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STOCHASTIC SYSTEMS, QUEUEING SYSTEMS

On Pareto Set in Control and Filtering Problems under Stochastic and Deterministic Disturbances

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Abstract—We consider two-criteria control or filtering problems for linear systems, where one criterion is the level of suppression for Gaussian white noise with unknown covariance, and another is the level of suppression for a deterministic signal of bounded power. We define a new criterion, the level of suppression for stochastic and deterministic disturbances that act jointly in the general case on different inputs. This criterion is characterized in terms of solutions of Riccati equations or linear matrix inequalities. We establish that for the choice of optimal controller or filter with respect to this criterion relative losses with respect to each of the original criteria compared to Pareto optimal solutions do not exceed the value $1 - \sqrt{2}/2$. We extend these results to dual control and filtering problems for systems with one input and two outputs, generalize them to the case of N criteria with loss estimate $1 - \sqrt{N}/N$, and also apply them for systems with external and initial disturbances. We show a numerical example.

Key words: multi-criteria optimization, Pareto set, control, filtering, stochastic disturbances, deterministic disturbances, H_∞-optimal solutions, γ_0-optimal solutions, Pareto suboptimal solutions, H_2/H_∞-norm.

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1. INTRODUCTION

Mathematical models of processes in control or filtering problems can be divided into stochastic and deterministic. In models of the first type, disturbances and measurement noise are described as random, and criteria are usually established variances of errors. If statistical characteristics of input random processes are known, then in the linear–quadratic case this leads to the so-called H_2-optimal controllers or filters [1] (optimal Gaussian control, or Wiener–Kalman filter), where the H_2-norm of the transfer matrix of the system of disturbances to the error is an integral parameter of system reaction to harmonic disturbances on all frequencies. In deterministic problems with unknown disturbances, the criterion is usually the so-called H_∞-norm of this transfer matrix that characterized the maximal possible ratio of error energy to disturbance energy. This theory leads to H_∞-optimal controllers and filters. Another important and developing direction combines stochastic H_2- and H_∞-theories in the concept of anisotropic norm; see, e.g., [2, 3]. Optimal solutions with respect to each of the above criteria differ from each other and in many cases are conflicting.

At the same time, the division between stochastic and deterministic processes is quite relative, it is often more natural to assume that some disturbances (e.g., the external disturbance) are deterministic, and some (e.g., measurement noise) are stochastic in the same problem. Besides, if one of the disturbances is absent, and control or filtering quality is evaluated with the corresponding criterion, then the same controller or filter must operate in the best possible way under each of the disturbances. Similar considerations have led to problems with the so-called H_2/H_∞-criterion [4–11]. In particular, the works [12, 13] assume that one input is subject to a deterministic disturbance of bounded power, another is subject to random white noise with unit covariance matrix, and the quality criterion is the maximal power of the output among all deterministic signals. The authors of these works were able to design filters and controllers that minimize an upper bound on the output power, but at the same time it remained unknown how well the resulting systems perform compared to the optimal with respect to individual criteria for deterministic and stochastic disturbances.

A development of this approach is the concept of the so-called multi-criteria control [14–16], where the objective is formulated in terms of the general Lyapunov function, although optimal solutions with respect to each criterion correspond to different Lyapunov functions. This has led to certain conservatism, but has allowed to design a controller that in some sense combines the properties of optimal solutions with respect to individual criteria. The problem of how conservative the resulting solutions are or, in other words, to what extent values of individual quality criteria in the resulting systems differ from their optimal values, also remains open.

A multi-criteria optimization problem with H_2-criteria has been considered in [17, 18]; with H_2- and H_∞-criteria, in [19], where the authors used the so-called Q-parametrization of controllers and suboptimal solutions were found with finite-dimensional controllers. As far as we know, there has been no significant progress in solving multi-criteria control problems since then, and publications of the last decade on this topic are chiefly related to computational aspects (see, e.g., [20]) or specific applications.

In the present work (see also [21]), the central problem is to characterize Pareto sets in control and filtering problems with deterministic and stochastic disturbances. Unlike the problems described above, we model stochastic signals as stationary Gaussian white noises with unknown covariance, and as the quality criterion we introduce the level of suppression for random disturbances as a maximal value of the ratio of limit variances (powers) of output and input signals, averaged over time, over all nonzero input covariances (control and filtering problems with this criterion were considered in [22–24]). This lets us construct, under jointly acting stochastic and deterministic disturbances, a parameterized "ideal" criterion that possesses the following property: the set of points on the plane of criteria corresponding to values of individual criteria under the optimal solution with respect to the ideal criterion includes the Pareto set.

Further, since it appears impossible to find optimal solutions with respect to the ideal criterion, we introduce instead a different criterion that lets us obtain lower and upper bounds on the Pareto set. This criterion represents the level of suppression for jointly acting stochastic and deterministic disturbances equal to the maximal ratio of output power to the square root of the weighted sum of squares of powers of input signals over all admissible deterministic and stochastic disturbances. Note that a counterpart of this criterion in the suppression problem for deterministic disturbances of bounded energy and disturbances generated by a nonzero initial state is the generalized H_∞-norm [25–29]. In this work we show (see also [24]) that the level of suppression for stochastic disturbances and level of suppression for jointly acting stochastic and deterministic disturbances can be expressed in terms of solutions of Riccati equations or linear matrix inequalities [30, 31]. This lets us design optimal filters and controllers with respect to different parameters in a unified context and evaluate losses with the ratio to the Pareto optimal solutions.

Using the duality idea, we have been able to establish that this approach can be extended to multi-criteria problems in systems with one input and two target outputs, criteria for which are the generalized H_2-norm and H_∞-norm corresponding to the channels. We show how the proposed approach can be extended to problems with N criteria, and also to control and filtering problems with external and initial disturbances. We show examples that illustrate our results.

2. PROBLEM SETTING

2.1. Control

Consider a controllable system

$$x(t+1) = Ax(t) + B_w w(t) + B_v v(t) + B_u u(t), \quad x(0) = x_0,$$
$$z(t) = C_1 x(t) + D_w w(t) + D_v v(t) + D_u u(t), \tag{2.1}$$

where $x(t) \in R^{n_x}$ is the state, $u(t) \in R^{n_u}$ is the control, $z(t) \in R^{n_z}$ is the controllable output, and x_0 is a random initial system state with zero expectation and unknown covariance matrix. We assume that the disturbance $w(t) \in R^{n_w}$ is an input stationary Gaussian random sequence of vectors with zero expectation and covariance matrix $Ew(t)w^{\mathrm{T}}(t) \equiv K_w$, and that $w(t)$ for all t is uncorrelated with x_0. The disturbance $v(t) \in R^{n_v}$ is assumed to be deterministic with bounded power, which is defined by the value

$$\|s\|_{\mathcal{P}}^2 = \lim_{N\to\infty} (1/N) \sum_{t=0}^{N-1} |s(t)|^2.$$

We denote the set of sequences with bounded power by $\mathcal{P} = \{s : \|s\|_{\mathcal{P}}^2 < \infty\}$. Note that all signals from l_2 with $\|s\|_2^2 = \sum_{t=0}^{\infty} |s(t)|^2 < \infty$ have zero power, so the power is a seminorm.

Suppose that the control law has the form of linear state feedback

$$u(t) = \Theta x(t), \tag{2.2}$$

under which the closed system

$$x(t+1) = A_c x(t) + B_w w(t) + B_v v(t), \quad x(0) = x_0,$$
$$z(t) = C_z x(t) + D_w w(t) + D_v v(t), \tag{2.3}$$

where

$$A_c = A + B_u \Theta, \quad C_z = C_1 + D_u \Theta, \tag{2.4}$$

is asymptotically stable with no disturbances.

If system (2.3) is subject only to a stochastic disturbance with unknown covariance matrix K_w, we will characterize the control quality by the level of suppression for stochastic disturbances whose square is the largest value of the ratio of established variances, averaged over time, for the output z and input w with nonzero covariance matrix K_w that belongs to the set $\mathcal{G}_{n_w}$ of vector Gaussian white noises of dimension n_w, i.e.,

$$\gamma_0(\Theta) = \sup_{w\in\mathcal{G}_{n_w}, v\equiv 0} \frac{\sqrt{J_z}}{\sqrt{J_w}}, \tag{2.5}$$

where

$$J_z = \lim_{N\to\infty} (1/N) \sum_{t=0}^{N-1} E|z(t)|^2, \quad J_w = \lim_{N\to\infty} (1/N) \sum_{t=0}^{N-1} E|w(t)|^2 = \operatorname{tr} K_w.$$

In this case, the output will be a stationary Gaussian sequence, i.e., it will be ergodic, so individual trajectories z have (with probability one) finite power and, consequently,

$$\gamma_0(\Theta) = \operatorname{ess} \sup_{w\in\mathcal{G}_{n_w}, v\equiv 0} \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}}, \tag{2.6}$$

where ess denotes essential supremum, i.e., the smallest upper bound with probability one. This value will be the induced norm of a linear operator generated by system (2.3) for $v \equiv 0$ and zero initial conditions and mapping $w(t) \in \mathcal{G}_{n_w}$ to $z(t) \in \mathcal{G}_{n_z}$. Below we will show how the level of suppression for stochastic disturbances can be expressed in terms of solutions of Lyapunov equations or linear matrix inequalities and also via the transfer matrix H_w of this system from w to z.

If only deterministic disturbances are present, the control quality is characterized by the level of suppression for deterministic disturbances which is equal to the maximal value of the ratio of powers of target output and disturbances, i.e., induced norm of a linear operator mapping $v(t) \in \mathcal{P}$ to $z(t) \in \mathcal{P}$. It has been shown in [4] that this norm equals the H_∞-norm of the transfer matrix H_v of this system from v to z, i.e.,

$$\gamma_\infty(\Theta) = \sup_{v \in \mathcal{P}, w \equiv 0} \frac{\|z\|_\mathcal{P}}{\|v\|_\mathcal{P}} = \|H_v\|_\infty.$$

For each of the above criteria, there exists its own optimal matrix of feedback parameters. The Pareto optimal solutions in the considered two-criteria problem is the set of parameters

$$\Theta_P = \arg\min_\Theta \{\gamma_0(\Theta), \gamma_\infty(\Theta)\},$$

i.e., such that there exists no other linear feedback under which the level of suppression for one of the disturbances would be smaller without increasing another. This set does exist hypothetically, but it is extremely hard to find. In general, the problem is to characterize this set in some way. Namely, we define a family of control laws under which the relative quality deterioration with respect to each of the criteria compared to a Pareto optimal control does not exceed a certain value. This family of control laws is also interesting by itself since it includes control laws that are optimal with respect to the level of suppression for jointly acting stochastic and deterministic disturbances.

2.2. Filtering

Consider a system defined by equations

$$\begin{aligned} x(t+1) &= Ax(t) + B_1 w(t) + B_2 v(t), \quad x(0) = x_0, \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} v(t), \\ z(t) &= C_z x(t), \end{aligned} \tag{2.7}$$

where $x(t) \in R^{n_x}$ is the state, $y(t) \in R^{n_y}$ is the measured output, $z(t) \in R^{n_z}$ is the target output, $w(t) \in R^{n_v}$ is a stationary Gaussian white noise with zero expectation and unknown covariance $Ew(t)w^{\mathrm{T}}(t) \equiv K_w$, $v(t) \in \mathcal{P}$ is a deterministic disturbance, x_0 is a random initial system state with zero expectation and unknown covariance matrix. We assume that the initial state is not correlated with $w(t)$ for all t.

To get an estimate of the target output, we choose a filter of the form

$$\begin{aligned} x_f(t+1) &= Ax_f(t) + \Theta[y(t) - C_2 x_f(t)], \quad x_f(0) = 0, \\ z_f(t) &= C_z x_f(t), \end{aligned} \tag{2.8}$$

where $x_f(t) \in R^{n_x}$ is the filter state, and Θ is its matrix of parameters. Then errors in the estimates of the state $e(t) = x(t) - x_f(t)$ and target output $e_z(t) = z(t) - z_f(t)$ satisfy equations

$$\begin{aligned} e(t+1) &= A_c e(t) + B_w w(t) + B_v v(t), \quad e(0) = x_0, \\ e_z(t) &= C_z e(t), \end{aligned} \tag{2.9}$$

where

$$A_c = A - \Theta C_2, \quad B_w = B_1 - \Theta D_{21}, \quad B_v = B_2 - \Theta D_{22}. \tag{2.10}$$

We define for system (2.9) levels of suppression for stochastic and deterministic disturbances as

$$\gamma_0(\Theta) = \sup_{w\in\mathcal{G}_{n_w}, v\equiv 0} \frac{\sqrt{J_{e_z}}}{\sqrt{J_w}}, \quad \gamma_\infty(\Theta) = \sup_{v\in\mathcal{P}, w\equiv 0} \frac{\|e_z\|_{\mathcal{P}}}{\|v\|_{\mathcal{P}}}$$

and pose the problem of characterizing Pareto optimal filters whose parameters satisfy condition

$$\Theta_P = \arg\min_L \{\gamma_0(\Theta), \gamma_\infty(\Theta)\}.$$

In the following section we will show that in each of the problems we can define an "ideal" criterion, depending on a scalar parameter, such that the Pareto set on the plane of criteria is contained in the set of points corresponding to the criteria's values under optimal solutions with this ideal criterion.

3. THE PARETO SET

Let us consider both problems formulated above in a single unified scheme. Namely, on system trajectories

$$\begin{gathered} x(t+1) = A_c(\Theta)x(t) + B_w(\Theta)w(t) + B_v(\Theta)v(t), \quad x(0) = x_0, \\ z(t) = C_z(\Theta)x(t) + D_w(\Theta)w(t) + D_v(\Theta)v(t), \end{gathered} \tag{3.1}$$

whose matrices depend on the matrix of parameters Θ, and where all eigenvalues of matrix $A(\Theta)$ are strictly less than one in absolute values, we define two criteria, $\gamma_0(\Theta)$ and $\gamma_\infty(\Theta)$, similar to above, that characterize levels of suppression for stochastic and deterministic disturbances separately. Suppose that Θ_P belongs to the Pareto set P_0 and $\gamma_0(\Theta_P) = \gamma_1$, $\gamma_\infty(\Theta_P) = \gamma_2$. Due to the definition of a level of suppression for stochastic disturbances, for $v \equiv 0$ we have

$$\sqrt{J_z} \leqslant \gamma_1 \sqrt{J_w} \quad \forall\, w \in \mathcal{G}.$$

In this case individual trajectories z_w have (with probability one) finite power and, consequently,

$$\|z_w\|_{\mathcal{P}} \leqslant \gamma_1 \|w\|_{\mathcal{P}} \quad \forall w \in \mathcal{G}. \tag{3.2}$$

By definition of a level of suppression for deterministic disturbances, we have that

$$\|z_v\|_{\mathcal{P}} \leqslant \gamma_2 \|v\|_{\mathcal{P}} \quad \forall v \in \mathcal{P}, \tag{3.3}$$

where z_v is the output of the closed system (3.1) for $w \equiv 0$.

We define on the trajectories of system (3.1), under jointly acting stochastic and deterministic disturbances, the criterion

$$J_\rho(\Theta) = \operatorname{ess} \sup_{w\in\mathcal{G}, v\in\mathcal{P}} \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}} + \rho\|v\|_{\mathcal{P}}}, \tag{3.4}$$

where $\rho > 0$ is the weight coefficient. Using the obvious inequality that follows from (3.2), (3.3),

$$\|z\|_{\mathcal{P}} = \|z_w + z_v\|_{\mathcal{P}} \leqslant \gamma_1(\|w\|_{\mathcal{P}} + \rho\|v\|_{\mathcal{P}}) \quad \forall\, w \in \mathcal{G}, \forall\, v \in \mathcal{P},$$

where $\rho = \gamma_2/\gamma_1$, we get that $J_\rho(\Theta_P) \leqslant \gamma_1$. Let Θ_ρ be a matrix of parameters minimizing $J_\rho(\Theta)$ for a given ρ. Then $J_\rho(\Theta_\rho) = \gamma_\rho \leqslant \gamma_1$ and, consequently, for $\Theta = \Theta_\rho$ we have

$$\|z\|_{\mathcal{P}} \leqslant \gamma_\rho(\|w\|_{\mathcal{P}} + \rho\|v\|_{\mathcal{P}}) \quad \forall\, w \in \mathcal{G}, \forall\, v \in \mathcal{P}.$$

This inequality implies that

$$\|z\|_{\mathcal{P}} \leqslant \gamma_\rho\|w\|_{\mathcal{P}}, \quad v \equiv 0 \quad \forall w \in \mathcal{G},$$
$$\|z\|_{\mathcal{P}} \leqslant \gamma_\rho\, \rho\|v\|_{\mathcal{P}}, \quad w \equiv 0 \quad \forall v \in \mathcal{P}.$$

Thus, we have that

$$\gamma_\rho \leqslant \gamma_1, \quad \gamma_\rho\, \rho = \gamma_\rho\, \gamma_2/\gamma_1 \leqslant \gamma_2.$$

This means that $\gamma_0(\Theta_\rho) \leqslant \gamma_1$ and $\gamma_\infty(\Theta_\rho) \leqslant \gamma_2$, i.e., $\gamma_0(\Theta_\rho) \leqslant \gamma_0(\Theta_P)$ and $\gamma_\infty(\Theta_\rho) \leqslant \gamma_\infty(\Theta_P)$. Since Θ_P belongs to the Pareto set, $\gamma_0(\Theta_\rho) = \gamma_0(\Theta_P)$, $\gamma_\infty(\Theta_\rho) = \gamma_\infty(\Theta_P)$ and, consequently, Θ_ρ also belongs to P_0. Summarizing, we formulate necessary Pareto optimality conditions for the problem at hand.

Theorem 3.1. *If* (γ_1, γ_2) *is a Pareto optimal point on the plane of criteria* $\gamma_0(\Theta)$ *and* $\gamma_\infty(\Theta)$ *for system* (3.1)*, then there exists a matrix of parameters* $\Theta_\rho \in P_0$ *minimizing criterion* (3.4) *for* $\rho = \gamma_2/\gamma_1$ *such that* $\gamma_0(\Theta_\rho) = \gamma_1$*,* $\gamma_\infty(\Theta_\rho) = \gamma_2$*.*

Thus, we should be looking for the Pareto set only among optimal solutions with respect to criterion $J_\rho(\Theta)$. However, since solving this one-criterion problem presents many obstacles, in the next section we will consider another one-criterion problem whose solution can be found efficiently and that lets us estimate the boundaries of the Pareto set.

4. ESTIMATING THE PARETO SET ON THE PLANE OF CRITERIA

We denote

$$\gamma_1^- = \min_\Theta \gamma_0(\Theta) = \gamma_0(\Theta_0), \quad \gamma_2^- = \min_\Theta \gamma_\infty(\Theta) = \gamma_\infty(\Theta_\infty),$$

where Θ_0 and Θ_∞ are the optimal matrices of parameters with respect to criteria γ_0 and γ_∞ respectively. Besides, we denote

$$\gamma_1^+ = \gamma_0(\Theta_\infty), \quad \gamma_2^+ = \gamma_\infty(\Theta_0).$$

Obviously, the Pareto set belongs to the set

$$D = \Big\{(\gamma_1, \gamma_2) : \gamma_1^- \leqslant \gamma_1 \leqslant \gamma_1^+, \quad \gamma_2^- \leqslant \gamma_2 \leqslant \gamma_2^+\Big\}.$$

Let us refine these estimates. To do that, we introduce on trajectories of system (3.1) a new criterion

$$\gamma_{0,\infty}(\Theta) = \text{ess} \sup_{w \in \mathcal{G}, v \in \mathcal{P}} \frac{\|z\|_{\mathcal{P}}}{\sqrt{\|w\|^2_{\mathcal{P}} + \rho^2\|v\|^2_{\mathcal{P}}}}, \tag{4.1}$$

which we call the level of suppression for (jointly acting) stochastic and deterministic disturbances. Now (4.1) immediately implies that

$$\gamma_{0,\infty}(\Theta) \geqslant \max\Big\{\gamma_0(\Theta), \rho^{-1}\gamma_\infty(\Theta)\Big\}. \tag{4.2}$$

We compare the corresponding values of original criteria $\gamma_0(\Theta)$ and $\gamma_\infty(\Theta)$ on the optimal solutions with respect to criteria $J_\rho(\Theta)$ and $\gamma_{0,\infty}(\Theta)$. Since

$$\frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}+\rho\|v\|_{\mathcal{P}}} \geqslant \frac{\|z\|_{\mathcal{P}}}{\sqrt{2}\sqrt{\|w\|_{\mathcal{P}}^2+\rho^2\|v\|_{\mathcal{P}}^2}},$$

we get that $J_\rho(\Theta) \geqslant (1/\sqrt{2})\gamma_{0,\infty}(\Theta)$. Let Θ_P belong to the Pareto set and $\gamma_0(\Theta_P)=\gamma_1$, $\gamma_\infty(\Theta_P)=\gamma_2$. For the matrix Θ_ρ, which is optimal with respect to criterion $J_\rho(\Theta)$ for $\rho=\gamma_2/\gamma_1$, it holds that $J_\rho(\Theta_\rho)=\gamma_\rho \leqslant \gamma_1$. Consequently, it holds that

$$\gamma_1 \geqslant J_\rho(\Theta_\rho) \geqslant (1/\sqrt{2})\gamma_{0,\infty}(\Theta_\rho) \geqslant (1/\sqrt{2})\gamma_{0,\infty}(\Theta_\rho^*),$$

where Θ_ρ^* is the optimal solution with respect to criterion (4.1). Taking into account (4.2), we derive from this conditions

$$(1/\sqrt{2})\gamma_0(\Theta_\rho^*) \leqslant \gamma_1, \quad (1/\sqrt{2})\gamma_\infty(\Theta_\rho^*) \leqslant \gamma_1\rho = \gamma_2. \tag{4.3}$$

Since $\Theta_\rho \in P_0$, one of the following three cases is possible:

$$\begin{aligned} &\gamma_0(\Theta_\rho^*) \geqslant \gamma_1, \quad \gamma_\infty(\Theta_\rho^*) \geqslant \gamma_2;\\ &\gamma_0(\Theta_\rho^*) > \gamma_1, \quad \gamma_\infty(\Theta_\rho^*) \leqslant \gamma_2;\\ &\gamma_0(\Theta_\rho^*) \leqslant \gamma_1, \quad \gamma_\infty(\Theta_\rho^*) > \gamma_2. \end{aligned}$$

In the first case, due to (4.3) we get that

$$\frac{\gamma_0(\Theta_\rho^*)-\gamma_1}{\gamma_0(\Theta_\rho^*)} \leqslant 1-\frac{\sqrt{2}}{2}, \quad \frac{\gamma_\infty(\Theta_\rho^*)-\gamma_2}{\gamma_\infty(\Theta_\rho^*)} \leqslant 1-\frac{\sqrt{2}}{2}, \tag{4.4}$$

i.e., when choosing optimal solutions Θ_ρ^* with respect to criterion $\gamma_{0,\infty}(\Theta)$ for a given ρ, relative losses with respect to each of the criteria $\gamma_0(\Theta)$ and $\gamma_\infty(\Theta)$ compared to the corresponding Pareto optimal solution, if it exists, do not exceed $1-\sqrt{2}/2$. In the other two cases, with respect to one of the criteria relative losses will also not exceed the value $1-\sqrt{2}/2$, and the value of the other criterion does not exceed its value at the Pareto optimal solution.

Let us finally show that loss estimates shown in (4.4) can be, generally speaking, refined with the properties of criterion $\gamma_{0,\infty}(\Theta)$. Namely, suppose that for a given ρ there exists a solution of problem

$$\min_{\Theta} \max_{w\in\mathcal{G}, v\in\mathcal{P}} \frac{\|z\|_{\mathcal{P}}}{\sqrt{\|w\|_{\mathcal{P}}^2+\rho^2\|v\|_{\mathcal{P}}^2}}, \tag{4.5}$$

which will be studied in detail in the next section. We denote by Θ_ρ^*, w_*, and v_* its optimal solution with respect to parameters Θ and the worst possible disturbances. Then

$$\gamma_{0,\infty}(\Theta_\rho^*) = \frac{\|z(\Theta_\rho^*, w_*, v_*)\|_{\mathcal{P}}}{\sqrt{\|w_*\|_{\mathcal{P}}^2+\rho^2\|v_*\|_{\mathcal{P}}^2}} \leqslant \frac{\|z(\Theta_\rho, w_*, v_*)\|_{\mathcal{P}}}{\sqrt{\|w_*\|_{\mathcal{P}}^2+\rho^2\|v_*\|_{\mathcal{P}}^2}} \leqslant \frac{\|z(\Theta_\rho, w_*, v_*)\|_{\mathcal{P}}}{\mu(\rho)(\|w_*\|_{\mathcal{P}}+\rho\|v_*\|_{\mathcal{P}})},$$

where $z(\Theta, w, v)$ is the target output of system (3.1) for the corresponding arguments, $\Theta_\rho \in P_0$, $\gamma_0(\Theta_\rho)=\gamma_1$, $\gamma_\infty(\Theta_\rho)=\gamma_2$, $\rho=\gamma_2/\gamma_1$, and

$$\mu(\rho) = \frac{\sqrt{\|w_*\|_{\mathcal{P}}^2+\rho^2\|v_*\|_{\mathcal{P}}^2}}{\|w_*\|_{\mathcal{P}}+\rho\|v_*\|_{\mathcal{P}}}. \tag{4.6}$$

We further have that

$$\mu(\rho)\gamma_{0,\infty}(\Theta_\rho^*) \leqslant \max_{w\in\mathcal{G},v\in\mathcal{P}} \frac{\|z(\Theta_\rho, w, v)\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}} + \rho\|v\|_{\mathcal{P}}} = J_\rho(\Theta_\rho) \leqslant \gamma_1.$$

This implies that

$$\mu(\rho)\gamma_0(\Theta_\rho^*) \leqslant \gamma_1, \quad \mu(\rho)\gamma_\infty(\Theta_\rho^*) \leqslant \rho\,\gamma_1 = \gamma_2. \tag{4.7}$$

Since $1/\sqrt{2} \leqslant \mu(\rho) \leqslant 1$, instead of estimates (4.3) we have obtained more exact estimates (4.7), which similarly lead to the following result.

Theorem 4.1. *Relative losses under the optimal solution with respect to the level of suppression for stochastic and deterministic disturbances for a given ρ compared to the Pareto optimal solution Θ_P for which $\gamma_0(\Theta_P) = \gamma_1$, $\gamma_\infty(\Theta_P) = \gamma_2$ and $\gamma_2/\gamma_1 = \rho$, do not exceed for each criterion $1 - \mu(\rho)$, i.e., it holds that*

$$\frac{\gamma_0(\Theta_\rho^*) - \gamma_1}{\gamma_0(\Theta_\rho^*)} \leqslant 1 - \mu(\rho), \quad \frac{\gamma_\infty(\Theta_\rho^*) - \gamma_2}{\gamma_\infty(\Theta_\rho^*)} \leqslant 1 - \mu(\rho), \tag{4.8}$$

where $1/\sqrt{2} \leqslant \mu(\rho) \leqslant 1$ has been defined in (4.6).

5. COMPUTING LEVELS OF SUPPRESSION FOR THE DISTURBANCES

Before we characterize the level of suppression for stochastic and deterministic disturbances, let us show how the level of suppression for stochastic disturbances can be expressed in terms of matrices of system Eqs. (3.1) or via its transfer matrix. In cases where it does not lead to confusion we will omit the arguments of matrix functions. Let $H(s) = (H_w(s)\ H_v(s))$ be the transfer matrix of this system, where

$$H_w(s) = C_z(sI - A_c)^{-1}B_w + D_w, \quad H_v(s) = C_z(sI - A_c)^{-1}B_v + D_v.$$

Theorem 5.1. *The level of suppression for stochastic disturbances in system* (3.1) *for $v \equiv 0$ can be found as*

$$\gamma_0^2(\Theta) = \lambda_{\max}(B_w^{\mathrm{T}}P_0B_w + D_w^{\mathrm{T}}D_w) = \lambda_{\max}\left\{\frac{1}{2\pi}\int_0^{2\pi} H_w^{\mathrm{T}}(e^{-j\varphi})H_w(e^{j\varphi})\,d\varphi\right\}, \tag{5.1}$$

where $P_0 = P_0^{\mathrm{T}} \geqslant 0$ is the solution of Lyapunov equation

$$A_c^{\mathrm{T}}PA_c - P + C_z^{\mathrm{T}}C_z = 0. \tag{5.2}$$

Here the covariance matrix of the worst possible disturbance equals $K_* = e_{\max}e_{\max}^{\mathrm{T}}$, where $e_{\max}$ is the eigenvector of matrix $B_w^{\mathrm{T}}P_0B_w + D_w^{\mathrm{T}}D_w$ corresponding to the maximal eigenvalue $\gamma_0^2(\Theta)$.

Note that, by substituting into (5.2) the corresponding expressions for matrices $A_c(\Theta)$ and $C_z(\Theta)$ and extracting the full square with respect to Θ, we can find for each of the control and filtering problems the optimal matrix of parameters Θ_0 under which the level of suppression for stochastic disturbances $\gamma_0^2(\Theta)$ is minimal. For instance, for the control problem we get

$$\Theta_0 = -\left(B_u^{\mathrm{T}}P_*B_u + D_u^{\mathrm{T}}D_u\right)^{-1}\left(B_u^{\mathrm{T}}P_*A + D_u^{\mathrm{T}}C_z\right), \tag{5.3}$$

where P_* is the stabilizing solution of Riccati equations

$$A^{\mathrm{T}}PA - P + C_z^{\mathrm{T}}C_z - \Theta_0^{\mathrm{T}}\left(B_u^{\mathrm{T}}P_*B_u + D_u^{\mathrm{T}}D_u\right)\Theta_0 = 0.$$

We now return to considering the level of suppression for stochastic and deterministic disturbances (4.1) for system (3.1), whose square is the largest value of the ratio of the square of output power to the weighted sum of squares of powers of deterministic and stochastic disturbances over all admissible disturbances. This definition immediately implies that

$$\gamma_{0,\infty} \geqslant \max\left\{\gamma_0, \rho^{-1}\gamma_\infty\right\}.$$

This value represents the induced norm of a linear operator that maps, due to system (3.1), a pair $(w(t), v(t))$, where $w(t) \in \mathcal{G}_{n_w}$ and $v(t) \in \mathcal{P}$, to $z(t) \in \mathcal{P}$; it can be called the H_∞/γ_0-norm of this system.

Theorem 5.2. *For system* (3.1), *condition* $\gamma_{0,\infty} < \gamma$ *holds if and only if*

$$\lambda_{\max}\left(B_w^{\mathrm{T}}X_\gamma B_w + D_w^{\mathrm{T}}D_w\right) < \gamma^2, \tag{5.4}$$

where $X_\gamma = X_\gamma^{\mathrm{T}} > 0$ *is the stabilizing solution of Riccati equations*

$$\begin{aligned} &A_c^{\mathrm{T}}XA_c - X + C_z^{\mathrm{T}}C_z \\ &\quad + \left(B_v^{\mathrm{T}}XA_c + D_v^{\mathrm{T}}C_z\right)^{\mathrm{T}}\left(\rho^2\gamma^2 I - B_v^{\mathrm{T}}XB_v - D_v^{\mathrm{T}}D_v\right)^{-1}\left(B_v^{\mathrm{T}}XA_c + D_v^{\mathrm{T}}C_z\right) = 0 \end{aligned} \tag{5.5}$$

such that $\rho^2\gamma^2 I - B_v^{\mathrm{T}}X_\gamma B_v - D_v^{\mathrm{T}}D_v > 0$ *and*

$$A_v = A_c + B_v\left(\rho^2\gamma^2 I - B_v^{\mathrm{T}}X_\gamma B_v - D_v^{\mathrm{T}}D_v\right)^{-1}\left(B_v^{\mathrm{T}}X_\gamma A_c + D_v^{\mathrm{T}}C_z\right) \tag{5.6}$$

is an asymptotically stable matrix.

Corollary. *For system* (3.1), *condition* $\gamma_{0,\infty} < \gamma$ *holds if and only if there exists a matrix* $X = X^{\mathrm{T}} > 0$ *satisfying linear matrix inequalities*

$$\begin{pmatrix} -X & XA_c & XB_v & 0 \\ \star & -X & 0 & C_z^{\mathrm{T}} \\ \star & \star & -\rho^2\gamma^2 I & D_v^{\mathrm{T}} \\ \star & \star & \star & -I \end{pmatrix} < 0, \quad \begin{pmatrix} -X & XB_w & 0 \\ \star & -\gamma^2 I & D_w^{\mathrm{T}} \\ \star & \star & -I \end{pmatrix} < 0. \tag{5.7}$$

Note that it follows from (5.7) that $\gamma_{0,\infty}^2 > \rho^{-2}\lambda_{\max}(D_v^{\mathrm{T}}D_v)$. Besides, inequality $\gamma_\infty < \gamma$ holds if and only if the first inequality in (5.7) is feasible for $\rho = 1$, and $\gamma_0 < \gamma$ when the first inequality in (5.7), where we exclude the third block row and column, and the second inequality holds.

Remark. It follows from the first inequality in (5.7) that for sufficiently large ρ inequality (5.7) becomes an inequality that defines γ_0^2, i.e., for $\rho \to \infty$ we have $\gamma_{0,\infty} \to \gamma_0$. For sufficiently small $\rho > 0$ the value $\gamma_{0,\infty}$ will coincide with $\rho^{-1}\gamma_\infty$. Indeed, let $X_\infty > 0$ satisfy the first inequality in (5.7) for the minimal value of $\rho^2\gamma^2 = \gamma_\infty^2 + \varepsilon$ for sufficiently small $\varepsilon > 0$. If at the same time $B_w^{\mathrm{T}}X_\infty B_w + D_w^{\mathrm{T}}D_w < (\gamma_\infty^2/\rho^2)I$, then $X_\infty > 0$ will satisfy the second inequality in (5.7), and then $\gamma_{0,\infty} = \rho^{-1}\gamma_\infty$. Thus, in order for the value $\gamma_{0,\infty}$ to actually reach a compromise between γ_0 and $\rho^{-1}\gamma_\infty$, it is necessary for the weight coefficient to satisfy condition

$$\rho^2 \geqslant \rho_*^2 = \gamma_\infty^2\lambda_{\max}^{-1}\left(B_w^{\mathrm{T}}X_\infty B_w + D_w^{\mathrm{T}}D_w\right). \tag{5.8}$$

Let us now consider the problem of the worst jointly acting stochastic and deterministic disturbances and show, in particular, that under a certain condition the supremum in (4.1) is reached.

Theorem 5.3. *Under* (5.8) *the maximal value of* $\gamma_{0,\infty}$ *is achieved when* $w_*(t)$ *is a white noise with covariance* $K_* = e_{\max}e_{\max}^{\mathrm{T}}$*, where* $e_{\max}$ *is the eigenvector of matrix* $B_w^{\mathrm{T}}X_*B_w + D_w^{\mathrm{T}}D_w$ *corresponding to the maximal eigenvalue* $\gamma_{0,\infty}^2$*, and*

$$v_*(t) = \left(\rho^2\gamma_{0,\infty}^2 I - B_v^{\mathrm{T}}X_*B_v - D_v^{\mathrm{T}}D_v\right)^{-1}\left(B_v^{\mathrm{T}}X_*A_c + D_v^{\mathrm{T}}C_z\right)x(t), \tag{5.9}$$

where X_* *is the stabilizing solution of Eqs.* (5.5) *for* $\gamma = \gamma_{0,\infty}$*,* $x(t)$ *is a solution of system*

$$x(t+1) = A_v x(t) + B_w w_*(t), \quad x(0) = x_0,$$

where matrix A_v *is defined in* (5.6) *for* $\gamma = \gamma_{0,\infty}$*.*

In case when both criteria are H_∞-norms, the level of suppression for two jointly acting deterministic disturbances takes the form

$$\gamma_{\infty,\infty}(\Theta) = \sup_{w\in\mathcal{P},v\in\mathcal{P}} \frac{\|z\|_{\mathcal{P}}}{\sqrt{\|w\|_{\mathcal{P}}^2 + \rho^2\|v\|_{\mathcal{P}}^2}} = \sup_{\zeta\in\mathcal{P}} \frac{\|z\|_{\mathcal{P}}}{\|\xi\|_{\mathcal{P}}}, \tag{5.10}$$

where $\xi = \operatorname{col}(w, \rho^{-1}v)$. This is the H_∞-norm of the system's transfer matrix from ξ to z, which is characterized by the corresponding linear matrix inequality.

In case when there are two stochastic disturbances, and both criteria are γ_0-norms, the level of suppression for these jointly acting disturbances takes the form

$$\gamma_{0,0}(\Theta) = \operatorname{ess}\sup_{w\in\mathcal{G},v\in\mathcal{G}} \frac{\|z\|_{\mathcal{P}}}{\sqrt{\|w\|_{\mathcal{P}}^2 + \rho^2\|v\|_{\mathcal{P}}^2}} = \sup_{\xi\in\mathcal{G}} \frac{\|z\|_{\mathcal{P}}}{\|\xi\|_{\mathcal{P}}}, \tag{5.11}$$

where $\xi = \operatorname{col}(w, \rho^{-1}v)$. This level equals the γ_0-norm of the corresponding transfer matrix of the system.

6. $\gamma_{0,\infty}$-OPTIMAL CONTROLLER AND FILTER

We now return to the controllable system

$$\begin{aligned} x(t+1) &= Ax(t) + B_w w(t) + B_v v(t) + B_u u(t), \quad x(0) = x_0, \\ z(t) &= C_1 x(t) + D_w w(t) + D_v v(t) + D_u u(t). \end{aligned} \tag{6.1}$$

We define the $\gamma_{0,\infty}$-optimal control in the class of linear feedbacks $u(t) = \Theta x(t)$ for which the closed system will be asymptotically stable, and the level of suppression for stochastic and deterministic disturbances for a given ρ will be minimal, i.e.,

$$\min_{\Theta} \gamma_{0,\infty}(\Theta) = \gamma_{0,\infty}(\Theta_\rho^*).$$

Substituting matrices of the closed system into inequality (5.7), multiplying the first and second inequality from the left and from the right by blockdiag (X^{-1}, X^{-1}, I, I) and blockdiag (X^{-1}, I, I) respectively, introducing new variables $Y = X^{-1}$ and $Z = \Theta Y$, due to corollary we arrive at the following.

Theorem 6.1. *The matrix of parameters for a* $\gamma_{0,\infty}$*-optimal controller can be computed as* $\Theta^*_\rho = ZY^{-1}$*, where* (Y, Z) *is the solution of problem* $\min \gamma^2$ *under constraints*

$$\begin{pmatrix} -Y & AY + B_u Z & B_v & 0 \\ \star & -Y & 0 & YC_1^{\mathrm{T}} + Z^{\mathrm{T}} D_u^{\mathrm{T}} \\ \star & \star & -\rho^2\gamma^2 I & 0 \\ \star & \star & \star & -I \end{pmatrix} < 0,$$
$$\begin{pmatrix} -Y & B_w & 0 \\ \star & -\gamma^2 I & D_w^{\mathrm{T}} \\ \star & \star & -I \end{pmatrix} < 0. \tag{6.2}$$

Let $\gamma_{0,\infty}(\Theta)$ denote the level of suppression for stochastic and deterministic disturbances in system

$$\begin{gathered} e(t+1) = (A - \Theta C_2)e(t) + (B_1 - \Theta D_{21})w(t) + (B_2 - \Theta D_{22})v(t), \quad e(0) = x_0, \\ e_z(t) = C_z e(t), \end{gathered} \tag{6.3}$$

which describes the dynamics of error filtering with output e_z. We define a $\gamma_{0,\infty}$-optimal filter for which $\gamma_{0,\infty}(\Theta)$ is minimal, i.e.,

$$\min_{\Theta} \gamma_{0,\infty}(\Theta) = \gamma_{0,\infty}(\Theta^*_\rho).$$

Substituting matrices of this system into inequality (5.7) and introducing auxiliary variables $Z = X\Theta$, we immediately get the following result.

Theorem 6.2. *The matrix of parameters for a* $\gamma_{0,\infty}$*-optimal filter can be found as* $\Theta^*_\rho = X^{-1}Z$*, where* (X, Z) *is the solution of problem* $\min \gamma^2$ *under constraints*

$$\begin{pmatrix} -X & XA - ZC_2 & XB_2 - ZD_{22} & 0 \\ \star & -X & 0 & C_z^{\mathrm{T}} \\ \star & \star & -\rho^2\gamma^2 I & 0 \\ \star & \star & \star & -I \end{pmatrix} < 0,$$
$$\begin{pmatrix} -X & XB_1 - ZD_{21} \\ \star & -\gamma^2 I \end{pmatrix} < 0. \tag{6.4}$$

According to Theorem 4.1, the relative losses of a $\gamma_{0,\infty}$-optimal filter do not exceed the value $1 - \mu(\rho)$, where

$$\mu(\rho) = \frac{\sqrt{\|w_*\|^2_{\mathcal{P}} + \rho^2 \|v_*\|^2_{\mathcal{P}}}}{\|w_*\|_{\mathcal{P}} + \rho \|v_*\|_{\mathcal{P}}}, \tag{6.5}$$

$v_*(t) = Lx(t)$ and $w_*(t)$ with $\|w_*\|^2_{\mathcal{P}} = \operatorname{tr} K_* = 1$ have been defined in Theorem 5.3. Note that $\|v_*\|^2_{\mathcal{P}} = \operatorname{tr}(L^{\mathrm{T}} L X_0)$, where $X_0 = X_0^{\mathrm{T}} \geqslant 0$ is the solution of Lyapunov equations

$$A_v X A_v^{\mathrm{T}} - X + B_w K_* B_w^{\mathrm{T}} = 0 \tag{6.6}$$

for matrices A_v and B_w that corresponds to the $\gamma_{0,\infty}$-optimal filter.

7. RELATION TO THE H_2/H_∞-OPTIMIZATION PROBLEM

Let us point out a difference in this context between criterion $\gamma_{0,\infty}$ and quality criterion H_2/H_∞. Suppose that in system (3.1) with deterministic and stochastic disturbances the white noise covariance matrix is known and is the unit matrix. Consider the value

$$\sup_{v\in\mathcal{P}}\left(\|z\|_{\mathcal{P}}^2-\rho^2\gamma^2\|v\|_{\mathcal{P}}^2\right),$$

assuming that constraint $\|H_v\|_\infty<\rho\,\gamma$ holds. By the proof of Theorem 5.2, we get that it is equal to $\operatorname{tr}(B_w^{\mathrm T}X_\gamma B_w+D_w^{\mathrm T}D_w)$, where X_γ is the stabilizing solution of Eqs. (5.5). It is well known that $\|H_w\|_2^2=\operatorname{tr}(B_w^{\mathrm T}P_0B_w+D_w^{\mathrm T}D_w)$, where P_0 is the solution of Lyapunov equations (5.2). By the monotonicity property of the solutions of Lyapunov equations we have that $0\leqslant P_0\leqslant X_\gamma\leqslant X$, where $X=X^{\mathrm T}>0$ satisfy the first inequality in (5.7). Consequently, $\|H_w\|_2^2\leqslant\operatorname{tr}(B_w^{\mathrm T}XB_w+D_w^{\mathrm T}D_w)$. Therefore, the ratio H_2/H_∞ can be found as

$$\gamma_{2,\infty}^2(\Theta)=\inf\left\{\mu^2:\operatorname{tr}\left(B_w^{\mathrm T}XB_w+D_w^{\mathrm T}D_w\right)\leqslant\mu^2\right\},\qquad(7.1)$$

where $X=X^{\mathrm T}>0$ satisfies the first inequality in (5.7). This definition implies that $\gamma_{2,\infty}$ depends on γ and that $\|H_w\|_2\leqslant\gamma_{2,\infty}$ and $\lim_{\gamma\to\infty}\gamma_{2,\infty}=\|H_w\|_2$.

Thus, while $\gamma_{0,\infty}$ serves as a criterion characterizing the worst possible system reaction to random disturbance with unknown covariance in one channel and deterministic disturbance in the other channel, $\gamma_{2,\infty}$ characterizes the worst possible system reaction to white noise with unit covariance matrix in one channel under a given constraint on the level of suppression for deterministic disturbances in the other channel. This means that in situations when statistical characteristics of random disturbances are unknown, as a criterion for filter or controller synthesis it makes sense to choose $\gamma_{0,\infty}$ rather than $\gamma_{2,\infty}$.

Besides, the most important point is as follows. It is well known that in the paradigm of multicriteria H_2/H_∞-control, objectives are usually formulated in terms of the general Lyapunov's function. This introduces conservatism but allows one to design a controller. However, the problem of how conservative the resulting solutions are or, in other words, how much the quality criteria of the resulting systems differ from Pareto optimal values, has remained open. Taking into account (7.1) and corollary, where we show a procedure for computing $\gamma_{0,\infty}$, it is easy to see that both these parameters for a corresponding value of ρ lead to the same optimal solution. This lets us conclude that relative losses in replacing an unknown Pareto optimal control with the H_2/H_∞-optimal control do not exceed the value $1-\sqrt{2}/2$.

8. DUAL CONTROLS AND FILTERING PROBLEMS

We have already considered two-criteria control and filtering problems that reduce to analyzing levels of suppression for disturbances in systems with a single objective output and two inputs: one for stochastic disturbances and another for deterministic disturbances. One can also consider two-criteria control and filtering problems for a system with one input that receives either stochastic or deterministic disturbance, and two target outputs so that we evaluate the level of suppression for stochastic disturbances with respect to one of them and deterministic disturbances with respect to the other. These considerations lead to a system of the form

$$\begin{aligned}x(t+1)&=A(\Theta)x(t)+B(\Theta)\zeta(t),\quad x(0)=x_0,\\ z_1(t)&=C_1(\Theta)x(t)+D_1(\Theta)\zeta(t),\\ z_2(t)&=C_2(\Theta)x(t)+D_2(\Theta)\zeta(t),\end{aligned}\qquad(8.1)$$

where $z_1 \in R^{n_{z_1}}$ and $z_2 \in R^{n_{z_2}}$ are two target outputs. With respect to disturbances $\zeta(t) \in R^{n_\xi}$, we assume that they represent a deterministic sequence with bounded energy, i.e., $\zeta(t) \in l_2$.

The level of suppression for disturbances with respect to the output z_1 can be estimated with

$$J_1(\Theta) = \sup_{\zeta(t)\in l_2} \frac{\sup_{t\geqslant 0} |z_1|}{\|\zeta\|_2} = \|H_1\|_{g2}, \tag{8.2}$$

where $H_1(s) = C_1(sI - A)^{-1}B + D_1$, $\|\cdot\|_{g2}$ is one of the ways to generalize the H_2-norm considered in [32, 33]. We remind that

$$\|H_1\|_{g2} = \lambda_{\max}^{1/2}\left(C_1 Y C_1^{\mathrm{T}} + D_1 D_1^{\mathrm{T}}\right), \tag{8.3}$$

where $Y = Y^{\mathrm{T}} > 0$ is a solution of Lyapunov equations

$$AYA^{\mathrm{T}} - Y + BB^{\mathrm{T}} = 0. \tag{8.4}$$

The level of suppression for disturbances with respect to the output z_2 is estimated by

$$J_2(\Theta) = \sup_{\zeta(t)\in \mathcal{P}} \frac{\|z_2\|_{\mathcal{P}}}{\|\zeta\|_{\mathcal{P}}} = \|H_2\|_\infty, \tag{8.5}$$

where $H_2(s) = C_2(sI - A)^{-1}B + D_2$. The problem is to characterize the Pareto set for these two criteria

$$\Theta_P = \arg\min_{\Theta}\{\|H_1(\Theta)\|_{g2}, \|H_2(\Theta)\|_\infty\}. \tag{8.6}$$

We write equations of the system dual to system (8.1), with two inputs and one output:

$$\begin{aligned} x(t+1) &= A^{\mathrm{T}}(\Theta)x(t) + C_1^{\mathrm{T}}(\Theta)w(t) + C_2^{\mathrm{T}}(\Theta)v(t), \quad x(0) = x_0,\\ z(t) &= B^{\mathrm{T}}(\Theta)x(t) + D_1^{\mathrm{T}}(\Theta)w(t) + D_2^{\mathrm{T}}(\Theta)v(t). \end{aligned} \tag{8.7}$$

In this system, transfer matrices of the two channels are symmetric to transfer matrices corresponding to channels in the system. One can immediately check that $\gamma_0(H_1^{\mathrm{T}}) = \|H_1\|_{g2}$, and since $\|H_2^{\mathrm{T}}\|_\infty = \|H_2\|_\infty$, we arrive at the two-criteria problem considered above.

9. GENERALIZATION FOR N CRITERIA

Consider a system with N vector inputs and one vector objective output

$$\begin{aligned} x(t+1) &= A_c(\Theta)x(t) + \sum_{i=1}^{r} B_w^{(i)}(\Theta)w_i(t) + \sum_{j=r+1}^{N} B_v^{(j)}(\Theta)v_j(t), \quad x(0) = x_0,\\ z(t) &= C_z(\Theta)x(t) + \sum_{i=1}^{r} D_w^{(i)}(\Theta)w_i(t) + \sum_{j=r+1}^{N} D_v^{(j)}(\Theta)v_j(t), \end{aligned} \tag{9.1}$$

where $w_i(t) \in R^{n_{w_i}}$ are stationary Gaussian random sequences of vectors with zero expectation and unknown covariance matrices that for all t are uncorrelated, where x_0 and $v_j(t) \in R^{n_{v_j}}$ are deterministic sequences of vectors with bounded power. If the disturbance acts only on the ith input, and there are no disturbances on the rest of the inputs, the level of suppression for the ith stochastic disturbances can be found as

$$\gamma_0^{(i)}(\Theta) = \sup_{w_i\in\mathcal{G}_{n_{w_i}}} \frac{\sqrt{J_z}}{\sqrt{J_{w_i}}}, \quad i = 1, \ldots, r,$$

and if the disturbance acts only on the jth input, the level of suppression for the jth deterministic disturbance can be found as

$$\gamma_\infty^{(j)}(\Theta) = \sup_{\|v_j\|_\mathcal{P} \neq 0} \frac{\|z\|_\mathcal{P}}{\|v_j\|_\mathcal{P}}, \quad j = r+1, \ldots, N.$$

Under jointly acting stochastic and deterministic disturbances, we define on trajectories of system (9.1) the criterion

$$J_\alpha(\Theta) = \operatorname{ess} \sup_{w_i \in \mathcal{G}, v_j \in \mathcal{P}, \forall\, i,j} \frac{\|z\|_\mathcal{P}}{\sum_{i=1}^r \alpha_i \|w_i\|_\mathcal{P} + \sum_{j=r+1}^N \alpha_j \|v_j\|_\mathcal{P}}, \tag{9.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\sum_1^N \alpha_k = 1$, $\forall\, \alpha_k > 0$.

Similar to Theorem 3.1, we can show that necessary Pareto optimality conditions in the above multi-criteria problem are as follows: if $(\gamma_1, \ldots, \gamma_N)$ is a Pareto optimal point in the space of criteria $\gamma_0^{(1)}, \ldots, \gamma_0^{(r)}, \gamma_\infty^{(r+1)}, \ldots, \gamma_\infty^{(N)}$ for system (9.1), then there exists a matrix of parameters Θ_α that minimizes criterion (9.2) for $\alpha_k = \gamma_k / \sum_1^N \gamma_k$, $k = 1, \ldots, N$ such that $\gamma_0^{(i)}(\Theta_\alpha) = \gamma_i$, $\gamma_\infty^{(j)}(\Theta_\alpha) = \gamma_j$, $i = 1, \ldots, r$, $j = r+1, \ldots, N$.

We define on trajectories of system (9.1) the level of suppression for stochastic and deterministic disturbances

$$\gamma_\alpha(\Theta) = \operatorname{ess} \sup_{w_i \in \mathcal{G}, v_j \in \mathcal{P}, \forall\, i,j} \frac{\|z\|_\mathcal{P}}{\sqrt{\sum_{i=1}^r \alpha_i^2 \|w_i\|_\mathcal{P}^2 + \sum_{j=r+1}^N \alpha_j^2 \|v_j\|_\mathcal{P}^2}}. \tag{9.3}$$

We can establish in a similar way that under the assumption that there exists a solution of the corresponding minimax problem (see (4.5)), the relative losses in quality for the choice of Θ_α^*-optimal solutions with respect to criterion $\gamma_\alpha(\Theta)$ compared to the corresponding Pareto optimal solutions do not exceed $1 - \mu(\alpha)$ with respect to each of the criteria, where

$$\mu(\alpha) = \frac{\sqrt{\sum_{i=1}^r \alpha_i^2 \|w_i^*\|_\mathcal{P}^2 + \sum_{j=r+1}^N \alpha_j^2 \|v_j^*\|_\mathcal{P}^2}}{\sum_{i=1}^r \alpha_i \|w_i^*\|_\mathcal{P} + \sum_{j=r+1}^N \alpha_j \|v_j^*\|_\mathcal{P}}, \quad \frac{1}{\sqrt{N}} \leqslant \mu(\alpha) \leqslant 1,$$

and w_i^*, v_j^* are the worst possible disturbances in (9.3) for $\Theta = \Theta_\alpha^*$. The optimal solution and the worst possible disturbances with respect to the suppression level for joint disturbances can be found by Theorem 5.2, its corollary, and Theorem 5.3, where $\rho = 1$, as

$$B_w = \left(\alpha_1^{-1} B_w^{(1)} \ \alpha_2^{-1} B_w^{(2)} \ \ldots \ \alpha_r^{-1} B_w^{(r)}\right), \quad D_w = \left(\alpha_1^{-1} D_w^{(1)} \ \alpha_2^{-1} D_w^{(2)} \ \ldots \ \alpha_r^{-1} D_w^{(r)}\right),$$
$$B_v = \left(\alpha_{r+1}^{-1} B_v^{(r+1)} \ \alpha_{r+2}^{-1} B_v^{(r+2)} \ \ldots \ \alpha_N^{-1} B_v^{(N)}\right), \quad D_v = \left(\alpha_{r+1}^{-1} D_v^{(r+1)} \ \alpha_{r+2}^{-1} D_v^{(r+2)} \ \ldots \ \alpha_N^{-1} D_v^{(N)}\right).$$

10. CONTROL AND FILTERING FOR EXTERNAL AND INITIAL DISTURBANCES

Consider an internally stable system

$$\begin{gathered} x(t+1) = A_c(\Theta) x(t) + B_w(\Theta) w(t) + B_v(\Theta) v(t), \quad x(0) = 0, \\ z(t) = C_z(\Theta) x(t) + D_w(\Theta) w(t) + D_v(\Theta) v(t), \end{gathered} \tag{10.1}$$

where $v(t) \in l_2$ is an external deterministic disturbance with bounded norm $\|v\|_2 = (\sum_{t=0}^\infty |v(t)|^2)^{1/2}$, $w(0) = w_0$, $w(t) = 0$, $t = 1, 2, \ldots$, and w_0 is an unknown initial deterministic disturbance.

If only disturbance $v(t)$ is present, we characterize the system's quality with the level of suppression for l_2-disturbances equal to the maximal value of the ratio of the l_2-norms of target output and disturbances, or, in other words, the H_∞-norm of the transfer matrix of this system from v to z, i.e.,

$$\gamma_\infty(\Theta) = \sup_{v\in l_2, w_0=0} \frac{\|z\|_2}{\|v\|_2}.$$

The H_∞-optimal controller and filter are optimal with respect to this criterion.

If only the initial disturbance is present, we define the level of suppression for the initial disturbance as the maximal value of the ratio of the l_2-norm of the target output to the Euclidean norm of initial disturbances that have caused this output, i.e.,

$$\gamma_0(\Theta) = \sup_{w_0\in R^{n_w}, v\equiv 0} \frac{\|z\|_2}{|w_0|}.$$

Now (10.1) immediately implies that for $v(t)\equiv 0$ we have that

$$z(0) = D_w w_0, \quad z(t) = C_z A_c^{t-1} B_w w_0, \quad t = 1, 2, \ldots,$$

so

$$\|z\|_2^2 = w_0^{\mathrm{T}} \left\{ D_w^{\mathrm{T}} D_w + B_w^{\mathrm{T}} \left[\sum_{t=1}^{\infty} (A_c^{\mathrm{T}})^{t-1} C_z^{\mathrm{T}} C_z A_c^{t-1} \right] B_w \right\} w_0.$$

Consequently, $\gamma_0^2(\Theta) = \lambda_{\max}(B_w^{\mathrm{T}} P_0 B_w + D_w^{\mathrm{T}} D_w)$, where $P_0 = P_0^{\mathrm{T}} \geqslant 0$ is a solution of Lyapunov equations (5.2). In particular, as the initial disturbances one can take an unknown initial system state, and in this case $w_0 = x(0)$, $B_w = A_c$, $D_w = C_z$. The so-called γ_0-optimal controller [34] and filter are optimal with respect to this criterion. If the initial state plays the role of unknown initial disturbances, this is a problem about an optimal linear–quadratic controller that ensures the best transition process under the worst initial conditions.

Thus, two-criteria control or filtering problems under external and initial disturbances can be reduced to the problems considered in the work, and all our conclusions regarding Pareto optimal solutions hold for these problems as well. Here the level of suppression for jointly acting external and initial disturbances can be found as

$$\gamma_{0,\infty}(\Theta) = \sup_{w_0\in R^{n_w}, v\in l_2} \frac{\|z\|_2}{\sqrt{|w_0|^2 + \rho^2\|v\|_2^2}},$$

computed according to Theorem 5.2 and its corollary. Parameters of the $\gamma_{0,\infty}$-optimal controller, which has been called in [29] a generalized H_∞-optimal controller, and a filter that provide a tradeoff between the quality of the transition process and the level of suppression for an external disturbance can be found according to Theorems 6.1 and 6.2.

11. EXAMPLE

For an illustration, we consider two bicriterial control problems for system

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 1 & 0.1 \\ -1 & 0.99 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} \zeta(t) + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} u(t), \\ z_1(t) &= (1 \quad 0) x(t), \\ z_2(t) &= u(t) \end{aligned} \tag{11.1}$$

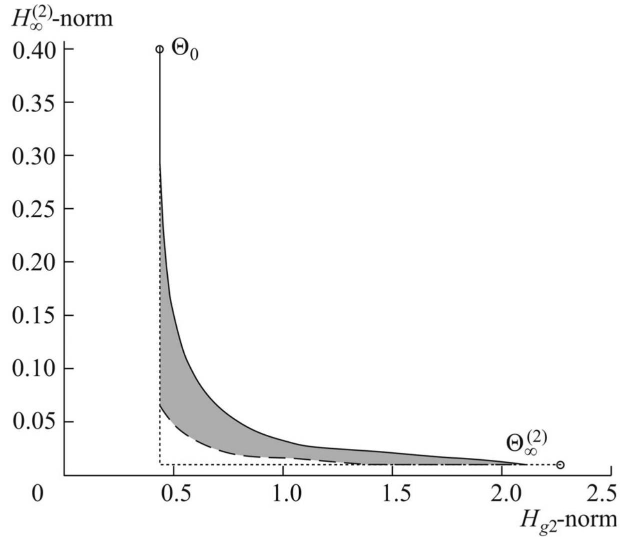

Fig. 1. The Pareto set on the plane of H_{g2}- and H_∞-criteria.

in the class of linear feedbacks $u = \Theta x$. We compose the following three criteria corresponding to classes of disturbances:

$$J_1(\Theta) = \sup_{\zeta \in \mathcal{P}} \frac{\|z_2\|_\mathcal{P}}{\|\zeta\|_\mathcal{P}} = \|H^{(2)}\|_\infty,$$
$$J_2(\Theta) = \sup_{\zeta \in \mathcal{P}} \frac{\|z_1\|_\mathcal{P}}{\|\zeta\|_\mathcal{P}} = \|H^{(1)}\|_\infty, \tag{11.2}$$
$$J_3(\Theta) = \sup_{\zeta(t) \in l_2} \frac{\sup\limits_{t \geqslant 0} |z_2|}{\|\zeta\|_2} = \|H^{(2)}\|_{g2},$$

where $H^{(1)}$ and $H^{(2)}$ are transfer matrices of the closed system from ζ to z_1 and z_2 respectively.

The first problem is to characterize the Pareto set

$$\Theta_P = \arg\min_\Theta \{J_3(\Theta),\ J_2(\Theta)\}. \tag{11.3}$$

Optimal values with respect to each of these criteria are achieved under the following parameters:

$$\Theta_0 = (0.826 \quad -1.643), \quad \Theta_\infty^{(2)} = (-90.0 \quad -19.9),$$

and they are equal to $J_3(\Theta_0) = 0.434$ and $J_2(\Theta_\infty^{(2)}) = 0.01$. Points denoted by Θ_0 and $\Theta_\infty^{(2)}$ on the plane of criteria (J_3, J_2) correspond to these controllers (see Fig. 1). According to Section 8, we find the level of suppression for jointly acting disturbances for the dual system. The Pareto set is located in the shaded area shown on Fig. 1. The boundary of this region consists of two straight lines corresponding to minimal values of each criterion and two curved lines. Coordinates of the solid line are $(J_3(\Theta_\alpha^*), J_2(\Theta_\alpha^*))$, where Θ_α^* are $\gamma_{0,\infty}$-optimal solutions, and coordinates of the dashed line are defined as $(\mu(\alpha)J_3(\Theta_\alpha^*), \mu(\alpha)J_2(\Theta_\alpha^*))$.

The second problem is to characterize the Pareto set

$$\Theta_P = \arg\min_\Theta \{J_1(\Theta),\ J_2(\Theta)\}. \tag{11.4}$$

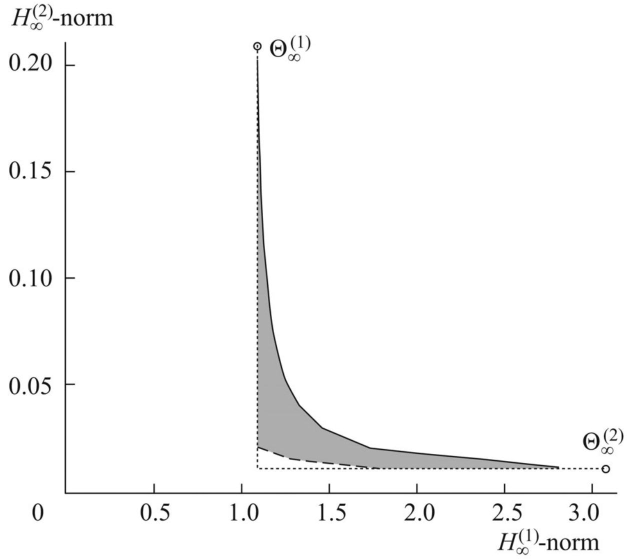

Fig. 2. The Pareto set on the plane two H_∞-criteria.

The optimal values for each of these criteria are

$$\Theta_\infty^{(1)} = (5.215 \quad -10.379), \quad \Theta_\infty^{(2)} = (-90.0 \quad -19.9),$$

for which $J_1(\Theta_\infty^{(1)}) = 1.09$ and $J_2(\Theta_\infty^{(2)}) = 0.01$. Points denoted by $\Theta_\infty^{(1)}$ and $\Theta_\infty^{(2)}$ on the plane of criteria (J_1, J_2) correspond to these controllers (see Fig. 2). The Pareto set for this problem is located in the shaded area shown on this figure.

12. CONCLUSION

In this work, we have developed a novel approach that lets one design Pareto suboptimal solutions in multi-criteria control and filtering problems under deterministic and stochastic disturbances. We have introduced a scalar objective function that reflects the level of suppression for jointly acting disturbances whose optimal solutions are characterized in terms of solutions for Riccati equations or linear matrix inequalities. We have shown that relative losses of these solutions with respect to each of the original criteria compared to the Pareto optimal solutions do not exceed the value $1 - \sqrt{N}/N$ for N criteria.

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APPENDIX

We begin by proving a technical lemma.

Lemma A.1. *Inequality* $\mathrm{tr}\,(\Xi K_w) \leqslant 0$ *holds for all* $K_w = K_w^{\mathrm{T}} \geqslant 0$ *if and only if* $\Xi \leqslant 0$.

Proof of Lemma A.1. We represent a symmetric matrix as $\Xi = \sum_{i=1}^{n_w} \lambda_i e_i e_i^{\mathrm{T}}$, where λ_i are its eigenvalues, and eigenvectors e_i, $i = 1, \ldots, n_w$, form an orthonormal basis. If we assume that there exist $\lambda_j \geqslant 0$, then by choosing $K_w = \lambda_j e_j e_j^{\mathrm{T}}$ we get that $\mathrm{tr}\,(K_w \Xi) = \lambda_j^2 \leqslant 0$. This means that $\lambda_j = 0$ and, consequently, $\Xi \leqslant 0$.

Conversely, if $\Xi \leqslant 0$ and the covariance matrix is represented as $K_w = \sum \mu_i f_i f_i^{\mathrm{T}}$, where $\mu_i \geqslant 0$ and f_i are its eigenvalues and eigenvectors, then $\mathrm{tr}\,(\Xi K_w) = \sum \mu_i f_i^{\mathrm{T}} \Xi f_i \leqslant 0$.

Proof of Theorem 5.1. By definition, we have to find a minimal value of γ^2 such that

$$J_z \leqslant \gamma^2 \mathrm{tr}\, K_w \quad \forall\, K_w \neq 0.$$

For system (3.1), for $v \equiv 0$ it holds that

$$J_z = \lim_{t\to\infty} E|z(t)|^2 = \mathrm{tr}\, \left[K_w(B_w^{\mathrm{T}} P_0 B_w + D_w^{\mathrm{T}} D_w)\right],$$

where $P_0 = P_0^{\mathrm{T}} \geqslant 0$ satisfies Eq. (5.2). Therefore, we have

$$\mathrm{tr}\, \left[K_w(B_w^{\mathrm{T}} P_0 B_w + D_w^{\mathrm{T}} D_w - \gamma^2 I)\right] \leqslant 0 \quad \forall\, K_w \neq 0.$$

Applying Lemma A.1, we get that $\gamma_0^2 = \lambda_{\max}(B_w^{\mathrm{T}} P_0 B_w + D_w^{\mathrm{T}} D_w)$. Here for $K_* = e_{\max} e_{\max}^{\mathrm{T}}$ we have that

$$\mathrm{tr}\, \left[K_*(B_w^{\mathrm{T}} P_0 B_w + D_w^{\mathrm{T}} D_w - \gamma_0^2 I)\right] = 0.$$

Besides, since

$$J_z = \frac{1}{2\pi} \int\limits_0^{2\pi} \mathrm{tr}\, \{H_w(e^{j\varphi}) H_w^{\mathrm{T}}(e^{-j\varphi})\}\, d\varphi,$$

γ_0^2 equals the minimal value of γ^2 for which

$$J_z - \gamma^2 J_w = \mathrm{tr}\, \left\{ \left[\frac{1}{2\pi} \int\limits_0^{2\pi} H_w^{\mathrm{T}}(e^{-j\varphi}) H_w(e^{j\varphi})\, d\varphi - \gamma^2 I \right] K_w \right\} \leqslant 0 \quad \forall\, K_w \neq 0.$$

Applying Lemma A.1, we arrive at the necessary conclusion.

Proof of Theorem 5.2. The proof is based on the following auxiliary statement.

Lemma A.2. *Let $V(x) = x^{\mathrm{T}} X_\gamma x$, where $X_\gamma = X_\gamma^{\mathrm{T}} > 0$ is a stabilizing solution of Eqs.* (5.5). *Then*

$$\begin{aligned} \|z\|_{\mathcal{P}}^2 - \rho^2\gamma^2 \|v\|_{\mathcal{P}}^2 &= \lim_{N\to\infty} (1/N) \sum_{t=0}^{N-1} w^{\mathrm{T}}(t)(B_w^{\mathrm{T}} X_\gamma B_w + D_w^{\mathrm{T}} D_w) w(t) \\ &- \lim_{N\to\infty} (1/N) \sum_{t=0}^{N-1} [v(t) - v_*(t)]^{\mathrm{T}} (\gamma^2 I - B_v^{\mathrm{T}} X_\gamma B_v - D_v^{\mathrm{T}} D_v)[v(t) - v_*(t)] \\ &\leqslant \mathrm{tr}\, \left[K_w(B_w^{\mathrm{T}} X_\gamma B_w + D_w^{\mathrm{T}} D_w)\right], \end{aligned} \tag{A.1}$$

where $v_*(t) = (\rho^2\gamma^2 I - B_v^{\mathrm{T}} X_\gamma B_v - D_v^{\mathrm{T}} D_v)^{-1} (B_v^{\mathrm{T}} X_\gamma A + D_v^{\mathrm{T}} C) x(t)$.

Lemma A.2 can be proven by immediate computation of the increment ΔV along the trajectory of system (3.1), using operation $\|\cdot\|_{\mathcal{P}}$ and extracting a full square.

We now proceed to proving the theorem. According to Lemma A.1, condition (5.4) means that $\mathrm{tr}\,[K_w(B_w^{\mathrm{T}} X_\gamma B_w + D_w^{\mathrm{T}} D_w)] \leqslant \gamma^2 \mathrm{tr}\, K_w$ for every K_w, and, consequently, by Lemma A.2 we get that $\gamma_{0,\infty}^2 < \gamma^2$.

Suppose now that $\gamma_{0,\infty} < \gamma$. Since $\rho^{-1}\gamma_\infty \leqslant \gamma_{0,\infty}$, we have that $\|H_v\|_\infty < \rho\gamma$. By the frequency theorem [35] it follows that there exists a stabilizing solution $X_\gamma = X_\gamma^{\mathrm{T}} > 0$ for Eqs. (5.5), and it holds that

$$\|z\|_{\mathcal{P}}^2 - \rho^2\gamma^2\|v_*\|_{\mathcal{P}}^2 = \mathrm{tr}\,\left[K_w(B_w^{\mathrm{T}}X_\gamma B_w + D_w^{\mathrm{T}}D_w)\right]. \qquad \text{(A.2)}$$

Let us show that matrix X_γ satisfies inequality (5.4). Assume the opposite, i.e., suppose there exists a $a \neq 0$ such that $a^{\mathrm{T}}(B_w^{\mathrm{T}}X_\gamma B_w + D_w^{\mathrm{T}}D_w)a \geqslant \gamma^2|a|^2$. We take the covariance matrix $\bar{K}_w = aa^{\mathrm{T}}$ in such a way that $\|z\|_{\mathcal{P}}^2 - \rho^2\gamma^2\|v_*\|_{\mathcal{P}}^2 = a^{\mathrm{T}}(B_w^{\mathrm{T}}X_\gamma B_w + D_w^{\mathrm{T}}D_w)a \geqslant \gamma^2\mathrm{tr}\,\bar{K}_w$. This implies that $\gamma_{0,\infty}^2 \geqslant \gamma^2$, which contradicts the assumption. Thus, $B_w^{\mathrm{T}}X_\gamma B_w + D_w^{\mathrm{T}}D_w < \gamma^2 I$ and, consequently, $\lambda_{\max}(B_w^{\mathrm{T}}X_\gamma B_w + D_w^{\mathrm{T}}D_w) < \gamma^2$.

Proof of Theorem 5.3. Due to remark, in the considered case $\rho^{-1}\gamma_\infty < \gamma_{0,\infty}$ and, consequently, for $\gamma = \rho\gamma_{0,\infty}$ there exists a X_*, a stabilizing solution for Eqs. (5.5). Then equality

$$\|z\|_{\mathcal{P}}^2 - \rho^2\gamma_{0,\infty}^2\|v_*\|_{\mathcal{P}}^2 = \mathrm{tr}\,\left[K_w(B_w^{\mathrm{T}}X_* B_w + D_w^{\mathrm{T}}D_w)\right]$$

holds under disturbance (5.9). This implies that

$$\frac{\|z\|_{\mathcal{P}}^2}{\mathrm{tr}\,K_w + \rho^2\|v_*\|_{\mathcal{P}}^2} = \gamma_{0,\infty}^2 + \frac{\mathrm{tr}\,[K_w(B_w^{\mathrm{T}}X_* B_w + D_w^{\mathrm{T}}D_w - \gamma_{0,\infty}^2 I)]}{\mathrm{tr}\,K_w + \rho^2\|v_*\|_{\mathcal{P}}^2}.$$

Note that $\mathrm{tr}\,[K_w(B_w^{\mathrm{T}}X_* B_w + D_w^{\mathrm{T}}D_w - \gamma_{0,\infty}^2 I)] \leqslant 0$, since

$$\frac{\|z\|_{\mathcal{P}}^2}{\mathrm{tr}\,K_w + \rho^2\|v_*\|_{\mathcal{P}}^2} \leqslant \gamma_{0,\infty}^2. \qquad \text{(A.3)}$$

Let us show that for K_w in the form $K_w = aa^{\mathrm{T}}$ it holds that

$$\mathrm{tr}\,\left[K_w(B_w^{\mathrm{T}}X_* B_w + D_w^{\mathrm{T}}D_w - \gamma_{0,\infty}^2 I)\right] = a^{\mathrm{T}}\left(B_w^{\mathrm{T}}X_* B_w + D_w^{\mathrm{T}}D_w - \gamma_{0,\infty}^2 I\right)a = 0.$$

Indeed, suppose that $a^{\mathrm{T}}(B_w^{\mathrm{T}}X_* B_w + D_w^{\mathrm{T}}D_w - \gamma_{0,\infty}^2 I)a < 0$ for all $a \neq 0$. Then by Theorem 5.2 the level of suppression for joint disturbances will be less than $\gamma_{0,\infty}$, which contradicts the assumption. Thus, for $K_w = e_{\max}e_{\max}^{\mathrm{T}}$, where $(B_w^{\mathrm{T}}X_* B_w + D_w^{\mathrm{T}}D_w)e_{\max} = \gamma_{0,\infty}^2 e_{\max}$, and inequality (A.3) turns into an equality. Note that the signal $v_*(t)$ is not unique since every signal of the form $v_*(t) + g(t)$ with $g(t) \in l_2$ leads to the same value $\gamma_{0,\infty}$.

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