ISSN 0005-1179, Automation and Remote Control, 2018, Vol. 79, No. 2, pp. 286–288. © Pleiades Publishing, Ltd., 2018.
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STOCHASTIC SYSTEMS

On One-Sided Convergence of a Modified Stochastic Approximation Process

T. P. Krasulina

St. Petersburg State University, St. Petersburg, Russia
e-mail: agelig@yandex.ru

Received March 24, 2017

Abstract—We study the one-sided convergence of the modified Anbar's process in previously unexplored cases of the choice of the sequence of steps.

Keywords: stochastic approximation, Anbar's process, one-sided convergence, slowly changing sequence of numbers, Kroneker's lemma

DOI: 10.1134/S0005117918020078

1. INTRODUCTION

Stochastic approximation is interesting for a wide range of researchers. It is being used in computational mathematics to solve stochastic programming problems and by researchers in biology, chemistry, and medicine. Stochastic approximation is also used in problems of recognition, identification, learning, and adaptation. In a number of problems it is important that the estimates converge to the solution from only one side (one-sided convergence). In such problems researchers often use a modified Robbins–Monro process, namely Anbar's process [1]. In this work we study one-sided convergence of the Anbar's process for some new cases of the choice of the sequence of steps.

2. PROBLEM SETTING AND MAIN RESULTS

Let $Y(x)$ be a random value depending on the parameter x,

$$H(y|x) = P(Y(x) < y)$$

and suppose that regression function $\mathcal{M}(x) = \int_{-\infty}^{+\infty} ydH(y/x)$ is continuous. The problem is to estimate the root of the regression function θ, $\mathcal{M}(\theta) = 0$.

In a number of practical problems, we need to ensure that the estimate converges to the necessary parameter of the regression function from only one side.

Consider Anbar's process [1]

$$X_{n+1} = X_n - a_n(Y_n(X_n) + b_n),$$

where X_1 is the initial random value, $a_n \geqslant 0$, $b_n \geqslant 0$. The following theorem holds.

Theorem. *Suppose that the following conditions hold*:

1) $|\mathcal{M}(x)| \leqslant D|x| + B$, *where* $D \geqslant 0$, $B \geqslant 0$ *are certain constants,* $\mathcal{M}(x)(X - \theta) > 0$, $x \neq \theta$;

2) $\inf |\mathcal{M}(x)| > 0$ *for* $\varepsilon < |x - \theta| < \varepsilon^{-1}$, $0 < \varepsilon < 1$;

3) $\mathcal{M}(x) = \alpha(x - \theta) + \delta(x, \theta)$, $\delta(x, \theta) = o(x - \theta)$, $x \to \theta$, $\alpha > 0$;

4) $\mathrm{E}z^2(x) \leqslant c < \infty$ *for all* x, $Z_n(X_n) = Y_n(X_n) - \mathcal{M}(X_n)$, $\mathrm{E}(Z_n(X_n)|X_1, X_2, \ldots, X_n) = 0$, E *is expectation*;

5) $a_n = \dfrac{A}{n}$, $b_n = \dfrac{b}{n^\beta}$, $A > 0$, $b > 0$, $0 < \beta < \dfrac{1}{2}$, $2A\alpha \geqslant 1$.

Then $X_n \to \theta$ *for* $n \to \infty$ *with probability one, and* $X_n > \theta$ *only a finite number of times.*

Proof. Without loss of generality we assume that $\theta = 0$. By the theorem's conditions we get

$$X_{n+1} = X_n(1 - d_n/n) - a_n Z_n(X_n) - a_n b_n,$$

where $d_n = A(\alpha + \Psi(X_n))$, $\Psi(x) \to 0$ for $x \to 0$. Iterating, we get

$$X_{n+1} = X_1\beta_{1n} - \sum_{k=1}^{n} a_k Z_k(X_k)\beta_{kn} - \sum_{k=1}^{n} a_k \beta_{kn} b_k,$$

where $\beta_{kn} = \prod_{j=k}^{n}(1 - d_j/j)$, $\beta_{nn} = 1$, $1 \leqslant k \leqslant n-1$.

By definition, $d_n \to a = A\alpha$ for $n \to \infty$ with probability one. We denote $\gamma_n = \prod_{j=j_0}^{n}(1 - d_j/j) \approx n^{-a}\tau_n$, where $j_0 = \min(j : d_k/k < 1, k > j)$, τ_n is a slowly changing sequence [2]. We remind that a sequence of numbers $f(n)$ is called slowly changing in the sense of Siegmund if $f(n)n^\delta \uparrow$, $f(n)n^{-\delta} \downarrow$ for $n \geqslant n_0(\delta)$ for every $\delta > 0$.

Further, for $n \geqslant j_0$ it holds that

$$X_{n+1} = \gamma_n \left(\varepsilon_0 - \sum_{k=j_0}^{n} a_k Z_k(X_k)\gamma_k^{-1} - \sum_{k=j_0}^{n} a_k b_k \gamma_k^{-1} \right),$$

where

$$\varepsilon_0 = \left(\beta_{1(j_0-1)} X_1 + \sum_{k=1}^{j_0-1} a_k \beta_{k(j_0-1)} Z_k(X_k) + \sum_{k=1}^{j_0-1} a_k \beta_{k(j_0-1)} b_k \right),$$

j_0 and γ_n have been defined above.

The following representation holds:

$$X_{n+1} = \gamma_n \left(\bar{\varepsilon} - \sum_{k=1}^{n} a_k Z_k(X_k)\gamma_k^{-1} - \sum_{k=1}^{n} a_k b_k \gamma_k^{-1} \right),$$

where

$$\bar{\varepsilon} = \varepsilon_0 - \sum_{k=1}^{j_0-1} a_k Z_k(X_k) k^a - \sum_{k=1}^{j_0-1} a_k b_k k^a, \quad 1 \leqslant k \leqslant j_0.$$

Consider the equality

$$n^\beta X_{n+1} = n^\beta \gamma_n \varepsilon - n^\beta \gamma_n \sum_{k=1}^{n} a_k Z_k(X_k)\gamma_k^{-1} - n^\beta \gamma_n \sum_{k=1}^{n} a_k b_k \gamma_k^{-1}.$$

Since $A\alpha \geqslant \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$, $-A\alpha + \beta < 0$ and τ_n is a slowly changing sequence, then $\tau_n n^{-a+\beta} \to 0$ for $n \to \infty$ with probability one. Since $\gamma_n \approx n^{-a}\tau_n$, we get that $\gamma_n n^\beta \to 0$ with probability one.

Then $\lim_{n\to\infty} \varepsilon\gamma_n n^\beta = 0$ with probability one. By the theorem's conditions and Doob's theorem [3] it follows that $\sum_{k=1}^{\infty} k^{-1+\beta} Z_k(X_k)$ converges with probability one. By the above and Kroneker's lemma [4] we find that

$$\gamma_n n^\beta \sum_{k=1}^{n} a_k Z_k(X_k)\gamma_k^{-1} \to 0$$

for $n \to \infty$ with probability one. By the theorem's conditions and [2] we get the relation $\gamma_n n^\beta \sum_{k=1}^{n} a_k b_k \gamma_k^{-1} \to \frac{Ab}{a-\beta}$ for $n \to \infty$ with probability one. This implies that $\lim_{n\to\infty} n^\beta X_n = -\frac{Ab}{a-\beta}$ and $X_n \to 0$ from below for $n \to \infty$. This completes the proof of the theorem.

We now note the following defect in [5, 6]: these works omit the continuity condition for the regression function $\mathcal{M}(x)$, which is used to prove convergence of the Robbins–Monro algorithm. If function $\mathcal{M}(x)$ is piecewise continuous, and discontinuities of the first kind are allowed, then the Robbins–Monro process was modified in [7].

3. CONCLUSION

With slowly changing sequences in the sense of Siegmund, we have shown convergence from below of the modified Robbins–Monro process (Anbar's process) for cases of the choice of sequence of steps that had not been considered before.

ACKNOWLEDGMENTS

This work was supported by the St. Petersburg State University research topic 6.38.230.2015.

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This paper was recommended for publication by A.V. Nazin, a member of the Editorial Board