushner and E.M. Solnechnyi for a careful perusal of our work and valuable comments, as well as reviewers of the paper for constructive recommendations.

APPENDIX

1. Let us give a brief outline of the fractional calculus used in this article.

The left-sided Caputo's fractional derivative of order $\alpha \in (0, 1]$ with respect to time for a function $Q(x, t)$ is defined by the following expression [2, Ch. 2]:

$$
{}_0^C D_t^\alpha Q(x,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\partial Q}{\partial t} \frac{d\tau}{(t-\tau)^\alpha}.
$$

The right-hand Riemann–Liouville fractional derivative and integral of order $\alpha \in (0, 1]$ of a function $f(t)$ are defined as follows [2, Ch. 2]:

$$
{}_t^{RL} D_T^\alpha f(t) = -\frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int\limits_t^T \frac{f(\tau) d\tau}{(\tau - t)^\alpha}, \tag{A.1}
$$

$$
{}_t I_T^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^T \frac{f(\tau) d\tau}{(\tau - t)^{1-\alpha}}. \tag{A.2}
$$

We have the identity

$$
{}_t^{RL} I_T^{1-\{\alpha\}} = {}_t^{RL} D_T^{\{\alpha\}-1}. \tag{A.3}
$$

The formula of fractional integration by parts looks like [21]:

$$\int_0^T f(t) \times {}_0^C D_t^{\alpha} g(t) dt = \int_0^T g(t) \times {}_t^{RL} D_T^{\alpha} f(t) dt$$
$$+ \sum_{j=0}^{[\alpha]} \left[{}_t^{RL} D_T^{\{\alpha\}+j-1} f(t) \times {}_t^{RL} D_T^{[\alpha]-j} g(t) \right] \Big|_0^T, \tag{A.4}$$

where $[\alpha]$ and $\{\alpha\}$ are respectively the integer and fractional parts of α.

The Mittag-Leffler function plays an important role in the fractional calculus; representation of this function in the form of a power series [2, Ch. 1] is as follows:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{A.5}$$

This series converges in absolute value on the entire real axis.

2. Consider a piecewise constant function of the form

$$f(t) = \operatorname{sgn}\left(\sum_{k=1}^{\infty} \xi_k g_k(t) \right),$$

where ξ_i are numbers, $g_k(t)$ are the functions which monotonic, continuous and positive definite almost everywhere, $t \in [0, T]$. We assume that this series converges in absolute value on the interval $t \in [0, T]$. The functions $g_k(t)$ given by formula (7) have one breakpoint, turning to infinity at $t = T$, but on the semi-interval $t \in [0, T)$ they are continuous, monotonic, and positive definite. Consequently, each such function belongs to a class of functions with a finite number of monotonicity intervals, and a linear combination of a countable number of such functions will change its sign at most a countable number of times. Then function $f(t)$ will have at most a countable number of jumps in the semi-interval $t \in [0, T)$ (and on the segment $t \in [0, T]$, where it will be defined almost everywhere).

Then the derivative of the function $f(t)$ will be defined in the class of generalized functions and can be represented in the form

$$\frac{df(t)}{dt} = \sum_{m=1}^{\infty} l_m \delta(t - t_m),$$

where t_m are breakpoints of the function $f(t)$, l_m are coefficients. Consequently, the Caputo derivative of the function $f(t)$ will look like

$${}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{\infty} l_m (t - t_m)^{-\alpha}.$$

3. We give a detailed derivation of the formulas (8) and (9). Consider the integral in (6) that contains the moments of fractional order derivatives of the boundary controls. Using the fractional integration by parts formula (A.4), we can obtain for the integral in question (6):

$$\int_0^T g_n(t, T) \left[v_{1n} \times {}_0^C D_t^{\alpha} u_1(t) + v_{2n} \times {}_0^C D_t^{\alpha} u_2(t) \right] dt$$
$$= \int_0^T [v_{1n} u_1(t) + v_{2n} u_2(t)] \times {}_t^{RL} D_T^{\alpha} g_n(t, T) dt \tag{A.6}$$
$$+ \left[[v_{1n} u_1(t) + v_{2n} u_2(t)] \times {}_t^{RL} I_T^{1-\alpha} g_n(t, T) \right] \Big|_0^T.$$

Here we have used identity (A.3). According to the definitions of the right-hand Riemann–Liouville fractional differentiation and integration operators (A.1), (A.2), and the above (see the explanation for formula (6)) expression for $g_n(t,T)$, we get that

$$
{}_t^{RL}D_T^{\alpha} g_n(t,T) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T (\tau-t)^{-\alpha}(T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[-\lambda_n(T-\tau)^{\alpha}\right]d\tau,
$$

$$
{}_t^{RL}I_T^{1-\alpha} g_n(t,T) = \frac{1}{\Gamma(1-\alpha)}\int_t^T (\tau-t)^{-\alpha}(T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[-\lambda_n(T-\tau)^{\alpha}\right]d\tau.
$$

The integral in the right-hand side of the latter two expressions can be calculated with a representation of the Mittag-Leffler function in the form of a power series (A.5). Since this series converges in absolute value on the entire real axis, one can immediately change the order of integration and summation in the formula and write:

$$
\begin{aligned}
&\int_t^T (\tau-t)^{-\alpha}(T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[-\lambda_n(T-\tau)^{\alpha}\right]d\tau \\
&= \sum_{k=0}^{\infty}\frac{(-\lambda_n)^k}{\Gamma[\alpha(k+1)]}\int_t^T (\tau-t)^{-\alpha}(T-\tau)^{\alpha(k+1)-1}d\tau \\
&= \sum_{k=0}^{\infty}\frac{(-\lambda_n)^k}{\Gamma[\alpha(k+1)]}(T-t)^{\alpha k}B(1-\alpha,\alpha(k+1)) \\
&= \Gamma(1-\alpha)E_{\alpha}\left[-\lambda_n(T-t)^{\alpha}\right],
\end{aligned}
$$

where $B(\alpha,\beta)$ is the Euler beta function, $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

We now compute the derivative of the resulting expression, using also the representation of the Mittag-Leffler function in the form of a power series (A.5):

$$
\begin{aligned}
\frac{d}{dt}\Gamma(1-\alpha)E_{\alpha}\left[-\lambda_n(T-t)^{\alpha}\right] &= \frac{1}{T-t}\sum_{k=0}^{\infty}\frac{\left[-\lambda_n(T-t)^{\alpha}\right]^k \alpha k}{\Gamma(\alpha k+1)} \\
&= \frac{1}{T-t}\sum_{k=1}^{\infty}\frac{\left[-\lambda_n(T-t)^{\alpha}\right]^k}{\Gamma(\alpha k)}.
\end{aligned}
$$

The sum in the resulting expression can be (by making the substitution $k=m+1$ in the subscript) rewritten as

$$
\sum_{k=1}^{\infty}\frac{\left[-\lambda_n(T-t)^{\alpha}\right]^k}{\Gamma(\alpha k)} = \sum_{m=0}^{\infty}\frac{\left[-\lambda_n(T-t)^{\alpha}\right]^{m+1}}{\Gamma(\alpha m+\alpha)} = -\lambda_n(T-t)^{\alpha}E_{\alpha,\alpha}\left[-\lambda_n(T-t)^{\alpha}\right].
$$

In the end we will have

$$
\begin{aligned}
{}_t^{RL}D_T^{\alpha} g_n(t,T) &= -\lambda_n(T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[-\lambda_n(T-\tau)^{\alpha}\right] = -\lambda_n g_n(t,T), \\
{}_t^{RL}I_T^{1-\alpha} g_n(t,T) &= E_{\alpha}\left[-\lambda_n(T-\tau)^{\alpha}\right].
\end{aligned}
$$

Substituting these expressions in the right-hand side of (A.6), we get:

$$\int_0^T g_n(t,T)\left[v_{1n}\times {}^C_0D^\alpha_t u_1(t)+v_{2n}\times {}^C_0D^\alpha_t u_2(t)\right]dt$$
$$=-\lambda_n\int_0^T \left[v_{1n}u_1(t)+v_{2n}u_2(t)\right]g_n(t,T)dt+\left[v_{1n}u_1(T)+v_{2n}u_2(T)\right]$$
$$-\left[v_{1n}u_1(0)+v_{2n}u_2(0)\right]E_\alpha\left[-\lambda_n T^\alpha\right].$$

Substituting the resulting formula in (6) and taking into account the explicit expression for $\tilde{c}_n$, we get formula (8) due to notation (9).

4. The initial, final, and boundary conditions chosen in Section 4 follow from (2)–(4) when the following relations hold:

$$a_1=a_2=1,\qquad b_1=b_2=0,$$
$$h_1(t)=h_2(t)=0,\qquad u_1(t)=u_2(t)=u(t),$$
$$v_1(x)=1-\frac{x}{L},\qquad v_2(x)=\frac{x}{L}.$$

We introduce in addition the function $f(x)=1$, $x\in[0,L]$. Then $\forall(x,t)\in\Omega$

$$V(x,t)=0,$$
$$v_1(x)+v_2(x)=f(x)=1, \tag{A.7}$$
$$v_1(x)u_1(t)+v_2(x)u_2(t)=f(x)u(t)=u(t). \tag{A.8}$$

Due to (A.7), (A.8), the expression in square brackets in (8) can be written as

$$v_{1n}u_1(t)+v_{2n}u_2(t)=(v_1(x)+v_2(x))_n u(t)=f_n u(t).$$

We now write the problem of moments (8) and the moments (9) for the example in question, taking into account the above relations:

$$\int_0^T g_n(t,T)\left[v_{1n}u_1(t)+v_{2n}u_1(t)\right]dt=f_n\int_0^T g_n(t,T)u(t)dt=c_n(T), \tag{A.9}$$
$$c_n(T)=\frac{Q^Tf_n-E_\alpha[-\lambda_nT^\alpha]\left(Q_0-u(0)\right)f_n-f_nu(0)E_\alpha[-\lambda_nT^\alpha]}{\lambda_n}$$
$$=f_n\frac{Q^T-Q_0E_\alpha[-\lambda_nT^\alpha]}{\lambda_n}.$$

As we can see from the resulting formulas, the left- and right-hand sides of expression (A.9) contain the factor f_n (which is not identically zero). Canceling it out, we get the expression (16).

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This paper was recommended for publication by P.V. Pakshin, a member of the Editorial Board