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Algorithms for Constructing Optimal N-Networks in Metric Spaces

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Abstract—We study optimal approximations of sets in various metric spaces with sets of balls of equal radius. We consider an Euclidean plane, a sphere, and a plane with a special non-uniform metric. The main component in our constructions of coverings are optimal Chebyshev n-networks and their generalizations. We propose algorithms for constructing optimal coverings based on partitioning a given set into subsets and finding their Chebyshev centers in the Euclidean metric and their counterparts in non-Euclidean ones. Our results have both theoretical and practical value and can be used to solve problems arising in security, communication, and infrastructural logistics.

Keywords: optimal Chebyshev network, optimal covering, Chebyshev center, metric, Voronoi diagram, Dirichlet cells.

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1. INTRODUCTION

The problem of approximating complex geometric figures with sets that are more convenient for processing is a classical problem in computational geometry [1] and is interesting both from a theoretical point of view and in relation to multiple applications in problems of cellular [2] and space communication [3], logistics [4], in constructing reachability sets for controllable systems [5].

The easiest and at the same time most convenient way to proceed is to replace the figure with a union of a fixed number of points. Such constructions were first introduced by A.L. Garkavi who defined, in particular, the notion of the optimal Chebyshev n-network [6, 7]. In previous work, we have already studied the constructions of approximations for objects with sets of a fixed number of points on the Euclidean plane [8, 9], on a sphere in a Euclidean space [10], and on a plane with non-uniform metric [11, 12]. All of these problems admit a common mathematical formalization. Consider a metric space X with metric ρ. We pose the problem of minimizing, for a given compact set M, the value $\max_{\mathbf{m}\in M} \min_{\mathbf{s}_i\in S} \rho_f(\mathbf{m}, \mathbf{s}_i)$, where S is a set containing a given number of points. This problem can be solved with methods of computational geometry [1].

In this work, we apply new methods for constructing optimal n-networks and their generalizations in various metric spaces based on iterative computational procedures. Their key elements include partitioning sets into subsets lying in the influence zone of each point from the previous iteration and computing points that are centers of these zones in the considered metric.

2. CONSTRUCTING AN OPTIMAL NETWORK IN THE EUCLIDEAN METRIC

One of the main practical problems that reduce to finding an optimal Chebyshev n-network is the problem of placing communication towers [2]. If we assume that communication quality is directly related to the proximity of a user to the nearest tower, and assume the tower itself to be a material point, we get an optimal placement criterion which in this case is minimization of Hausdorff distance [13] between the compact set and a set with a given number of points. This problem is closely related to another problem, which grows in importance every year: the problem of optimal placement of sensor networks [14] created with high-precision sensors that control a certain territory. From the geometric point of view the domain of operation for each sensor is a circle. Sensor networks are applied to monitor natural phenomena, solve problems in biology, medicine, and security. Similar settings in the domain of architecture have been considered in [15, Chapter 3].

Let us formulate the optimal approximation problem for a compact set on a plane with Euclidean metric. We assume that the deviation of one bounded closed set A from another set B is defined in the Hausdorff metric [13] as

$$h(A, B) = \max_{\mathbf{a}\in A} \min_{\mathbf{b}\in B} \|\mathbf{a} - \mathbf{b}\|. \tag{1}$$

Now an n-network [7] is a nonempty set on the plane that consists of at most n points. We denote by Σ_n the set of all n-networks.

Problem 1. For a given compact set $M \subset \mathbb{R}^2$ and number $n \in \mathbb{N}$, find such an n-network for which the Hausdorff distance $h(M, S_n)$ is minimal among all elements of the set Σ_n.

This problem can be considered as an optimal covering problem for a compact set M by a union of n balls of equal radius r. The optimality criterion here is the value of r, and we would like to minimize it. Points $\mathbf{s}_i$, $i = 1, \ldots, n$, of the n-network S_n are centers of the balls $O(s_i, r)$ that constitute an optimal coverage, and their radius equals $r = h(M, S)$.

The problem of optimal approximation of a set with n points in the simplest case, for $n = 1$, reduces to finding the Chebyshev center of a set. This notion was introduced by A.L. Garkavi for a set M in Banach space [6]. In a Euclidean space of dimension m the Chebyshev center $\mathbf{c}(M)$ of a set is the center of a ball with smallest radius that completely contains M [7]. Algorithms for constructing it have been shown, in particular, in [8]. For $n > 1$ Problem 1 is to construct an optimal Chebyshev n-network for the set M [16].

Various methods for solving Problem 1 for polygons have been proposed and implemented by S.A. Piyavskii and V.F. Krotov [17, 18]. They studied the problems on covering flat cells for constructing networks of man-made satellites. Optimal Chebyshev n-networks have been considered for a square [19, 20] and a circle [21]. It has been shown for small n that these results are optimal. One of the authors has considered Problem 1 before for some classes of flat sets [9].

For a given set M and fixed n, the problem of constructing an optimal Chebyshev n-network S_n can be solve with various methods. We have already developed a software suite [10] based on applying iterative algorithms for stepwise improvement of an initial network S_n^0. In consists of several procedures that can be viewed as separate algorithms.

The main geometric method for constructing, based on the current n-network S_n, a new iteration $\widehat{S}_n$ that would in some sense more precisely reflect the geometry of set M, is the following scheme based on Voronoi diagrams [1].

Algorithm 1.

1. Construct the Voronoi diagram for the points of n-network S_n

$$W(S_n) = \left\{\mathbf{w} \in \mathbb{R}^2 \colon \exists \mathbf{s}_i \in S_n, \exists \mathbf{s}_j \in S_n, \|\mathbf{w} - \mathbf{s}_i\| = \|\mathbf{w} - \mathbf{s}_j\| = h(\{\mathbf{w}\}, S_n) (i \neq j)\right\},$$

i.e., find points that have two or more nearest elements from the set S_n. By construction, the Voronoi diagram consists of rays, segments, and their junction points.

2. For each point $\mathbf{s}_i \in S_n$, $i = \overline{1,n}$, construct with the Voronoi diagram a region on the plane $\Pi(S_n, \mathbf{s}_i) = \left\{\mathbf{p} \in \mathbb{R}^2 \colon \forall j = \overline{1,n}\ (\|\mathbf{p} - \mathbf{s}_i\| \leqslant \|\mathbf{p} - \mathbf{s}_j\|)\right\}$.

3. Find Dirichlet cells for the points $\mathbf{s}_i \in S_n$, $i = \overline{1,n}$: $M(S_n, \mathbf{s}_i) = P_i \cap M$, $i = \overline{1,n}$, i.e., subsets of M that lie no further from $\mathbf{s}_i$ than from other points in the n-network S_n.

4. Construct a new network $\widehat{S}_n = \{\widehat{\mathbf{s}}_i\}_{i=1}^n$ by the following rule:

$$\widehat{\mathbf{s}}_i = \begin{cases} \mathbf{c}(M(S_n, \mathbf{s}_i)), \text{ if } M(S_n, \mathbf{s}_i) \neq \varnothing \\ \mathbf{s}_i, \text{ if } M(S_n, \mathbf{s}_i) = \varnothing, \end{cases} \qquad i = \overline{1,n}. \tag{2}$$

The algorithm is applied multiple times until the deviation of the new network from the previous becomes less than a given value δ.

Theorem 1. *Let M be a closed bounded set in $\mathbb{R}^2$. Then for every n-network S_n and the set M obtained for it after running Algorithm 1, the n-network $\widehat{S}_n$ satisfies the estimate*

$$h(M, \widehat{S}_n) \leqslant h(M, S_n). \tag{3}$$

If, moreover, points of n-networks S_n and $\widehat{S}_n$ satisfy

$$\forall i = \overline{1,n}, \quad \mathbf{s}_i \neq \widehat{\mathbf{s}}_i, \tag{4}$$

then inequality (3) *is strict:*

$$h(M, \widehat{S}_n) < h(M, S_n). \tag{5}$$

Proof of Theorem 1 is given in the Appendix.

The algorithm 1 is completely similar to, e.g., algorithm A1 from [22]. That work indicates that in the general case there is no convergence to the globally optimal solution for Algorithm 1. However, for each generation of initial conditions, under Hausdorff iterations the distance between the current and next value of the n-network in the limit tends to zero, which is also supported by the results of [17, 18]. Therefore, some approximation of the optimal network will always be found in finite time. Due to stochastic choice of initial conditions we can find the optimal among obtained results.

An important problem in the development of a software suite is to generate the initial iteration of an n-network $\overline{S}_n$ to which we can then apply Algorithm 1. This generation is supposed to, on one hand, provide a relatively uniform distribution of points across the entire region of the compact set M, while not deviating too far from this region (although note that there may exist optimal n-networks part of whose points lie outside the set M and even outside its convex hull). On the other hand, for every run of the software suite the initial n-network $\overline{S}_n$ must differ from the previous ones so that we would be able to choose the best approximation out of the ones it obtains. Naturally, figures with different geometry may require different generation schemes. In particular, for a square centered at the origin and with sides of length $2l$ parallel to the coordinate axes we can use the following scheme.

Algorithm 2.

1. Specify a number $\gamma \in (0, 1)$ as a parameter for generating stochastic components of the coordinates.

2. Find the least natural number m that satisfies inequality $m^2 \geqslant n$.

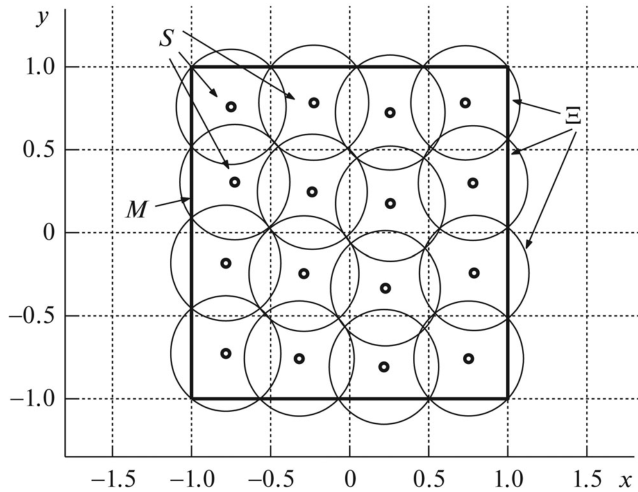

Fig. 1. Set M in Example 1, its optimal 16-network approximation and set of circles Ξ.

3. Divide the square M into m^2 equal squares $M_i, i = \overline{1, m^2}$, with sides of length $2l/m$ parallel to the coordinate axes. Squares are numbered. The top row contains squares numbered 1 to m from left to right; the second row from above, from $m + 1$ to $2m$, and so on. We denote by x_i^m and y_i^m respectively the horizontal and vertical coordinate of the center of square M_i with number i.

4. Generate two arrays of random numbers each of which consists of n elements: $X^* = \{x_i^*\}_{i=1}^n$ and $Y^* = \{y_i^*\}_{i=1}^n$.

5. If $n > m^2 - m$, as the initial n-network $\overline{S}_n$ take the set of points with coordinates

$$\overline{S}_n = \left\{\overline{\mathbf{s}}_i = \left(x_i^m + \gamma l(x_i^* - 0.5)/m, y_i^m + \gamma l(y_i^* - 0.5)/m\right) : i = \overline{1, n}\right\}. \tag{6}$$

6. If $n \leqslant m^2 - m$, as the initial n-network $\overline{S}_n$ take the set of points with coordinates

$$\overline{S}_n = \left\{\overline{\mathbf{s}}_i = \left(x_{i-m}^m + \gamma l(x_i^* - 0.5)/m, y_{i-m}^m + \gamma l(y_i^* - 0.5)/m\right) : i = \overline{1, n}\right\}. \tag{7}$$

Remark 1. Algorithm 2 can be used to generate an initial approximation of an n-network not only for a square but also for a figure which is close in geometry and is embedded into the square.

Example 1. Let us solve Problem 1 for $n = 16$ for the set M, a square with side 2 and center at the origin.

We solved it numerically, with multiple applications of Algorithm 1 for initial generation of n-networks obtained with Algorithm 2.

Approximation S_{16} for the optimal 16-network has the form

$$\begin{aligned} S_{16} \approx \{&(-0.7834; -0.7273), (-0.7834; -0.1820), (-0.7259; 0.3049), (-0.7495; 0.7591), \\ &(-0.3188; -0.7577), (-0.2889; -0.2458), (-0.2364; 0.2469), (-0.2269; 0.7840), \\ &(0.2168; -0.8078), (0.2263; -0.3343), (0.2579; 0.1756), (0.2563; 0.7249), \\ &(0.7524; -0.7573), (0.7879; -0.2420), (0.7801; 0.2992), (0.7318; 0.7833)\}. \end{aligned}$$

The Hausdorff distance between square M and the optimal 16-network approximation is $r = h(M, S_{16}) \approx 0.3482$. Note that the resulting 16-network improves over the result found by the authors in [10] with a 16-network whose Hausdorff distance from the square M was $\widetilde{r} = 0.3521$.

The set M, optimal 16-network S, and the set of circles Ξ covering set M with minimal radius are shown on Fig. 1. Although the number of points in the optimal n-network approximation is a

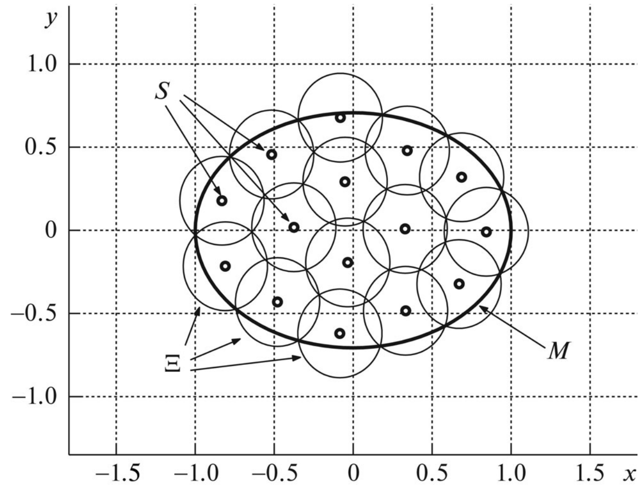

Fig. 2. Set M in Example 2, its optimal 15-network approximation, and set of circles Ξ.

square of 4, they are not placed at the centers of squares $M_i,\ i = \overline{1,16}$, from which the figure M is composed, but form a rather complex structure. Note that the optimal 9-network approximations for the square found in [21] also do not form a rectangular grid.

Example 2. Let us solve Problem 1 for $n = 15$ for the set M, an ellipse bounded by the curve $x^2 + 2y^2 = 1$.

We solve this problem in the same way as in the previous example. One characteristic feature of the set M is that its boundary is a curve of degree two. Therefore, in the construction of a Dirichlet region we need to find intersections of line segments that occur in the Voronoi diagram with an ellipse.

Approximation S_{15} for the optimal 15-network has the form

$$S_{15} \approx \{(-0.4787; -0.4315), (-0.5162; 0.4561), (-0.0340; -0.1933), (-0.3757; 0.0182),$$
$$(0.3318; -0.4839), (-0.0512; 0.2927), (-0.0815; 0.6776), (-0.0839; -0.6200),$$
$$(0.8428; -0.0088), (-0.8308; 0.1779), (0.6704; -0.3234), (0.3294; 0.0086),$$
$$(0.3426; 0.4801), (-0.8092; -0.2161), (0.6869; 0.3190)\}.$$

Hausdorff distance between the ellipse M and the approximation for the optimal 15-network is $r = h(M, S_{15}) \approx 0.2668$. The set M, optimal 15-network S, and set of circles Ξ covering set M with minimal radius are shown on Fig. 2.

3. CONSTRUCTING AN OPTIMAL NETWORK ON A SPHERE

In the design of control systems for underwater robots we need to solve the problem of placing the sensors on a hemisphere that would ensure covering of the entire hemisphere for a given radius of sensor operation [23]. From the mathematical point of view this is a problem of finding the set of points on a sphere such that a given region lies inside the union of spherical segments of equal radius with centers in these points [24]. In other words, the problem reduces to constructing a counterpart of an optimal Chebyshev n-network in the spherical metric. Similar problems also arise in the design of networks of man-made Earth satellites intended for communication, in the monitoring of ecological processes, or for navigation. In this case we need to account for the shape of the Earth, which in this case can be assumed to be a ball.

Let us formulate the problem of optimal approximation for a compact set on a sphere of unit radius, which we denote by Θ. We introduce a metric on a sphere.

Definition 1. The distance $\sigma(\mathbf{a}, \mathbf{b})$ between points $\mathbf{a} \in \Theta$ and $\mathbf{b} \in \Theta$ on a sphere is the minimal length of a curve $\Gamma \subset \Theta$ connecting points $\mathbf{a}$ and $\mathbf{b}$.

Definition 2. Spherical distance $h_\sigma(A, B)$ between a closed set $A \subset \Theta$ and a closed set $B \subset \Theta$ is the value

$$h_\sigma(A, B) = \max_{\mathbf{a} \in A} \min_{\mathbf{b} \in B} \sigma(\mathbf{a}, \mathbf{b}). \tag{8}$$

We call a spherical n-network a nonempty set that consists of at most n points on a sphere Θ. We denote by Σ_n^σ the set of all spherical n-networks.

Problem 2. For a given compact set $M \subseteq \Theta$ and number $n \in \mathbb{N}$, find such a set of n points $S_n \subset \Theta$ for which the value $h_\sigma(M, S_n)$ is minimal among all possible sets.

Problem 2 can be considered as a problem of optimal covering for a compact set $M \subseteq \Theta$ by a set of a fixed number of spherical segments of equal radius. Here the centers of segments coincide with points from the set S_n. In what follows we will call S_n a spherical n-network. Similar problems of sphere coverings have been considered in [24].

Unlike the plane, the sets on a sphere, generally speaking, do not have a well-defined Chebyshev center since for some sets M the point $\mathbf{x}^*$ where function $h(M, \{\mathbf{x}\})$ takes minimal value is not unique. For instance, if M is a circle of unit radius centered at the origin and lying in the plane xOy, there will be two such points: $(0, 0, 1)$ and $(0, 0, -1)$. However, for sufficiently small compact sets M bounded by arcs of circles we can find a point where the value of $h(M, \{\mathbf{x}\})$ is minimal. This lets us implement, for stepwise improvement of a spherical n-network, a modification of Algorithm 1. Dirichlet cells in this case correspond to cells on a sphere lying with respect to the spherical metric no further from one of the points $\mathbf{s}_i \in S_n$ than from the rest of the points from S_n. Instead of center perpendiculars to segments that form a Voronoi diagram, on a sphere we construct arcs of large circles (centered at the origin) that are equidistant from two points in the n-network. The algorithm is applied multiple times until the deviation of the new network from the previous one becomes less than a given value δ.

To solve Problem 2 with a stepwise iterative algorithm, the choice of initial placement of points is very important. In this work, we have studied mostly coverings of a spherical segment centered at point $(0, 0, 1)$ of radius $r^* \in (0, \pi)$. For this case, we have developed the following scheme.

Algorithm 3.

1. Generate an array $P = \{p_i\}_{i=1}^n$ of n random numbers.

2. Construct an array of distances from the points of the n-network to the center of the covered spherical segment $D = \{d_i = (i+1)r^*/(n+1)\}_{i=1}^n$.

3. Find coordinates of points $\mathbf{s}_i = (x_i, y_i)$, $i = \overline{1, n}$, for the initial approximation by formula $x_i = d_i \cos(\pi p_i)$, $y_i = d_i \sin(\pi p_i)(-1)^{i+1}$, $i = \overline{1, n}$.

The set of points constructed with Algorithm 3 is embedded into the segment M, for which we propose to solve Problem 2. Here the points (by construction) are at distance at least $r^*/(n+1)$ from each other (in the metric on the sphere's surface). At the same time, there is a significant amount of randomness in the generation of the coordinates for each element in the set S_n, which lets us get significantly different results for every new run of the program.

Example 3. Let us solve Problem 2 for the set $M = \{(x, y, z)\colon x^2 + y^2 + z^2 = 1, z \geqslant 0\}$, which is an upper hemisphere of the sphere Θ for $n = 18$.

This problem was solved with the software suite we developed, by multiple runs of Algorithm 3. Among the resulting approximate 18-networks we have chosen the one for which the value of

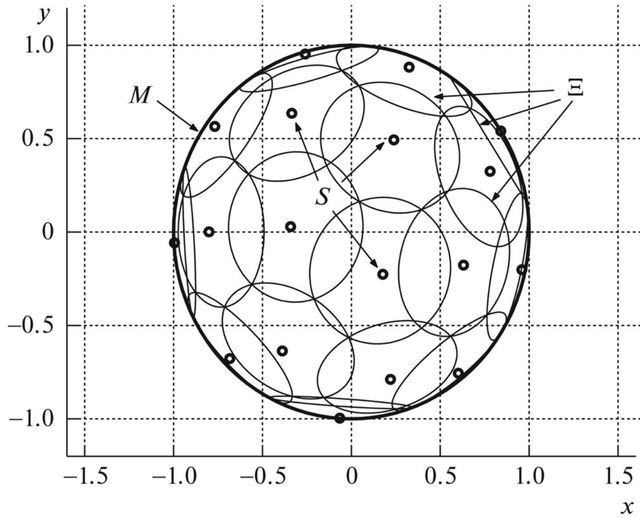

Fig. 3. Projections on the plane xOy for the set M, its optimal spherical 18-network S, and set of spherical segments Ξ in Example 3.

$h_\sigma(M, S_n)$ is minimal. This approximation S_{18} for the optimal spherical 18-network has the form

$$S_{18} \approx \{(0.3246; 0.8831; 0.3389), (-0.0657; -0.9965; 0.0508), (-0.3417; 0.0296; 0.9394), (0.8413; 0.5405; 0), (0.7809; 0.3241; 0.5340), (-0.2595; 0.9531; 0.1559), (-0.7676; 0.5660; 0.3007), (-0.8000; 0.0004; 0.6), (0.6018; -0.7563; 0.2568), (-0.6838; -0.6770; 0.2723), (-0.3353; 0.6365; 0.6946), (-0.9960; -0.0579; 0.0676), (0.1772; -0.2254; 0.9580), (0.2385; 0.4936; 0.8364), (0.2203; -0.7885; 0.5742), (0.6312; -0.1779; 0.7549), (-0.3878; -0.6366; 0.6666), (0.9587; -0.2020; 0.2004)\}.$$

The spherical distance between set M and S_{18} is $r = h_\sigma(M, S_{18}) \approx 0.4143$. Projections on the plane xOy for the hemisphere M, approximation S for the optimal spherical 18-network, and the set of spherical segments Ξ of smallest radius centered at the points of the 18-network and covering M are shown on Fig. 3.

4. CONSTRUCTING AN OPTIMAL NETWORK IN A NON-UNIFORM METRIC

In spatial economics and transportation logistics, the problem of placing servicing centers [11] that minimizes the costs of delivering goods to consumers from the nearest center [12, 25] is very important [4]. This problem reduces to minimization of a functional that defines a non-uniform metric that characterizes transportation costs in different parts of the region [12]. We have considered practical problems that lead to settings of this kind and segmented logistical servicing zones on the territory of Sverdlovks [26] and Irkutsk [27] regions. There, the metric was constructed with regard to geographical features and non-uniform population density of the territory.

Consider a vector space on a plane with a metric where the distance between points $\mathbf{a}$ and $\mathbf{b}$ is defined as follows:

$$\rho_f(\mathbf{a}, \mathbf{b}) = \min_{\Gamma \in \Gamma(\mathbf{a},\mathbf{b})} \int_\Gamma \frac{d\Gamma}{f(x,y)}, \tag{9}$$

where $0 < f(x, y) < K$ is a piecewise continuous function; $\Gamma(\mathbf{a}, \mathbf{b})$ is the set of continuous curve connecting $\mathbf{a}$ and $\mathbf{b}$. If $f(x, y) \equiv 1$, we have a Euclidean metric.

This metric arises in problems of transportation and infrastructural logistics [12]. For example, if we need to find optimal placement for a fixed number n of logistical centers (warehouses, stores) in case when the consumers are distributed continuously but non-uniformly. If we represent all objects as points, we can formalize this problem in geometric terms.

Problem 3. For a given compact set M, function $f(x, y)$ with domain $(-\infty, \infty) \times (-\infty, \infty)$ and number $n \in \mathbb{N}$, find such a set of n points S_n for which the value

$$\max_{\mathbf{m} \in M} \min_{j=\overline{1,n}} \rho_f(\mathbf{m}, \mathbf{s}_i) \tag{10}$$

will be minimal among all possible sets.

To solve Problem 3, we propose an approach based on an analogy between the propagation of light in an optically non-uniform medium and finding the minimum of an integral functional (the optical–geometric approach) [11, 12]. It is known that light, in its motion, chooses the path that it can travel in minimal time (Fermat's principle), and also that every point reached by the light becomes, in turn, a secondary light source (Huygens' principle). This implies that the front of a light wave at any moment of time represents a sphere in the metric space with metric (9), where $f(x, y)$ is the optical permeability of the medium (local speed of light at the corresponding point), and the set of "illuminated" points is the ball bounded by the front. Here the ball's radius, generally speaking, increases with time. We can consider this procedure of running a wave in a medium in a space of any finite dimension, but for applications it is most interesting to consider the case of dimension two, so we restrict ourselves to this case in the present work. The procedure of running a wave has been shown by the authors in [11, 12], so here we omit their formal description. Further we show an algorithm for solving Problem 3.

We assume that we have some initial n-network S_n^0, which can be constructed, for instance, by random sampling of the points.

Algorithm 4.

1. Construct a counterpart of the Voronoi diagram for the points of the current n-network S_n, i.e., find points that have two or more nearest elements from the set S_n. We call the points that have three or more such elements "corner" points. The construction can be done by running simultaneous light waves from all points s_i $(i = \overline{1, n})$ and finding those points in the set M which two or more waves reach at the same time. We also consider the points on the boundary of the set M where two or more waves arrive corner points.

2. Find Dirichlet cells of the points $\mathbf{s}_i \in S_n$, i.e., subsets of M that lie no further from the point $\mathbf{s}_i$ than from other points in the n-network S_n. For this purpose, for each point in the set M we establish which number wave (the numbering of waves corresponds to the numbering of points $\mathbf{s}_i$) has arrived to this point first.

3. Construct a new network $\widehat{S}_n = \{\widehat{\mathbf{s}}_i\}_{i=1}^n$. For this purpose, for each Dirichlet region number i we find corner points at maximal distance from each other (there can be, obviously, two or more such points). From these points, we run waves inside the Dirichlet region and find the point that will be "illuminated" the last. This point is taken as the element $\widehat{\mathbf{s}}_i$ in the new $\widehat{S}_n$-network.

4. Go to step 1.

The algorithm is applied multiple times until the deviation of the new network from the previous one becomes less than a given value δ.

Since with this algorithm we can, obviously, find only local extremal points, we have to run the generation procedure of the initial n-network multiple times (multistart). Note that developing methods for directed generation of initial positions (with algorithms of type 2 and 3) in this case meets significant obstacles since we have to account both for the geometry set M and for the properties of function $f(x, y)$.

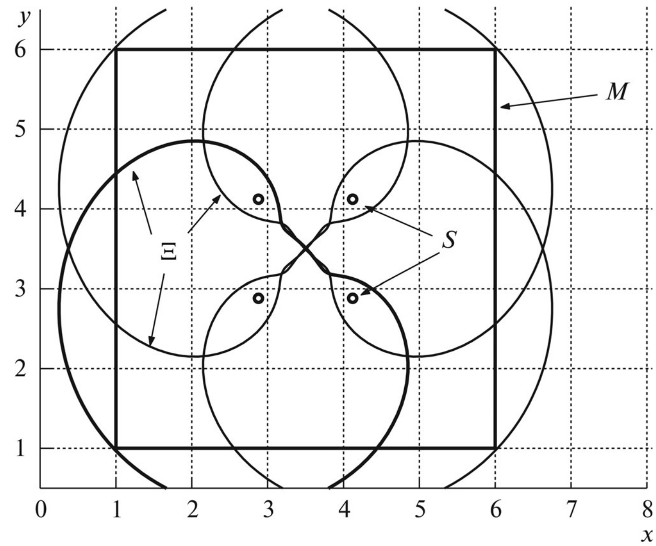

Fig. 4. Set M, approximation of the optimal 4-network S, and set of "circles" in the variational metric Ξ in Example 4.

Example 4. Solve Problem 3 for $n = 4$ for the set $M = \{1 \leqslant x \leqslant 6; 1 \leqslant y \leqslant 6\}$, which is a square, and medium function

$$f(x,y) = \begin{cases} 0.2, \ a(x,y) \leqslant 0.2 \\ a(x,y), \ 0.2 < a(x,y) < 0.8 \\ 0.8, \ a(x,y) \geqslant 0.8, \end{cases} \qquad a(x,y) = \frac{(x-3.5)^2 + (y-3.5)^2}{1 + (x-3.5)^2 + (y-3.5)^2}.$$

We have solved this problem with the developed software suite by multiple runs of Algorithm 4. From the resulting approximations to 4-networks we have chosen the one with minimal radius of covering "circles" in the metric (9). Coordinates of its points are

$$S_4 = \{\mathbf{s}_i\}_{i=1}^4 \approx \{(2.88; 2.88), (2.88; 4.12), (4.12; 2.88), (4.12; 4.12)\}.$$

The value of expression (10) equals $r \approx 3.76$. Figure 4 shows the set M, network S, and set Ξ of covering "circles." One of the "circles" is shown with a thick line (to show its form), and the other three, located symmetrically, are shown with thin lines. Note that the "circles" are nonconvex, and their boundaries have complex wave-like geometry in the neighborhood of the point (3.5; 3.5). This is due to the small value of the speed $f(x,y)$ of propagation in its neighborhood, which curves the lines of light propagation and makes them significantly different from line segments.

5. QUALITY ESTIMATION FOR THE ALGORITHMS

Our software has been implemented in the MATLAB R2012a software suite and has total size 389 KBytes. Modeling was done on a desktop computer with Intel(R) Core(TM)2 Duo CPU E4500 @ 2.21 GHz with 2.00 GB RAM.

In solving Problem 1, we have set various parameters for the time limit on the operation of the software suite. The number of iterations performed by Algorithm 1 is approximately inversely proportional to the square root of the accuracy parameter δ. In particular, when computing the optimal 16-network for a square on a plane for $\delta = 0.001$ the number of iterations was $I = 21$–23; for $\delta = 0.0001$, $I = 67$–83; for $\delta = 0.00001$, the number of iterations was in the range of $I = 161$–189. The running time of the software was on average about 5–7 minutes for the smallest accuracy parameter.

In solving Problem 2, we similarly performed modeling for different parameters. When constructing an optimal spherical 18-network for a hemisphere of unit radius for $\delta = 0.001$ the number of iterations was $I = 54$–88; for $\delta = 0.0001$, $I = 123$–132; for $\delta = 0.00001$, the number of iterations was in the range $I = 206$–343. The running time of the software was on average about 12–15 minutes for the smallest accuracy parameter.

6. CONCLUSION

The iterative algorithms proposed in this work have been implemented and are successfully used to construct optimal Chebyshev n-networks and circle approximations for compact sets on a plane. Similar algorithms let one construct networks and coverings made of spherical segments on a sphere, i.e., surface with non-Euclidean geometry. These problems are important for technical applications; in particular, they are used in the design of sensor networks.

An important direction for further applications of these studies is the construction of transportation networks and placing logistical centers for the servicing. In such problems, there often arises a conceptually different metric that reflects the fact that the environment is non-uniform. To solve these problems, one uses algorithms previously developed by the authors based on the optical–geometric approach. Here we can see that in Example 4 the number of points and accuracy of computations is smaller than in Examples 1–3. This is due to the fact that the implementation of the optical–geometric approach that we have done requires quite a lot of computational resources. One possible way to solve this problem appears to be parallelizing the computations.

APPENDIX

Proof of Theorem 1. We denote $r = h(M, S_n)$ Suppose that inequality (3) does not hold. Then there exists a point $\mathbf{m}^* \in M$ that satisfies

$$\min\{\|\mathbf{m}^* - \widehat{\mathbf{s}}_i\| \colon i = \overline{1, n}\} > r.$$

Next we find the point $\mathbf{s}_j$ from n-network S_n nearest to $\mathbf{m}^*$ in the Euclidean metric (if there are two or more such points, we can take any one of them). By construction, $\mathbf{m}^*$ lies in Dirichlet cells $M(S_n, \mathbf{s}_j)$ and, respectively, $M(S_n, \mathbf{s}_j)$ is a nonempty set. Consequently, its Chebyshev center is a point $\mathbf{s}_j^*$ that belongs to the n-network $\widehat{S}_n$. By definition of a Chebyshev center it follows that

$$h\left(M(S_n, \mathbf{s}_j), \{\mathbf{s}_j^*\}\right) \leqslant h(M(S_n, \mathbf{s}_j), \{\mathbf{s}_j\}).$$

At the same time, by definition of Dirichlet cells $M(S_n, \mathbf{s}_j)$ it follows that its Hausdorff distance from the points $\mathbf{s}_j$ does not exceed the Hausdorff distance of the set M from $S_n, h(M(S_n, \mathbf{s}_j), \{\mathbf{s}_j\}) \leqslant r$. Consequently, for the set $M(S_n, \mathbf{s}_j)$ and points $\mathbf{s}_j^*$ it holds that $h(M(S_n, \mathbf{s}_j), \{\mathbf{s}_j^*\}) \leqslant r$. Since by construction $\mathbf{m}^* \in M(S_n, \mathbf{s}_j)$, it also holds that $\|\mathbf{m}^* - \widehat{\mathbf{s}}_j\| \leqslant r$, so we arrive at a contradiction.

Let us now show that (4) implies (5). Note that condition (4) means, as a consequence of formula (2), that all Dirichlet cells $M(S_n, \mathbf{s}_i)$, $i = \overline{1, n}$, are nonempty (otherwise at least one point in the new n-network $\widehat{S}_n$ would coincide with a point from the old n-network S_n with the same index). Let us now show that

$$\forall i = \overline{1, n}\ h\big(M(S_n, \mathbf{s}_i), \{\mathbf{s}_i^*\}\big) < h\big(M(S_n, \mathbf{s}_i), \{\mathbf{s}_i\}\big). \tag{A.1}$$

By formula (2), points $\mathbf{s}_i^*$, $i = \overline{1, n}$ are Chebyshev centers of sets $M(S_n, \mathbf{s}_i)$, $i = \overline{1, n}$. Condition (4) means, respectively, that points $\mathbf{s}_i$, $i = \overline{1, n}$ do not coincide with them. By uniqueness of Chebyshev centers [6] it follows that for every point that does not coincide with it the Hausdorff distance from a given compact set is strictly larger. Consequently, estimate (A.1) holds. By the definition of a Dirichlet region, $r = \max\{h(M(S_n, \mathbf{s}_i), \{\mathbf{s}_i\})\colon i = \overline{1, n}\}$. At the same time, for an n-network $\widehat{S}_n$ we can write an estimate $h(M, \widehat{S}_n) \leqslant \max\{h(M(S_n, \mathbf{s}_i), \{\mathbf{s}_i^*\})\colon i = \overline{1, n}\}$. This, together with inequalities (A.1), implies $h(M, \widehat{S}_n) < r$, which coincides with (5).

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