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MATHEMATICAL GAME THEORY AND APPLICATIONS

# Altruistic Behavior in a Nonantagonistic Positional Differential Game

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**Abstract**—We consider a two-person nonantagonistic positional differential game (NPDG) whose dynamics is described by an ordinary nonlinear vector differential equation. Constraints on values of players' controls are geometric. Final time of the game is fixed. Payoff functionals of both players are terminal. The formalization of positional strategies in an NPDG is based on the formalization and results of the general theory of antagonistic positional differential games (APDGs) (see monographs by N.N. Krasovskii and A.I. Subbotin [3, 4]). Additionally, in the present paper we assume that each player, together with the usual, *normal* (*nor*), type of behavior aimed at maximizing his own functional, can use other behavior types introduced in [2, 5]. In particular, these may be *altruistic* (*alt*), *aggressive* (*agg*), and *paradoxical* (*par*) types. It is assumed that in the course of the game players can switch their behavior from one type to another. Using the possibility of such switches in a repeated bimatrix $2 \times 2$ game in [5, 6] allowed to obtain new solutions of this game. In the present paper, extension of this approach to NPDGs leads to a new formulation of the problem. In particular, of interest is the question of how players' outcomes at Nash solutions are transformed. An urgent problem is minimizing the time of "abnormal" behavior while achieving a good result. The paper proposes a formalization of an NPDG with behavior types (NPDGwBT). It is assumed that in an NPDGwBT each player, simultaneously with choosing a positional strategy, chooses also his own indicator function defined on the whole game horizon and taking values in the set {*normal*, *altruistic*, *aggressive*, *paradoxical*}. The indicator function of a player shows the dynamics of changes in the behavior type demonstrated by the player. Thus, in this NPDGwBT each player controls the choice of a pair {positional strategy, indicator function}. We define the notion of a BT-solution of such a game. It is expected that using behavior types in the NPDGwBT which differ from the normal one (so-called *abnormal* types) in some cases may lead to more favorable outcomes for the players than in the NPDG. We consider two examples of an NPDGwBT with simple dynamics in the plane in each of which one player keeps to altruistic behavior type over some time period. It is shown that in the first example payoffs of both players increase on a BT-solution as compared to the game with the normal behavior type, and in the second example, the sum of players' payoffs is increased.

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## 1. MOTION EQUATIONS, FUNCTIONALS, STRATEGIES, AND MOTIONS

The dynamics of a nonantagonistic positional differential game (NPDG) is described by the equation

$$\dot{x} = f(t, x, u, v), \quad t \in [t_0, \vartheta], \quad x(t_0) = x_0, \tag{1.1}$$

where $x \in \mathbb{R}^n$ is a phase vector; controls of the first and second player, $u$ and $v$, are constrained by the conditions $u \in P \in \operatorname{comp} \mathbb{R}^p$ and $v \in Q \in \operatorname{comp} \mathbb{R}^q$. In the space of variables $t, x$ there is given a compact set $G$ whose projection onto the $t$ axis is an interval $[t_0, \vartheta]$, where $\vartheta$ is a fixed final time of the game. The function $f\colon G \times P \times Q \to \mathbb{R}^n$ is assumed to be continuous, Lipschitz in $x$, and sublinearly growing with $x$. Moreover, we assume that the saddle point condition in a small game [4, p. 56] is satisfied.

The players' payoff functionals are of the form

$$I_i = \sigma_i(x(\vartheta)), \quad i = 1, 2, \tag{1.2}$$

where $\sigma_i(\cdot)$ are continuous functions.

Both players are assumed to have complete information on the current game position $(t, x)$. The NPDG formalization used in the paper, which includes a description of the players' positional strategies as well as constructive and limiting motions generated by these strategies, is based on the formalization and results given in [3, 4] and is described in detail in [1]. Let us briefly recall basic elements of this formalization.

The positional strategy of player 1 is identified with a pair $U = \{u(t, x, \varepsilon), \beta_1(\varepsilon)\}$ where $u(\cdot)$ is an arbitrary function of the position $(t, x)$ and of a positive precision parameter $\varepsilon$ taking values in a set $P$. The function $\beta_1\colon (0, \infty) \to (0, \infty)$ is continuous, monotonic, and satisfies the condition $\beta_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. For a fixed $\varepsilon$, $\beta_1(\varepsilon)$ specifies an upper limit for the partition step used by player 1 to partition the interval $[t_0, \vartheta]$ when constructing approximated motions. The positional strategy of player 2 is defined similarly: $V = \{v(t, x, \varepsilon), \beta_2(\varepsilon)\}$.

We consider two types of motions generated by a pair of strategies $(U, V)$ from the initial position $(t_0, x_0)$: approximated (Euler polygons) and ideal (limiting) motions. An approximated motion

$$x^{\varepsilon}_{\Delta}[t] = x\left[t, t_0, x_0, U, \varepsilon_1, \Delta_1, V, \varepsilon_2, \Delta_2\right]$$

is defined for precision parameter values $\varepsilon_1$ and $\varepsilon_2$ and partitions $\Delta_1 = \{t_i^{(1)}\}$ and $\Delta_2 = \{t_k^{(2)}\}$ of the interval $[t_0, \vartheta]$ that are chosen by players 1 and 2. Player $i$, $i = 1, 2$, uses as his control a piecewise-constant function with values computed at nodes of his partition with a partition step satisfying the conditions $\delta(\Delta_i) \le \beta_i(\varepsilon_i)$, where $\delta(\Delta_i) = \max\limits_j (t^{(i)}_{j+1} - t^{(i)}_j)$. An ideal (limiting) motion $x(t) = x(t, t_0, x_0, U, V)$ is defined as a uniform limit of the sequence of approximated motions

$$\left\{x^{\varepsilon^s}_{\Delta^s}\left[t, t^s_0, x^s_0, U, \varepsilon^s_1, \Delta^s_1, V, \varepsilon^s_2, \Delta^s_2\right]\right\}$$

as $s \to \infty$, $\varepsilon^s_i \to 0$, $t^s_0 \to t_0$, $x^s_0 \to x_0$, and $\delta(\Delta^s_i) \le \beta_i(\varepsilon^s_i)$, $i = 1, 2$.

The set of limiting motions $X(t_0, x_0, U, V)$ is compact in the metric of the space $C[t_0, \vartheta]$.

## 2. SOME RESULTS OF NPDG THEORY

Below we need some results of NPDG theory [1].

**Definition 2.1.** A pair of strategies $(U^N, V^N)$ forms a Nash-equilibrium solution (*NE* solution) in NPDG (1.1), (1.2) if for any motion $\overline{x}(\cdot) \in X(t_0, x_0, U^N, V^N)$, any $\tau \in [t_0, \vartheta]$, and any strategies $U$ and $V$ we have the inequalities

$$\max_{x(\cdot)} \sigma_1\left(x\left[\vartheta, \tau, \overline{x}(\tau), U, V^N\right]\right) \le \min_{x(\cdot)} \sigma_1\left(x\left[\vartheta, \tau, \overline{x}(\tau), U^N, V^N\right]\right), \tag{2.1}$$

$$\max_{x(\cdot)} \sigma_2\left(x\left[\vartheta, \tau, \overline{x}(\tau), U^N, V\right]\right) \le \min_{x(\cdot)} \sigma_2\left(x\left[\vartheta, \tau, \overline{x}(\tau), U^N, V^N\right]\right), \tag{2.2}$$

where the max and min operations are over the corresponding sets of limiting motions.

**Definition 2.2.** An *NE*-solution $(U^P, V^P)$ which is Pareto-unimprovable with respect to $(I_1, I_2)$ (1.2) is called a $P^*$-solution.

Consider auxiliary antagonistic positional differential games $\Gamma_1$ and $\Gamma_2$. The dynamics of both games are described by Eqs. (1.1). In game $\Gamma_i$ player $i$ maximizes the functional $\sigma_i(x(\vartheta))$ (1.2), and player $(3-i)$ opposes. It follows from [3] that both games $\Gamma_1$ and $\Gamma_2$ have universal saddle points

$$u^{(i)}(t,x,\varepsilon),\ v^{(i)}(t,x,\varepsilon), \quad i=1,2 \tag{2.3}$$

and continuous cost functions $\gamma_1(t,x)$ and $\gamma_2(t,x)$. The universality property of strategies (2.3) means that they are optimal not only for a fixed initial positional $(t_0,x_0)$ but also for any position $(t_*,x_*) \in G$ considered as initial. Clearly, for a position $(t,x) \in G$ of the NPDG game, the quantity $\gamma_i(t,x)$ is the guaranteed payoff of player $i$ in this position.

In [1] it is shown that all *NE*- and $P^*$-solutions of the game can be found in the class of pairs of strategies $(U,V)$ generating a unique limiting motion (trajectory). Decisive strategies composing a pair that generates a trajectory $x^*(\cdot)$ are of the following form:

$$U^0 \div \left\{u^0(t,x,\varepsilon),\ \beta_1^0(\varepsilon)\right\}, \quad V^0 \div \left\{v^0(t,x,\varepsilon),\ \beta_2^0(\varepsilon)\right\}, \tag{2.4}$$

$$\begin{aligned} u^0(t,x,\varepsilon) &= \begin{cases} u^*(t,\varepsilon) & \text{if } \|x-x^*(t)\| < \varepsilon\varphi(t) \\ u^{(2)}(t,x,\varepsilon) & \text{if } \|x-x^*(t)\| \geq \varepsilon\varphi(t), \end{cases} \\ v^0(t,x,\varepsilon) &= \begin{cases} u^*(t,\varepsilon) & \text{if } \|x-x^*(t)\| < \varepsilon\varphi(t) \\ v^{(1)}(t,x,\varepsilon) & \text{if } \|x-x^*(t)\| \geq \varepsilon\varphi(t), \end{cases} \end{aligned} \tag{2.5}$$

for all $t \in [t_0,\vartheta]$, $\varepsilon > 0$. In (2.4), (2.5) by $u^*(t,\varepsilon)$ and $v^*(t,\varepsilon)$ we denote families of program controls generating the trajectory $x^*(t)$. The function $\varphi(\cdot)$ and the functions $\beta_1^0(\cdot)$ and $\beta_2^0(\cdot)$ are chosen in such a way that the Euler polygons generated by the pair $(U^0,V^0)$ from the initial position $(t_0,x_0)$ do not leave the $\varepsilon\varphi(t)$-neighborhood of the trajectory $x^*(t)$. The functions $u^{(2)}(\cdot\,,\cdot\,,\cdot)$ and $v^{(1)}(\cdot\,,\cdot\,,\cdot)$ are defined in (2.3).

Further, for each *NE*- and $P^*$-trajectory $x^*(t)$, the following property is satisfied: the point $t=\vartheta$ is the maximum point of the guaranteed payoff function of player $i$ computed along this trajectory, i.e.,

$$\max_{t\in[t_0,\vartheta]} \gamma_i(t,x^*(t)) = \gamma_i(\vartheta,x^*(\vartheta)), \quad i=1,2. \tag{2.6}$$

## 3. PLAYERS' BEHAVIOR TYPES

We additionally assume that along with the usual, *normal* (*nor*), behavior type, aimed at maximizing their own payoff functionals (1.2), players may use other behavior types, e.g., those introduced in [2, 5], namely *altruistic* ("the better for my rival, the better for me"), *aggressive* ("the worse for my rival, the better for me"), and *paradoxical* ("the worse for me, the better for me").

These three behavior types can be formalized as follows.

**Definition 3.1.** We say that player 1 demonstrates altruistic (*alt*) behavior type towards player 2 in an interval $[t_*,t^*]$ if actions of player 1 in this interval are aimed at maximizing the functional $I_2$ of player 2.

**Definition 3.2.** We say that player 1 demonstrates aggressive (*agg*) behavior type towards player 2 in an interval $[t_*,t^*]$ if actions of player 1 in this interval are aimed at minimizing the functional $I_2$ of player 2.

**Definition 3.3.** We say that player 1 demonstrates paradoxical (*par*) behavior type in an interval $[t_*,t^*]$ if actions of player 1 in this interval are aimed at minimizing his own functional $I_1$.

**Table**

| | *nor* | *alt* | *agg* | *par* |
|---|---|---|---|---|
| *nor* | $(I_1,I_2)$ | $(I_1,I_1)$ | $(I_1,-I_1)$ | $(I_1,-I_2)$ |
| *alt* | $(I_2,I_2)$ | $(I_2,I_1)$ | $(I_2,-I_1)$ | $(I_2,-I_2)$ |
| *agg* | $(-I_2,I_2)$ | $(-I_2,I_1)$ | $(-I_2,-I_1)$ | $(-I_2,-I_2)$ |
| *par* | $(-I_1,I_2)$ | $(-I_1,I_1)$ | $(-I_1,-I_1)$ | $(-I_1,-I_2)$ |

Altruistic and aggressive behavior types of player 2 towards player 1 and the paradoxical behavior type of player 2 are defined similarly. Note that the aggressive behavior type is actually used in an NPDG in the form of penalty strategies involved in the structure of game solutions (see, e.g., [1]).

The above definitions characterize extremal behavior types of the players. In real life, as a rule, individuals behave partly normally, partly altruistically, partly aggressively, and partly paradoxically. In other words, apparently, mixed behavior types better agree with reality.

If each player is restricted to "pure" behavior types, then in the considered two-player game (1.1) with functionals $I_1$ and $I_2$ (1.2) there are 16 possible combinations, which are shown in table. In four of the combinations, interests of the players coincide, and the players solve team control problems. In other four combinations, the players have opposite interests, and therefore antagonistic differential games are played. The remaining 8 pairs define nonantagonistic differential games.

The idea that players may use the possibility of switching their behavior from one behavior type to another in the course of the play was applied in [5] for a game with cooperative dynamics and in [6] for a repeated bimatrix $2 \times 2$ game, which made it possible to obtain new solutions in those games.

Extending this approach to nonantagonistic positional differential games leads to new problem settings. In particular, of interest is the question of how players' payoffs obtained on Nash solutions are transformed. The problem of minimizing the time of "abnormal" behavior while achieving a better result than under the normal behavior of the players becomes actual.

In what follows we assume that, simultaneously with choosing a positional strategy, each player also chooses his indicator function defined on the interval $t \in [t_0, \vartheta]$ and taking values in the set $\{nor,\ alt,\ agg,\ par\}$. Denote the indicator function of player $i$ by $\alpha_i \colon [t_0, \vartheta] \to \{nor, alt, agg, par\}$, $i = 1, 2$. If the indicator function of some player takes the value, say, *alt* in some time interval, then this player behaves as an altruist towards his rival within this interval.

Thus, in the considered game with various behavior types, player 1 controls the choice of a pair of *actions* {positional strategy, indicator function}: $(U,\ \alpha_1(\cdot))$, and player 2 controls the choice of a pair of actions $(V,\ \alpha_2(\cdot))$. Below such an NPDG with behavior types will be denoted by NPDGwBT.

Note that if the indicator functions of both players are identically equal to *nor* on the whole game horizon, then we have a classical NPDG.

## 4. *BT*-SOLUTION OF AN NPDGWBT

Now consider an NPDGwBT with classes of actions of players 1 and 2

$$(U,\ \alpha_1(\cdot)),\ \ (V,\ \alpha_2(\cdot)). \tag{4.1}$$

Clearly, the set of motions generated by a pair of actions (4.1) coincides with the set of actions generated by the pair $(U, V)$ in the corresponding NPDG.

**Definition 4.1.** A pair $\{(U^0, \alpha_1^0(\cdot)), (V^0, \alpha_2^0(\cdot))\}$ forms a strong $BT$-solution of an NPDGwBT if there exist a trajectory $x^{BT}(\cdot)$ generated by the pair and a $P^*$-solution in the corresponding

NPDG, generating a trajectory $x^P(\cdot)$, such that

$$\sigma_i\left(x^{BT}(\vartheta)\right) > \sigma_i\left(x^P(\vartheta)\right), \quad i = 1, 2. \tag{4.2}$$

**Definition 4.2.** A pair $\{(U^0, \alpha_1^0(\cdot)), (V^0, \alpha_2^0(\cdot))\}$ forms a weak $BT$-solution of an NPDGwBT if there exist a trajectory $x^{BT}(\cdot)$ generated by the pair and a $P^*$-solution in the corresponding NPDG, generating a trajectory $x^P(\cdot)$, such that

$$\sum \sigma_i\left(x^{BT}(\vartheta)\right) > \sum \sigma_i\left(x^P(\vartheta)\right). \tag{4.3}$$

Clearly, a strong $BT$-solution is a weak $BT$-solution; the converse is not true in general.

*Problem 4.1.* Find the set of (strong and weak) $BT$-solutions.

*Problem 4.2.* Find the set of $BT$-solutions minimizing the time during which the players use *abnormal* behavior types.

It is quite probable that using behavior types other than normal in an NPDGwBT can in a number of cases lead to outcomes more favorable for the players than those in the NPDG with the normal behavior type only.

Below we present two examples of a game with dynamics of simple plane motion, in each of which one of the players keeps to altruistic behavior type during some time period. We show that, as compared to the game with normal behavior of both players, in the first example payoffs of both players grow, whereas in the second example the altruistic player considerably increases the payoff of the other player, his own payoff being slightly reduced, but the aggregate payoff of the players grows. In both examples we solve the problem of finding $BT$-solutions minimizing the abnormal behavior duration.

## 5. EXAMPLE 1

Let the dynamics (1.1) be

$$\dot{x} = u + v, \quad x, u, v \in \mathbb{R}^2, \quad \|u\| \le 1, \quad \|v\| \le 1, \quad 0 \le t \le \vartheta, \quad x(0) = x_0, \tag{5.1}$$

and let the payoff functionals (1.2) be

$$I_i = 10 - \left\|(x(\vartheta) - a^{(i)}\right\|, \quad i = 1, 2, \tag{5.2}$$

i.e., player $i$ wants to bring the point $x(\vartheta)$ as close to his target point $a^{(i)}$ as possible. We set the following values of the game parameters: $\vartheta = 2.5$, $x_0 = (0, 0)$, $a^{(1)} = (7.0, 5.0)$, and $a^{(2)} = (-7.0, 5.0)$.

We also specify an additional condition. In the plane $(x_1, x_2)$, a circle $S$ of radius 2 centered at $m = (0, 2.5)$ depicts the following phase constraint in the problem: trajectories of system (5.1) are forbidden to enter the interior of $S$.

In Fig. 1, the curves $\hat{d}\hat{a}$ and $ad$ together with the arc $\hat{a}\hat{c}ca$ of the boundary of $S$ and the arc $\hat{d}qd$ of the circumference of a circle of radius 5 bound the reachability set of system (5.1) with the introduced phase constraints constructed for time $\vartheta = 2.5$. The line $Oc$ is tangent to the circle $S$. The point $b$ is a point of $S$ nearest to $a^{(1)}$. Computation results are as follows (hereinafter, we give approximate numerical values): $a = (1.36, 3.96)$, $c = (1.20, 0.90)$, and $b = (1.88, 3.17)$. Figure 1 is symmetric about the ordinate axis. The notation for a symmetric point differs in a circumflex mark only. Further description is mainly given for the right-hand half of the figure.

In the NPDG with the normal behavior type, the Nash trajectory (and at the same time the $P^*$-trajectory) is $x(t) \equiv 0$, $t \in [0, 2.5]$ (stationary point $O$), on which the players' payoffs are

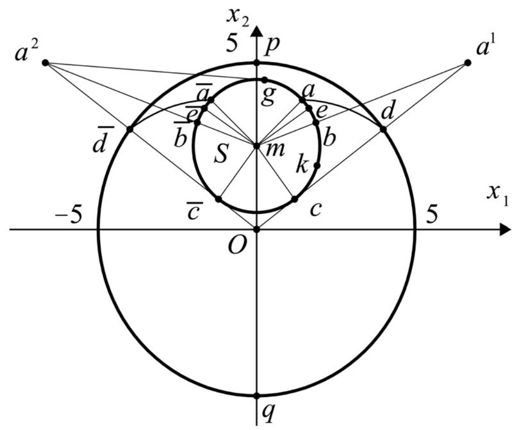


**Fig. 1.** To Example 1.

$I_1 = I_2 = 10 - \sqrt{74} \approx 1.40$. The trajectory given by the $Ocba$ line is not a Nash trajectory, since at point $b$ player 1 acquires the maximum payoff (equal to 4.57) and is not interested in further tracing the trajectory up to point $a$. The trajectory given by the $Ocb$ line is neither a Nash trajectory: now player 2 does not agree to trace the trajectory, since at point $b$ he gets the payoff of 0.53, which is less than that at point $O$.

Now we pass to the NPDGwBT and consider the case where player 1 demonstrates altruism towards player 2 during some time interval.

On the circumference $S$, find a point $k$ equidistant from $a^{(2)}$ when walking around $S$ both clockwise and counterclockwise. We obtain $k = (1.88, 1.83)$. Next, on the circumference $S$, find a point $e$ the distance from which to $a^{(2)}$ computed along the $ega^{(2)}$ line ($a^{(2)}g$ is a tangent to $S$) is equal to the length of the segment $Oa^{(2)}$. We get $e = (1.43, 3.90)$. The time of hitting the point $k$ when moving with the maximum velocity is $t = 1.34$, and that of hitting $e$ is $t = 2.45$. If at time $t = 2.45$ we continue moving along the arc $ea$ counterclockwise, then the distance to $a^{(2)}$ becomes less than the length of $Oa^{(2)}$. Therefore, it is profitable to player 2 to terminate his movement at some point of the arc $ea$.

Let $h$ be some fixed point of the arc $ea$, and let $t^*$ be the time of hitting the point $h$ when moving with the maximum velocity. Obviously, $t^* \in [2.45, 2.5]$.

Consider a pair of strategies $(U^*, V^*)$ generating a unique limiting motion shown by the $Oceh$ line for $t \in [0, t^*)$ and remaining stationary at $h$ for $t \in [t^*, 2.5]$. Next, consider the following indicator functions of the players: $\alpha_1^*(t) = \{nor,\ t \in [0, 2.01);\ alt,\ t \in [2.01, t^*);\ nor,\ t \in [t^*, 2.5]\}$ and $\alpha_2^*(t) = \{alt,\ t \in [0, 1.34);\ nor,\ t \in [1.34, 2.5]\}$. Here $t = 2.01$ is the time of hitting $b$ when moving with the maximum velocity.

It is easily seen that both players are interested in realizing the strategies $(U^*, V^*)$ and indicator function programs $\alpha_1^*(\cdot), \alpha_2^*(\cdot)$.

Thus, we obtain a one-parametric family of collections of players' actions

$$\{(U^*, \alpha_1^*(\cdot)),\ (V^*, \alpha_2^*(\cdot))\} \tag{5.3}$$

depending on the parameter $t^* \in [2.45, 2.5]$, each of which is a $BT$-solution.

If the parameter value is $t^* = 2.45$, we have a weak $BT$-solution; for other values of $t^* \in (2.45, 2.5]$, each of the collections (5.3) is a strong $BT$-solution.

If a $BT$-trajectory thus constructed terminates at point $a$ (this is the case for $t^* = 2.5$), the players' payoffs will be $I_1 = 4.2$ and $I_2 = 1.50$; i.e., both players gain as compared to the game with the normal behavior type.

The solution of problem 4.2 is attained at the collection (5.3) for $t^* = 2.45$.

## 6. EXAMPLE 2

Let the dynamics (1.1) be of the form

$$\dot{x} = u + v, \quad x, u, v \in \mathbb{R}^2, \quad \|u\| \le 1, \quad \|v\| \le 1, \quad 0 \le t \le \vartheta, \quad x(0) = x_0, \tag{6.1}$$

and let the payoff functionals (1.2) be

$$I_1 = \sigma_1(x(\vartheta)) = 9 - \left\| x(\vartheta) - a^{(1)} \right\|, \tag{6.2}$$

$$I_2 = \sigma_2(x(\vartheta)) = 5 + \sqrt{3}|x_1(\vartheta)| - x_2(\vartheta), \tag{6.3}$$

i.e., the goal of player 1 is to bring the vector $x(\vartheta)$ as close to the target point $a^{(1)}$ as possible.

We specify the following values of the game parameters: $\vartheta = 1.5$, $x_0 = (-2, 2\sqrt{3})$, and $a^{(1)} = (5, 7.5)$. In Fig. 2, on the plane $(x_1, x_2)$, a circle of radius 3 centered at the initial point $A(-2, 2\sqrt{3})$ depicts the reachability set of system (6.1) constructed for the time $\vartheta = 1.5$. The initial point $A$ lies on a level line of the function $\sigma_2(x_1, x_2) = 5 + \sqrt{3}|x_1| - x_2 = 5$.

Since $AO \perp Aa^{(1)}$, in the NPDG with the normal behavior type of the players, a unique Nash trajectory, and hence a unique trajectory generated by a $P^*$-solution, is the trajectory $x(t) = 0$, $t \in [0, 1.5]$, which coincides with the stationary point $A$. Computations show that the players' payoffs on this $P^*$-solution are $I_1 = 0.920$ and $I_2 = 5.000$; the aggregate payoff of the players is $I_1 + I_2 = 5.920$ (hereinafter, we give approximate values).

Passing to the NPDGwBT, we note first of all that existence of strong $BT$-solutions in the considered example is definitely impossible, since in the reachability set there are no points where both players obtain payoffs greater than those on the $P^*$-solution. Indeed, the intersection of the two subsets of the reachability set, each consisting of the points where the payoff of one of the players is greater than at point $A$, is empty.

We will look for weak $BT$-solutions and consider the case where player 2 demonstrates altruism towards player 1 during some time period. In the reachability set, we find all points where the sum $I_1 + I_2$ of the players' payoffs is greater than or equal to the sum of the payoffs on the $P^*$-solution. Computations show that the set $T$ of such points is nonempty, and its intersection

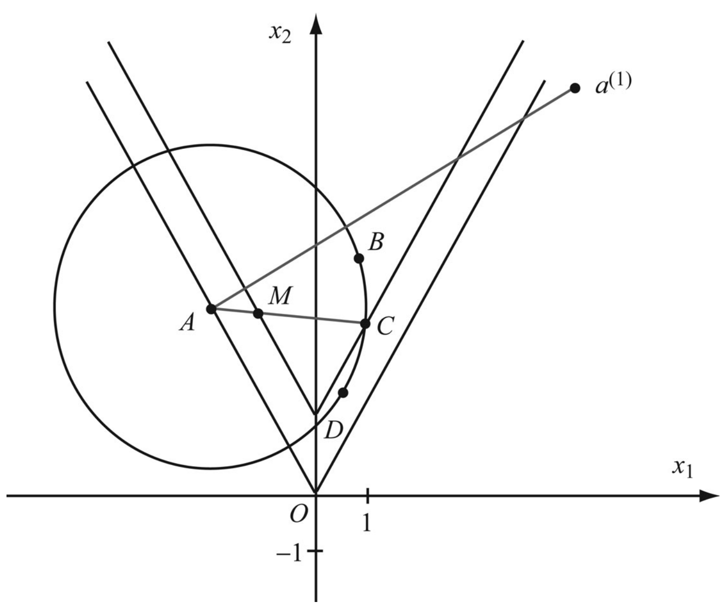


**Fig. 2.** To Example 2.

with the boundary of the reachability set is the arc $BD$ with $B(0.86, 4.37)$ and $D(0.56, 1.90)$. Points of the arc $BD$ dominate the other points of $T$ in the aggregate payoff. Next, find a point $C(0.984, 3.155)$ maximizing the sum of the players' payoffs on the arc $BD$. The level line of the function $\sigma_2(x_1, x_2) = 5 + \sqrt{3}|x_1| - x_2 = 3.549$ passing through $C$ intersects the segment $AC$ in one more point, $M(-1.109, 3.371)$.

It is easily seen that the segment $AC$ is a trajectory generated by a weak $BT$-solution of the game if, when moving with the maximum velocity from the initial point $A$ to $M$ (on this part of the trajectory, player 2 reduces his guaranteed payoff from 5 to 3.083), player 2 demonstrates altruism towards player 1, and after passing the point $M$ player 2 returns to the normal behavior type. Since the distance between $A$ and $M$ is 0.896, the duration of the altruistic behavior of player 2 is $0.5 \times 0.896 = 0.448$.

Consider the pair of strategies $(U^*, V^*)$ generating for $t \in [0, 1.5]$ the unique limiting motion depicted by the segment $AC$, and consider the following indicator functions of the players: $\alpha_1^*(t) = \{nor,\ t \in [0, 1.5]\}$ and $\alpha_2^*(t) = \{alt,\ t \in [0, 0.448);\ nor,\ t \in [0.448, 1.5]\}$. It is easily seen that both players are interested in realizing the strategies $(U^*, V^*)$ and indicator function programs $\alpha_1^*(\cdot), \alpha_2^*(\cdot)$.

Thus, the collection of players' actions $(U^*, \alpha_1^*(\cdot)), (V^*, \alpha_2^*(\cdot))$ generating the unique trajectory $AC$ is a weak $BT$-solution. On this $BT$-solution, the players' payoffs are $I_1 = 3.083$ and $I_2 = 3.549$; the aggregate payoff is $I_1 + I_2 = 6.632$. Thus, as compared to the game with the normal behavior type, player 2 has lost a little and player 1 has gained considerably, so that the aggregate payoff of the players has increased.

One can also construct other weak $BT$-solutions which generate unique limiting motions coinciding with segments joining the point $A$ with point of the arc $BD$ except for its endpoints. The corresponding collections of the players' actions are constructed in the way described above.

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