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MATHEMATICAL GAME THEORY AND APPLICATIONS

Strategic Stability in Linear-Quadratic Differential Games with Nontransferable Payoffs

A. V. Tur
St. Petersburg State University, St. Petersburg, Russia
e-mail: a.tur@spbu.ru
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Abstract—We address the problem of strategically supported cooperation for linear-quadratic differential games with nontransferable payoffs. As an optimality principle, we study Pareto-optimal solutions. It is assumed that players use a payoff distribution procedure guaranteeing individual rationality of a cooperative solution over the entire game horizon. We prove that under these conditions a Pareto-optimal solution can be strategically supported by an ε-Nash equilibrium. An example is considered.

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1. INTRODUCTION

Situations are often encountered where consideration of games with transferable payoffs imposes excessive constraints. It may happen that players cannot redistribute (or cannot transfer without losses) their payoffs obtained in the course of the play. This could be caused by several reasons; for example, lack of a common medium of exchange, restricted or forbidden transfers. In the present paper we consider linear-quadratic differential games with nontransferable payoffs. As a cooperative optimality principle, we suggest to consider a Pareto-optimal solution, which does not assume payoff distribution among players. Since the game evolves in time, even if at the beginning of the game a Pareto-optimal solution satisfies the condition of individual rationality, later on this condition can be violated. To avoid such a situation, we use the time-consistent payoff distribution procedure proposed by L.A. Petrosyan [8]. It is found that in this case the outcome of the cooperative agreement is attained at some Nash equilibrium, which guarantees strategic stability of the cooperative solution [2].

2. PROBLEM SETTING

Consider a linear-quadratic nonantagonistic n-person differential game $\Gamma(t_0, x_0)$ whose state at each time instant is characterized by a vector $x(t)$ changing in time according to the system of equations

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i u_i(t), \quad x(t_0) = x_0, \tag{2.1}$$

where $x \in R^m$ is a column vector; a column vector $u_i \in R^r$ is the control of player i, $i = 1, \ldots, n$; A and B_i are matrices of sizes $m \times m$ and $m \times r$, respectively; $x(t_0) = x_0$ is the initial state.

Denote by $N = \{1, \ldots, n\}$ the set of all players. The payoff of player $i \in N$ is denoted by $J_i(t_0, x_0, u)$, where $u = (u_1, \ldots, u_n)$.

We assume that the payoff of player i is of the form

$$J_i(t_0, x_0, u) = \int_{t_0}^{\infty} \left(x^T(t) P_i x(t) + u_i^T(t) R_i u_i(t) \right) dt, \tag{2.2}$$

where P_i are symmetric positive semidefinite $m \times m$ matrices and R_i are symmetric positive definite $r \times r$ matrices, $i = 1, \ldots, n$. Each player wants to maximize his payoff. Players' payoffs are assumed to be nontransferable.

Players choose their strategies in a class of admissible strategies.

Definition 2.1. A set of strategies

$$\{u_i(t, x) = M_i(t)x, \ i = 1, \ldots, n\} \tag{2.3}$$

is said to be admissible if

$$M_i(t) = \begin{cases} M_{i,1}, & \text{for } t \in [t_0, t_i) \\ M_{i,2}, & \text{for } t \in [t_i, \infty), \end{cases}$$

are piecewise constant matrix functions with at most one discontinuity point t_i and the systems

$$\dot{x}(t) = \left(A + \sum_{i=1}^{n} B_i \overline{M}_i \right) x(t) \tag{2.4}$$

are asymptotically stable. Here $\left\{ \overline{M}_1, \ldots, \overline{M}_n \right\}$ are all collections of constant matrices $\overline{M}_i$ with values $M_{i,1}$ or $M_{i,2}$.

By an asymptotically stable system we understand a system all solutions of which are asymptotically Lyapunov stable. According to [1], a homogeneous linear differential system with a constant matrix is asymptotically stable if and only if all eigenvalues of its matrix have negative real parts.

Note that for a one-player problem, an optimal solution in the class of controls with constant matrices M_i coincides with an optimal solution in the class of controls with piecewise constant matrices $M_i(t)$. Likewise, Nash-equilibrium strategies for the considered classes of strategies in an n-person game also coincide. However, introducing admissible strategies in this way, we allow the players to deviate from a trajectory chosen at the initial time, which might be actual if the considered cooperative optimality principle will happen to be time-inconsistent.

As an optimality principle in the cooperative game $\Gamma(t_0, x_0)$, we consider Pareto-optimal solutions [9].

Definition 2.2. A collection of strategies $u^* = (u_1^*, \ldots, u_n^*)$ in the class of admissible strategies is said to be Pareto-optimal if there is no admissible collection u of strategies for which

$$J_i(t_0, x_0, u^*) \geq J_i(t_0, x_0, u), \quad i = 1, \ldots, n,$$

where at least one of the inequalities is strict.

Let the players agree to use a weight vector

$$(\alpha_1, \ldots, \alpha_n): \quad 0 < \alpha_i < 1, \quad \sum_{i=1}^{n} \alpha_i = 1,$$

for finding an optimal solution.

Then (see [6]) optimal strategies of the players can be obtained as solutions of the following minimization problem:

$$\min_{(u_1,\ldots,u_n)} \sum_{i=1}^{n} \alpha_i J_i(t_0, x_0, u). \tag{2.5}$$

Let

$$(u_1^\alpha, \ldots, u_n^\alpha) = \arg \min_{(u_1,\ldots,u_n)} \sum_{i=1}^{n} \alpha_i J_i(t_0, x_0, u), \tag{2.6}$$

$$J^\alpha(t_0, x_0, u) = \sum_{i=1}^{n} \alpha_i J_i(t_0, x_0, u),$$

$$P^\alpha = \sum_{i=1}^{n} \alpha_i P_i,$$

$$R^\alpha = \begin{pmatrix} \alpha_1 R_1 & \mathbb{O} & \ldots & \mathbb{O} \\ \mathbb{O} & \alpha_2 R_2 & \ldots & \mathbb{O} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{O} & \mathbb{O} & \ldots & \alpha_n R_n \end{pmatrix}.$$

Then

$$J^\alpha(t_0, x_0, u) = \int_{t_0}^{\infty} \Big(x^T(t) P^\alpha x(t) + u^T(t) R^\alpha u(t) \Big)\, dt. \tag{2.7}$$

Finding a Pareto-optimal solution reduces to the linear-quadratic optimal control problem (2.1)–(2.5) with a single control $u(t)$.

According to [4], optimal strategies are of the form

$$\Big\{ u_i^\alpha(t) = M_i^\alpha x, \quad i = 1, \ldots, n \Big\},$$

where M_i^α is the ith block of the matrix $M^\alpha = -\left(R^\alpha\right)^{-1} B^T \Theta^\alpha$, and Θ^α is a solution to the matrix equation

$$A^T\Theta + \Theta A - \Theta B (R^\alpha)^{-1} B^T \Theta + P^\alpha = 0, \tag{2.8}$$

where $B = (B_1, \ldots, B_n)$.

Then a cooperative trajectory $x^\alpha(t)$ can be found by solving the system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i u_i^\alpha(t), \quad x(t_0) = x_0. \tag{2.9}$$

Players' payoffs under cooperation are

$$J_i^\alpha(t_0, x_0, u^\alpha) = \int_{t_0}^{\infty} \Big((x^\alpha(t))^T P_i x^\alpha(t) + (u_i^\alpha(t))^T R_i u_i^\alpha(t) \Big)\, dt. \tag{2.10}$$

3. INDIVIDUAL RATIONALITY

In games with nontransferable payoffs, time consistency of a cooperative solution reduces to validity of the following conditions:

1. Distribution remains Pareto-optimal in subgames along a cooperative trajectory;
2. Individual rationality condition is satisfied over the whole game horizon.

If, when finding a Pareto-optimal solution, players choose one and the same weight coefficient α for the whole game, condition 1 is satisfied. Therefore, analysis of dynamic stability of a Pareto-optimal solution reduces to checking the individual rationality condition, i.e., condition 2. We say that a Pareto-optimal solution is individually rational if every player does not lose under this outcome as compared to his payoff in the Nash-equilibrium situation.

There exists a vector α such that at the beginning of the game $\Gamma(t_0, x_0)$, on a cooperative trajectory $x^\alpha(t)$, the individual rationality condition for a Pareto-optimal solution is satisfied [10]:

$$J_i^\alpha(t_0, x_0, u^\alpha) \leq V_i(t_0, x_0), \quad i = 1, \ldots, n. \tag{3.1}$$

Here $V_i(t_0, x_0)$ is the payoff of player i in the Nash-equilibrium situation [7] in the game $\Gamma(t_0, x_0)$.

However, it may happen in the course of the play that at some time l, $l > t_0$, the individual rationality condition will not hold for some player $i \in N$:

$$J_i^\alpha\,(l, x^\alpha(l), u^\alpha) > V_i(l, x^\alpha(l)).$$

Here $V_i(l, x^\alpha(l))$ is the payoff of player i in the Nash-equilibrium situation in the subgame $\Gamma(l, x^\alpha(l))$ starting from state $x^\alpha(l)$.

To avoid instability of a Pareto-optimal solution, we use the payoff distribution procedure proposed by L.A. Petrosyan [8].

Definition 3.1. A vector function $\beta(t) = (\beta_1(t), \ldots, \beta_n(t))$ is called a payoff distribution procedure [8] if

$$J_i(t_0, x_0, u) = \int_{t_0}^{\infty} \beta_i(t)\, dt, \quad i = 1, \ldots, n. \tag{3.2}$$

Definition 3.2. A Pareto-optimal solution is said to be dynamically stable [8] if there exists a payoff distribution procedure $\beta(t)$ such that the individual rationality condition

$$\int_{l}^{\infty} \beta_i(t)\, dt \leq V_i(l, x^\alpha(l)), \quad \forall l \geq t_0, \quad i = 1, \ldots, n, \tag{3.3}$$

is satisfied, where $V_i(l, x^\alpha(l))$ is the payoff of player i in the Nash-equilibrium situation in the subgame $\Gamma(l, x^\alpha(l))$. Such a payoff distribution procedure is said to be time-consistent.

Let condition (3.1) be satisfied for some Pareto-optimal solution. Then there exist functions $\eta_i(t) \leq 0$ such that

$$J_i^\alpha(t_0, x_0, u^\alpha) - V_i(t_0, x_0) = \int_{t_0}^{\infty} \eta_i(t)\, dt. \tag{3.4}$$

In [8], a payoff distribution procedure was proposed for differential games with nontransferable payoffs, which allows to avoid instability of a Pareto-optimal solution.

Theorem 3.1 [8]. *If for some Pareto-optimal solution we have*

$$J_i^\alpha(t_0, x_0, u^\alpha) \le V_i(t_0, x_0), \quad i = 1, \ldots, n,$$

then the payoff distribution procedure $\beta(t)$ *of the form*

$$\beta_i(t) = \eta_i(t) - \frac{d}{dt} V_i(t, x^\alpha(t)), \quad i = 1, \ldots, n, \tag{3.5}$$

guarantees dynamic stability of this Pareto-optimal solution along the whole cooperative trajectory $x^\alpha(t)$*. Here* $\eta_i(t) \le 0$ *are functions satisfying* (3.4).

According to [5], a Nash equilibrium in the considered game (2.1), (2.2) is an admissible collection of strategies $\left\{u_i^{NE}(t) = M_i^{NE} x,\ i=1, \ldots, n\right\}$, where

$$M_i^{NE} = -(R_i)^{-1} B_i^T \Theta_i^{NE},$$

Θ_i^{NE} being a solution of the system of matrix equations

$$\begin{gathered}\left(A - \sum_{j \ne i} B_j R_j^{-1} B_j^T \Theta_j\right)^T \Theta_i + \Theta_i \left(A - \sum_{j \ne i} B_j R_j^{-1} B_j^T \Theta_j\right) \\ -\Theta_i B_i (R_i)^{-1} B_i^T \Theta_i + P_i = 0, \quad i = 1, \ldots, n.\end{gathered} \tag{3.6}$$

Then

$$\begin{gathered}V_i(t_0, x_0) = (x_0)^T \Theta_i^{NE} x_0, \\ V_i(t, x^\alpha(t)) = (x^\alpha(t))^T \Theta_i^{NE} x^\alpha(t).\end{gathered}$$

The payoff distribution procedure constructed according to the rule (3.5) is of the form

$$\begin{gathered}\beta_i(t) = \eta_i(t) - (x^\alpha(t))^T \left(\left(A + \sum_{i \in N} B_i M_i^{NE}\right)^T \Theta_i^{NE} \right. \\ \left. + \Theta_i^{NE} \left(A + \sum_{i \in N} B_i M_i^{NE}\right) \right) x^\alpha(t), \quad i = 1, \ldots, n.\end{gathered}$$

4. STRATEGIC SUPPORT OF A PARETO-OPTIMAL SOLUTION

Assume that players agree to use a weight vector $(\alpha_1, \ldots, \alpha_n)$ such that the individual rationality of a Pareto-optimal solution holds at the initial time. We also assume that a time-consistent payoff distribution procedure

$$\int_l^\infty \beta_i(t)\, dt \le V_i(l, x^\alpha(l)), \quad \forall l \ge t_0, \quad i = 1, \ldots, n, \tag{4.1}$$

is used.

In [2], the problem of strategic support of a cooperative solution was considered. It was proved that for a special class of differential games a dynamically stable cooperative solution can be supported by a Nash equilibrium.

Following [2], to punish those who deviate from the cooperative agreement, we consider a special game $\Gamma_\alpha(t_0, x_0)$, differing from the original one by players' payoffs on the cooperative trajectory only.

Let $\overline{J_i}(t_0, x_0, u)$ be the payoff of player i in the game $\Gamma_\alpha(t_0, x_0)$.

Then

$$\overline{J_i}(t_0, x_0, u) = \int\limits_{t_0}^{l} \beta_i(t)\, dt + J_i(l, x(l), u),$$

where l is the last time such that $x(\tau) = x^\alpha(\tau)$, for all $\tau \in [t_0; t]$. This means that at time l a deviation from the optimal trajectory occurs. Furthermore,

$$\overline{J_i}(t_0, x_0, u) = J_i(t_0, x_0, u)$$

if there is no $\tau > t_0$ such that $x(t) = x^\alpha(t)$ for $t_0 \le t \le \tau$.

If $x(t) \equiv x^\alpha(t)$ for $t \ge t_0$, then

$$\overline{J_i}(t_0, x_0, u^\alpha) = \int\limits_{t_0}^{\infty} \beta_i(t)\, dt = J_i^\alpha(t_0, x_0, u^\alpha).$$

Definition 4.1 [3]. A collection of strategies $u^\varepsilon = (u_1^\varepsilon, \ldots, u_n^\varepsilon)$ in the class of admissible strategies is called an ε-equilibrium if for any admissible collection of strategies

$$u^\varepsilon \mid u_i = (u_1^\varepsilon, \ldots, u_{i-1}^\varepsilon, u_i, u_{i+1}^\varepsilon, \ldots, u_n^\varepsilon)$$

and for any player i we have

$$J_i\left(t_0, x_0, u^\varepsilon \mid u_i\right) + \varepsilon \ge J_i(t_0, x_0, u^\varepsilon).$$

Theorem 4.1. *Let the inequalities*

$$J_i^\alpha(k_0, x_0, u^\alpha) \le V_i(k_0, x_0), \quad i = 1, \ldots, n,$$

be satisfied for some Pareto-optimal solution. Then in the game $\Gamma_\alpha(t_0, x_0)$ for any $\varepsilon > 0$ there exists an ε-equilibrium situation with players' payoffs

$$\left(J_1^\alpha(t_0, x_0, u^\alpha), \ldots, J_n^\alpha(t_0, x_0, u^\alpha)\right).$$

Proof. Consider the following strategy of player i in the game $\Gamma_\alpha(t_0, x_0)$:

$$u^* = \begin{cases} u_i^\alpha(t) = M_i^\alpha x(t), & \text{for } t \in [t_0, l+\delta), \text{ if there exists} \\ & l > t_0 : x(t) = x^\alpha(t),\ t_0 \le t \le l \\ \widehat{u}_i(t) = M_i^{NE} x(t), & \text{the } i\text{th component of the Nash equilibrium} \\ & \text{in the subgame } \Gamma(l+\delta, x(l+\delta)), \text{ if at time } l \\ & \text{a deviation from the optimal trajectory occurs,} \\ & \text{and at time } l+\delta \text{ player } i \text{ responses to this.} \end{cases} \tag{4.2}$$

One can show that for each $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that the constructed collection of strategies is an ε-equilibrium in the game $\Gamma_\alpha(t_0, x_0)$.

If player i deviates from the optimal trajectory at time l and after that uses a strategy $\tilde{u}_i = \tilde{M}_i x(t)$, and other players response to this at time $l + \delta$, then the payoff of player i is at least

$$\int\limits_{t_0}^{l} \beta_i(t)\, dt + \int\limits_{l}^{l+\delta} \left(\tilde{x}^T(t) P_i \tilde{x}(t) + \tilde{u}_i^T(t) R_i \tilde{u}_i(t)\right) dt + V_i(l+\delta, \tilde{x}(l+\delta)).$$

The second term is nonnegative here, since the matrices P_i are positive semidefinite and R_i are positive definite. Hence,

$$\int_{t_0}^{l} \beta_i(t)\,dt + \int_{l}^{l+\delta} \left(\tilde{x}^T(t)P_i\tilde{x}(t) + \tilde{u}_i^T(t)R_i\tilde{u}_i(t)\right)\,dt$$

$$+V_i(l+\delta,\tilde{x}(l+\delta)) \geq \int_{t_0}^{l} \beta_i(t)\,dt + V_i(l+\delta,\tilde{x}(l+\delta)).$$

Here $\tilde{x}(t)$ is a solution of the differential equation

$$\dot{x}(t) = \left(A + B_i\tilde{M}_i + \sum_{j\neq i} B_jM_j^{\alpha}\right)x(t), \quad \tilde{x}(l) = x^{\alpha}(l). \tag{4.3}$$

Furthermore,

$$V_i(l+\delta, x(l+\delta)) = (x^{\alpha}(l))^T\left(e^{\tilde{A}\delta}\right)^T\theta_i^{NE}e^{\tilde{A}\delta}x^{\alpha}(l),$$

where $\tilde{A} = A + B_i\tilde{M}_i + \sum_{j\neq i} B_jM_j^{\alpha}$.

Note that the matrix $\tilde{A}$ has negative eigenvalues, since players may use admissible strategies only. Therefore,

$$\lim_{\delta\to 0}\left(V_i(l,x^{\alpha}(l)) - V_i(l+\delta,\tilde{x}(l+\delta))\right) = 0,$$

and for each $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that

$$V_i(l,x^{\alpha}(l)) - V_i(l+\delta,\tilde{x}(l+\delta)) \leq \varepsilon.$$

Then for such a δ we have

$$\int_{t_0}^{l} \beta_i(t)\,dt + V_i(l+\delta,\tilde{x}(l+\delta)) \geq \int_{t_0}^{l} \beta_i(t)\,dt + V_i(l,x^{\alpha}(l)) - \varepsilon.$$

Since the players use a time-consistent payoff distribution procedure, we can choose a vector function $\beta(t)$ so that inequalities (4.1) are satisfied. Then

$$\int_{t_0}^{l} \beta_i(t)\,dt + V_i(l,x^{\alpha}(l)) - \varepsilon \geq \int_{t_0}^{l} \beta_i(t)\,dt + \int_{l}^{\infty} \beta_i(t)\,dt - \varepsilon = J_i^{\alpha}(t_0,x_0,u^{\alpha}) - \varepsilon.$$

Hence, if player i deviates from an optimal trajectory, his payoff will be at least $J_i^{\alpha}(t_0,x_0,u^{\alpha})-\varepsilon$, and the constructed collection of strategies is an ε-equilibrium with payoff $J_i^{\alpha}(t_0,x_0,u^{\alpha})$. □

5. EXAMPLE

Consider the two-person game (2.1), (2.2). Let the state of system be described by the equation

$$\dot{x}(t) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} x(t) + \sum_{i=1}^{2} \begin{pmatrix} -2 \\ 0 \end{pmatrix} u_i(t), \quad x(t_0) = x_0, \tag{5.1}$$

and let players' payoffs be of the form

$$J_1(t_0, x_0, u) = \int_{t_0}^{\infty} \left(x^T(t) \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} x(t) + \frac{1}{2} u_1^T(t) u_1(t) \right) dt, \tag{5.2}$$

$$J_2(t_0, x_0, u) = \int_{t_0}^{\infty} \left(x^T(t) \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} x(t) + \frac{1}{4} u_2^T(t) u_2(t) \right) dt. \tag{5.3}$$

Let

$$\theta_1 = \begin{pmatrix} q_1 & q_2 \\ q_2 & q_3 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}. \tag{5.4}$$

To find the Nash equilibrium, we have to solve system (3.6), which in this case takes the form

$$\begin{aligned}
-(2(1-16s_1))q_1 - 1 + 8q_1^2 = 0,\\
-(1-16s_1)q_2 - (-1-16s_2)q_1 + q_2 - 2 + 8q_1q_2 = 0,\\
-(2(-1-16s_2))q_2 + 2q_3 - 5 + 8q_2^2 = 0,\\
-(2(1-8q_1))s_1 - 1 + 16s_1^2 = 0,\\
-(1-8q_1)s_2 - (-1-8q_2)s_1 + s_2 - 2 + 16s_1s_2 = 0,\\
-(2(-1-8q_2))s_2 + 2s_3 - 5 + 16s_2^2 = 0.
\end{aligned}$$

By solving the system, we obtain

$$\begin{aligned}
\theta_1^{NE} &= \begin{pmatrix} 0.1463 & 0.2509 \\ 0.2509 & 0.9709 \end{pmatrix},\\
\theta_2^{NE} &= \begin{pmatrix} 0.2396 & 0.2557 \\ 0.2557 & 1.2079 \end{pmatrix},\\
M_1^{NE} &= \begin{pmatrix} 0.5852 & 1.0036 \end{pmatrix},\\
M_2^{NE} &= \begin{pmatrix} 1.9168 & 2.0456 \end{pmatrix},\\
V_1(t, x(t)) &= x^T(t)\theta_1^{NE}x(t),\\
V_2(t, x(t)) &= x^T(t)\theta_2^{NE}x(t).
\end{aligned}$$

If $\theta = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$, then, to find a Pareto-optimal solution, we have to solve system (2.8), which in this case takes the form

$$\begin{aligned}
2x_1 + 1 - x_1^2(8/\alpha + 16/(1-\alpha)) = 0,\\
-x_1 + 2 - x_1(8/\alpha + 16/(1-\alpha))x_2 = 0,\\
-2x_2 - 2x_3 + 5 - x_2^2(8/\alpha + 16/(1-\alpha)) = 0.
\end{aligned}$$

Then

$$\theta^\alpha = \begin{pmatrix} \dfrac{1}{8}\dfrac{-\alpha^2 + r + \alpha}{1+\alpha} & \dfrac{\alpha(-\alpha^3 + r\alpha - 14\alpha^2 - r - \alpha + 16)}{8(-\alpha^3 + r\alpha + r + \alpha)} \\ \dfrac{\alpha(-\alpha^3 + r\alpha - 14\alpha^2 - r - \alpha + 16)}{8(-\alpha^3 + r\alpha + r + \alpha)} & \dfrac{(-\alpha^4 + r\alpha^2 + 46\alpha^3 - 41r\alpha - 33\alpha^2 - 40r - 108\alpha - 32)}{16(\alpha^3 - r\alpha - 4\alpha^2 - r - 9\alpha - 4)} \end{pmatrix},$$

where $r = \sqrt{\alpha^4 - 10\alpha^3 + \alpha^2 + 8\alpha}$.

For $\alpha = 0.6$, we obtain

$$\theta^\alpha = \begin{pmatrix} 0.1569 & 0.2202 \\ 0.2202 & 0.9872 \end{pmatrix},$$
$$M_1^\alpha = \begin{pmatrix} 1.046 & 1.468 \end{pmatrix},$$
$$M_2^\alpha = \begin{pmatrix} 3.138 & 4.404 \end{pmatrix}.$$

By substituting the obtained values of optimal controls into the system dynamic equation, we find an optimal trajectory. One can find the payoff of each player along the optimal trajectory:

$$J_1(t_0, x_0, u^\alpha) = (x^\alpha(t))^T \begin{pmatrix} 0.1049 & 0.1709 \\ 0.1709 & 0.8612 \end{pmatrix} x^\alpha(t),$$
$$J_2(t_0, x_0, u^\alpha) = (x^\alpha(t))^T \begin{pmatrix} 0.2349 & 0.29415 \\ 0.29415 & 1.1762 \end{pmatrix} x^\alpha(t).$$

If $x_0 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then

$$J_1(t_0, x_0, u^\alpha) - V_1(t_0, x_0) = -0.1602 < 0,$$
$$J_2(t_0, x_0, u^\alpha) - V_2(t_0, x_0) = -0.2853 < 0.$$

However, at time $t = 0.08$

$$J_1(t, x^\alpha(t), u^\alpha) - V_1(t, x^\alpha(t)) = 0.055 > 0.$$

We chose the functions $\eta_i(t)$ according to the following rule:

$$\eta_1(t) = -0.0207(x^\alpha(t))^T x^\alpha(t),$$
$$\eta_2(t) = -0.0368(x^\alpha(t))^T x^\alpha(t).$$

Then

$$\beta_1(t) = (x^\alpha(t))^T \begin{pmatrix} 2.1352 & 3.964 \\ 3.964 & 8.316 \end{pmatrix} x^\alpha(t),$$
$$\beta_2(t) = (x^\alpha(t))^T \begin{pmatrix} 3.4939 & 5.1932 \\ 5.1932 & 8.8963 \end{pmatrix} x^\alpha(t).$$

Using the constructed payoff distribution procedure, the players obtain a strategically stable cooperative solution.

6. CONCLUSION

In the paper we have considered the problem of strategic support of a Pareto-optimal solution in linear-quadratic differential games. We have shown that in a game of a special type, which is constructed with the help of a time-consistent payoff distribution procedure and differs from the original game in payoffs on the cooperative trajectory only, no individual deviation from cooperation is profitable for the deviating player. The outcome of the cooperative solution is attained at some ε-Nash equilibrium in this game.

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